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A PROPERTY OF EIGENVECTORS OF NONNEGATIVE SYMMETRIC MATRICES AND ITS APPLICATION TO GRAPH THEORY

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A theorem relating the eigenvectors of a nonnegative symmetric matrix A with the degrees of reducibility of some principal submatrices of A is proved and applied in the theory of algebraic connectivity of non-directed graphs.

1. Notation and algebraic preliminaries. In the whole paper, all numbers will be real, n will be an integer, $n \geqq 2$, and N will denote the set $\{1, 2, \ldots, n\}$. If A is an $n \times n$ matrix and M a proper nonvoid subset of N, we shall denote by $A(M)$ that principal submatrix of A the indices of the rows (and columns) of which belong to M. The transpose matrix to A will be denoted by A^T, the identity matrix by I. A vector is always considered as a column vector. By the inner product of two vectors x and y, denoted by (x, y), we mean as usual the number $y^T x$. In the third section, e will always mean the vector $(1, 1, \ldots, 1)^T$, with n ones.

As usual, an $n \times n$ matrix $A = (a_{ik})$ is called *irreducible* if for no decomposition of N into two non-void subsets N_1, N_2, $a_{ik} = 0$ whenever $i \in N_1$, $k \in N_2$. We shall investigate here the case of symmetric real matrices only. For such matrices, we shall speak about *degree of reducibility* in the following sense:

A symmetric $n \times n$ matrix $A = (a_{ik})$ is of degree of reducibility s, $0 \leqq s \leqq n - 1$ if there exists a decomposition of N into $s + 1$ non-void subsets $N = N_1 \cup N_2 \cup \ldots \cup N_{s+1}$ such that

(i) $A(N_i)$ are irreducible, $i = 1, \ldots, s + 1$,

(ii) $a_{pq} = 0$ whenever $p \in N_i$, $q \in N_j$, $i \neq j$.

This means, of course, that there exists an $n \times n$ permutation matrix P such that PAP^T has the block-diagonal form

$$\begin{bmatrix} A(N_1) & 0 & \dots & 0 \\ 0 & A(N_2) & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & A(N_{s+1}) \end{bmatrix}.$$

An irreducible symmetric matrix is thus of degree of reducibility zero.

Given a symmetric matrix A, we shall define its signature $s(A)$ as the row vector $s(A) = (p, q)$ where p denotes the number of positive and q the number of negative eigenvalues of A. The following result is classical:

(1,1) *If A_1 is a principal submatrix of a symmetric matrix A then $s(A_1) \leqq s(A)$.*

A very easy consequence of this result is the following assertion:

(1,2) *Let A be an $n \times n$ symmetric matrix, $\lambda_1 \geqq \lambda_2 \geqq \dots \geqq \lambda_n$ its eigenvalues. Then no principal submatrix of the matrix $\lambda_s I - A$ $(s = 1, \dots, n)$ has more than $s - 1$ negative eigenvalues.*

We shall also need the notion of M-matrices or, equivalently, of matrices of class $\mathbf{K}$.

Let us recall (cf. [2]) that a real square matrix A belongs to the *class* $\mathbf{K}$ (or $\mathbf{K}_0$, respectively) iff all off-diagonal entries of A are nonpositive and all principal minors of A are positive (or nonnegative, respectively).

The following assertions have been proved in [2] and [7]:

(1,3) *A matrix from $\mathbf{K}_0$ belongs to $\mathbf{K}$ iff it is nonsingular.*

(1,4) *If $A \in \mathbf{K}$ then $A^{-1} \geqq 0$. If $A \in \mathbf{K}$ is irreducible then $A^{-1} > 0$.*

(1,5) *If $A \in \mathbf{K}_0$ is irreducible and singular then zero is a simple eigenvalue of A, there exists unique real vector (up to a factor) $u \neq 0$ such that $Au = 0$, and this vector is either positive, or negative.*

For symmetric matrices, from the definition of class $\mathbf{K}$ or $\mathbf{K}_0$ and the properties of positive definite or semidefinite matrices follows immediately:

(1,6) *A symmetric matrix belongs to $\mathbf{K}$(or $\mathbf{K}_0$, respectively) iff all its off-diagonal entries are nonpositive and all its eigenvalues positive (or nonnegative, respectively).*

From (1,6) and (1,5), the following assertion follows immediately:

(1,7) *If A symmetric irreducible has all off-diagonal entries nonpositive and $Az = 0$ for a real vector $z \neq 0$ which is neither positive nor negative then A is not positive semidefinite.*

2. Matrix-theoretical results. We shall prove first the following theorem:

(2,1) **Theorem.** *Let A be an $n \times n$ nonnegative irreducible symmetric matrix* *with eigenvalues* $\lambda_1 \geqq \lambda_2 \geqq \ldots \geqq \lambda_n$. *Let* $v = (v_i)$ *be a column vector such that for a fixed* $s \in N$, $s \geqq 2$, $Av \geqq \lambda_s v$.

If $M = \{i \in N \mid v_i \geqq 0\}$ *then* M *is non-void and the degree of reducibility of the matrix* $A(M)$ *does not exceed* $s - 2$.

Proof. Suppose first that M is void. Then $v < 0$ and the vector $z = -v$ satisfies $z > 0$ and

$$Az \leqq \lambda_s z \,. \tag{1}$$

Since $\lambda_1 I - A \in \boldsymbol{K}_0$ by (1,6) and is irreducible, there exists by (1,5) – or, of course, by the Perron-Frobenius theorem [5] – a vector $u > 0$ such that $A^T u = \lambda_1 u$, or equivalently,

$$u^T A = \lambda_1 u^T \,.$$

Then

$$u^T A z = \lambda_1 u^T z > \lambda_s u^T z \,,$$

since λ_1 is simple by (1,5). However, by (1)

$$u^T A z \leqq \lambda_s u^T z$$

which is a contradiction. Thus $M \neq \emptyset$.

If $M = N$, the theorem is true. Thus let $\emptyset \neq M \neq N$ and suppose that the degree of reducibility of $A(M)$ is at least $s - 1$. Without loss of generality, we can assume that $M = \{1, \ldots, m\}$, $m < n$, and that

$$A = \begin{bmatrix} A_{11}, & 0, & \ldots, 0, & A_{1,r+1} \\ 0, & A_{22}, & \ldots, 0, & A_{2,r+1} \\ \ldots & \ldots & \ldots & \ldots \\ 0, & 0, & \ldots, A_{rr}, & A_{r,r+1} \\ A^T_{1,r+1}, & A^T_{2,r+1} & \ldots, A^T_{r,r+1}, & A_{r+1,r+1} \end{bmatrix}$$

where $r \geqq s$ and A_{ii}, $i = 1, \ldots, r$, are irreducible, the sum of their dimensions being m. If the vector v is partitioned conformally,

$$v = \begin{bmatrix} v^{(1)} \\ v^{(2)} \\ \vdots \\ v^{(r)} \\ v^{(r+1)} \end{bmatrix}$$

then

$$v^{(i)} \geqq 0 \,, \quad i = 1, \ldots, r \,, \quad v^{(r+1)} < 0 \,. \tag{2}$$

According to the assumption in the theorem,

$$(3) \qquad (A_{ii} - \lambda_s I_i)\, v^{(i)} \geqq -A_{i,r+1} v^{(r+1)}, \quad i = 1, \ldots, r.$$

Since

$$\begin{bmatrix} \lambda_s I_1 - A_{11} & & & \\ & \cdot & & \\ & & \cdot & \\ & & & \cdot \\ & & & \lambda_s I_r - A_{rr} \end{bmatrix}$$

is a principal submatrix of $\lambda_s I - A$, it follows from (1,2) that it has at most $s - 1$ negative eigenvalues. Thus at least one of the matrices $\lambda_s I_i - A_{ii}$, say $\lambda_s I_1 - A_{11}$, has nonnegative eigenvalues only. By (1,6), $\lambda_s I_1 - A_{11} \in \mathbf{K}_0$ and is irreducible. Suppose first that $\lambda_s I_1 - A_{11}$ is nonsingular. Then it belongs to $\boldsymbol{K}$ by (1,3) and its inverse is positive by (1,4). Since (3) and (2) imply

$$(4) \qquad (\lambda_s I_1 - A_{11})\, v^{(1)} \leqq A_{1,r+1} v^{(r+1)} \leqq 0,$$

it follows that

$$v^{(1)} \leqq (\lambda_s I_1 - A_{11})^{-1} A_{1,r+1} v^{(r+1)} \leqq 0.$$

Consequently, $v^{(1)} = 0$ by (2) and $A_{1,r+1} v^{(r+1)} = 0$ which implies $A_{1,r+1} = 0$. This is a contradiction to irreducibility of A.

Thus $\lambda_s I_1 - A_{11}$ is singular. By (1,5), there exists a vector $u^{(1)} > 0$ such that

$$(5) \qquad u^{(1)T}(\lambda_s I_1 - A_{11}) = 0.$$

Thus

$$u^{(1)T}(\lambda_s I_1 - A_{11})\, v^{(1)} = 0.$$

Since

$$(\lambda_s I_1 - A_{11})\, v^{(1)} \leqq 0$$

by (4), it follows that

$$(\lambda_s I_1 - A_{11})\, v^{(1)} = 0.$$

By (4), $A_{1,r+1} v^{(r+1)} = 0$ so that, by (2), $A_{1,r+1} = 0$, a contradiction. The proof is complete.

From this theorem, two corollaries follow:

(2,2) **Corollary.** *Let A be an $n \times n$ nonnegative irreducible, symmetric matrix with eigenvalues $\lambda_1 \geqq \lambda_2 \geqq \ldots \geqq \lambda_n$. Let $s \in N$, $s \geqq 2$ and let $v = (v_i)$ be any eigenvector corresponding to λ_s. Then $M = \{i \in N \mid v_i \geqq 0\}$ is non-void and the degree of reducibility of $A(M)$ does not exceed $s - 2$.*

(2,3) **Corollary.** *Let A be an $n \times n$ nonnegative irreducible, symmetric matrix with eigenvalues $\lambda_1 \geqq \lambda_2 \geqq \ldots \geqq \lambda_n$. Let $u = (u_i) > 0$ be an eigenvector corresponding to λ_1 and $v = (v_i)$ an eigenvector corresponding to λ_2. Then for any $\alpha \geqq 0$, the submatrix $A(M_\alpha)$ is irreducible where $M_\alpha = \{i \in N \mid v_i + \alpha u_i \geqq 0\}$.*

The proof of (2,2) being immediate, (2,3) follows from (2,1) since $v + \alpha u$ satisfies the assumption for $s = 2$.

In the sequel, we shall also use (in different terminology) the following lemma which was proved in [4] (as Lemma (1, 12)):

(2,4) **Lemma.** *Let*

$$A = \begin{bmatrix} B & c \\ c^T & d \end{bmatrix}$$

be an $n \times n$ partitioned symmetric matrix, B an $(n-1) \times (n-1)$ matrix. If, for some vector u,

$$Bu = 0\,, \quad c^T u \neq 0$$

then

$$s(A) = s(B) + (1,1)\,.$$

3. Applications in the graph theory. In this section, under a graph we shall always understand a nondirected finite graph without loops and multiple edges. We shall denote the vertices of such a graph G by numbers, usually $1, 2, \ldots, n$, and the set of vertices will then be denoted by N. Thus $G = (N, E)$, E being the set of edges (i, k) of G (unordered pairs of different indices of N).

Under a *cut* in the graph G we shall understand, as usual, a set of edges C to which a decomposition $N = (N_1, N_2)$ of the vertex set N of G exists (i.e. $N_1 \neq \emptyset \neq N_2$, $N_1 \cup N_2 = N$, $N_1 \cap N_2 = \emptyset$) such that C consists exactly of all edges in G with one vertex in N_1 and the other in N_2. We shall call *bank* of the cut each of the subgraphs of G induced by the subsets N_1 and N_2.

It is clear that in this definition, the decomposition $N = (N_1, N_2)$ may not be uniquely determined by the cut C. However, it is easy to prove the following assertion:

(3,1) *Let C be a cut in a graph G. If there is a decomposition $N = (N_1, N_2)$ of the vertex set N of G corresponding to C such that both corresponding banks are connected then the decomposition of N corresponding to C is unique.*

If G is a graph, it is well known [5] that on its edge set E an equivalence relation R can be defined as follows: If $e_1 \in E$, $e_2 \in E$ then $e_1 R e_2$ iff there is a simple circuit in G containing both edges e_1 and e_2. Let $E = E_1 \cup E_2 \cup \ldots \cup E_r$ be the decomposition of E into classes of equivalence with respect to R. The subgraphs G_i $(i = 1, \ldots, r)$ of G consisting of all edges in E_i and of all vertices adjacent to them will be called *blocks* of G. As is well known, vertices in common to more than one such block are points of articulation of G. We shall say that two such points of articulation are neighbouring if there exists a block of G to which both of them belong. The following assertion is well known [6]:

(3,2) *Let G be a connected graph, B_1, B_2 its two different blocks. Then there is a unique sequence $p_1, \ldots, p_s$ $(s \geqq 1)$ of points of articulation such that $p_1 \in B_1$, $p_s \in B_2$ and p_k, p_{k+1} are neighbouring for $k = 1, \ldots, s - 1$.*

To a graph $G = (N, E)$, a matrix $A(G)$ which we shall call Laplacean of G (in accordance with Anderson [1]), is assigned as the matrix of the quadratic form

$$(A(G)\,x, x) = \sum_{(i,k)\in E} (x_i - x_k)^2 .$$

Thus, $A(G) = (a_{ik})$ where

$$\begin{aligned} a_{ik} &= 0\ (=a_{ki}) \quad \text{if} \quad i \neq k \quad \text{and} \quad (i, k) \notin E , \\ a_{ik} &= -1\ (=a_{ki}) \quad \text{if} \quad i \neq k \quad \text{and} \quad (i, k) \in E , \\ a_{ii} &= -\sum_{k \neq i} a_{ik} , \quad i, k \in N . \end{aligned}$$

In [3], the algebraic connectivity $a(G)$ of the graph G was defined as the second smallest eigenvalue of $A(G)$ (the smallest is always zero). Many properties of this notion have been proved and relations to other connectivity numbers found.

We shall recall just one property of $a(G)$ that $a(G) > 0$ iff G is connected. It is clear that G is connected iff $A(G)$ is irreducible.

We shall be interested here in graph-theoretical properties of the eigenvector of $A(G)$ corresponding to $a(G)$. The coordinates of this eigenvector are assigned to the vertices of G in a natural way and can be considered as valuations of the vertices of G. We shall call this valuation *characteristic valuation of G*. It is always non-zero and is determined uniquely up to a non-zero factor if $a(G)$ is a simple eigenvalue of G.

To obtain more general results, we shall investigate valuated graphs, i.e. graphs to each edge (i, k), $i \neq k$, of which a positive number $c_{ik} = c_{ki}$ is assigned. The generalized Laplacean $A_C(G)$ of this graph will then be defined by

$$(A_C(G)\,x, x) = \sum_{(i,k)\in E} c_{ik}(x_i - x_n)^2 , \tag{6}$$

i.e. $A_C(G) = (a_{ik})$ where

$$\begin{aligned} a_{ik} &= a_{ki} = -c_{ik} \quad \text{if} \quad i \neq k , \quad (i, k) \in E , \\ a_{ik} &= a_{ki} = 0 \quad \text{if} \quad i \neq k , \quad (i, k) \notin E , \\ a_{ii} &= \sum_{k \neq i, (i,k)\in E} c_{ik} . \end{aligned}$$

The second smallest eigenvalue $a_C(G)$ will be analogously called *algebraic connectivity of G*.

We shall prove first the following lemma:

(3,2) **Lemma.** *Let $G = (N, E)$ be a connected graph valuated by positive numbers c_{ik}. Then the algebraic connectivity of G is positive and equal to the minimum of the function*

$$\varphi(x) = n \frac{\sum_{(i,k)\in E} c_{ik}(x_i - x_k)^2}{\sum_{(i,k),i<k} (x_i - x_k)^2}$$

over all nonconstant n-tuples $x = (x_i)$ (i.e. n-tuples which are not of the form $x_i = c$, $i = 1, \ldots, n$). The corresponding characteristic valuations $y = (y_i)$ of G are then those nonconstant n-tuples for which the minimum of $\varphi(x)$ is attained and for which $\sum_{i=1}^{n} y_i = 0$.

Proof. Since $A_C(G)\, e = 0$ and $A_C(G)$ is positive semidefinite, zero is the smallest eigenvalue of $A_C(G)$. Since G is connected, it follows easily from (6) that e is the only linearly independent solution of $(A_C(G)\, x, x) = 0$. This means that zero is a simple eigenvalue, all remaining eigenvalues are positive and all eigenvectors which correspond to these eigenvalues are orthogonal to e. According to the well known Courant-Fischer principle [5], the second smallest eigenvalue $a_C(G)$ of $A_C(G)$ satisfies

$$a_C(G) = \min_{x \neq 0, (x,e)=0} \frac{\sum_{(i,k)\in E} c_{ik}(x_i - x_k)^2}{\sum_{i=1}^{n} x_i^2}$$

and the minimum is attained for any eigenvector corresponding to $a_C(G)$. By the Lagrange identity

$$n \sum_{i=1}^{n} x_i^2 - \left(\sum_{i=1}^{n} x_i\right)^2 = \sum_{(i,k)} (x_i - x_k)^2$$

we have, whenever $x \neq 0$ and $(x, e) = 0$,

$$\frac{\sum_{(i,k)\in E} c_{ik}(x_i - x_k)^2}{\sum_{i=1}^{n} x_i^2} = n \frac{\sum_{(i,k)\in E} c_{ik}(x_i - x_n)^2}{\sum_{(i,k)} (x_i - x_k)^2}$$

and the right-hand side is invariant with respect to adding a multiple of e to x. The proof is then easily completed.

(3,3) **Theorem.** *Let G be a finite connected graph with n vertices $1, \ldots, n$, to every edge (i, k) of which a positive number c_{ik} is assigned. Let $y = (y_i)$ be a characteristic valuation of G. For any $r \geqq 0$, let*

$$M(r) = \{i \in N \mid y_i + r \geqq 0\}\,.$$

Then the subgraph $G(r)$ induced by G on $M(r)$ is connected.

Proof. Denote by B the symmetric matrix (b_{ik}), $i, k \in N$, defined by

$$\begin{aligned} b_{ik} &= c_{ik} \quad \text{if} \quad i \neq k \quad \text{and} \quad (i, k) \in E\,, \\ b_{ik} &= 0 \quad \text{if} \quad i \neq k \quad \text{and} \quad (i, k) \notin E\,, \\ b_{ii} &= -\sum_{k, k \neq i} b_{ik}\,. \end{aligned}$$

Since

$$-(Bx, x) = \sum_{(i,k) \in G} c_{ik}(x_i - x_k)^2\,,$$

we have

$$B = -A_C(G)\,.$$

On the other hand, the off-diagonal part of B being nonnegative, $B + \sigma I$ is nonnegative for a sufficiently large σ. The eigenvectors of $A_C(G)$ are identical with those of $B + \sigma I$ and to the second smallest eigenvalue of $A_C(G)$ corresponds the second largest eigenvalue of $B + \sigma I$. By Corollary (2,3), $y + re$ where y is the vector of characteristic valuation of G, has the property that the submatrix of $B + \sigma I$ with indices in $M(r)$ is irreducible. Thus, the subgraph $G(r)$ is connected. The proof is complete.

(3,4) Remark. *A similar statement can be proved for $r \leqq 0$ and the set $M'(r)$ of all those i's for which $y_i + r \leqq 0$.*

(3,5) **Corollary.** *Let G be a valuated connected graph with vertices $1, 2, \ldots, n$, let $y = (y_i)$ be a characteristic valuation of G. If c is a number such that $0 \leqq c < \max y_i$ and $c \neq y_i$ for all i then the set of all those edges (i, k) of G for which $y_i < c < y_k$ forms a cut C of G. If $N_1 = \{k \in N \mid y_k > c\}$ and $N_2 = \{k \in N \mid y_k < c\}$ then $N = (N_1, N_2)$ is a decomposition of N corresponding to C and the bank $G(N_2)$ is connected.*

(3,6) **Corollary.** *Let G be a valuated connected graph with vertices $1, 2, \ldots, n$, let $y = (y_i)$ be a characteristic valuation.*

If $y_i \neq 0$ for all $i \in N$ the the set of all alternating edges, i.e. edges (i, k) for which $y_i y_k < 0$, forms a cut C of G such that both banks of G are connected.

(3,7) Remark. *By Theorem (3,1), the decomposition and the banks are in this case uniquely determined. If*

$$N_1 = \{i \in N \mid y_i > 0\}\,, \quad N_2 = \{i \in N \mid y_i < 0\}$$

then $N = (N_1, N_2)$ is the decomposition corresponding to C.

In the following theorem, we shall show that, in this manner, all cuts with connected banks in a connected graph can be obtained.

(3,8) **Theorem.** *Let G be a connected graph, let C be a cut of G such that both banks of C are connected. Then there exists a positive valuation of edges of G such that the corresponding characteristic valuation $y = (y_i)$ is unique (up to a factor), $y_i \neq 0$ for all i, and that C is formed exactly by alternating edges (as in (3,6)) of the valuation y.*

Proof. Let $N = (N_1, N_2)$ be a decomposition of the set of vertices N of $G = (N, E)$ corresponding to the cut C. For $\varepsilon \geqq 0$, define an $n \times n$ symmetric matrix $C(\varepsilon)$ as the matrix of the quadratic form

$$(7) \quad (C(\varepsilon)\, x, x) = \sum_{\substack{i \in N_1, k \in N_1 \\ (i,k) \in E}} (x_i - x_k)^2 + \sum_{\substack{i \in N_2, k \in N_2 \\ (i,k) \in E}} (x_i - x_k)^2 + \varepsilon \sum_{\substack{i \in N_1, k \in N_2 \\ (i,k) \in E}} (x_i - x_k)^2 .$$

The matrix $C(\varepsilon)$ is clearly positive semidefinite. For $\varepsilon > 0$, the only vector $x \neq 0$ for which $(C(\varepsilon)\, x, x) = 0$, is (up to a multiple) the vector $e = (1, 1, \ldots, 1)^T$. Thus for $\varepsilon > 0$ the smallest eigenvalue zero is simple and the second eigenvalue $\gamma_2(\varepsilon)$ is positive, equal to

$$(8) \qquad \min \{(C(\varepsilon)\, x, x) \mid x,\ (x, x) = 1,\ (x, e) = 0\} .$$

Clearly, $\gamma_2(\varepsilon)$ is a continuous function of ε and $\gamma_2(0) = 0$. For $\varepsilon = 0$, the third eigenvalue $\gamma_3(0)$ of $C(0)$ is already positive since $(C(0)\, x, x) = 0$ iff x is a linear combination of the vectors $z^{(1)} = (z_k^{(1)})$, $z^{(2)} = (z_k^{(2)})$ where

$$z_k^{(1)} = 1 \quad \text{if} \quad k \in N_1 , \qquad z_k^{(2)} = 0 \quad \text{if} \quad k \in N_1 ,$$
$$z_k^{(1)} = 0 \quad \text{if} \quad k \in N_2 , \qquad z_k^{(2)} = 1 \quad \text{if} \quad k \in N_2 .$$

(Here, we used the fact that both banks are connected.) Since $\gamma_3(\varepsilon)$ is also a continuous function of ε,

$$\gamma_2(\varepsilon) < \gamma_3(\varepsilon)$$

in some open interval $I_1 = (0, \eta)$ where $\eta > 0$. For $\varepsilon \in I_1 \cup \{0\}$, there exists, up to multiplication by -1, a unique eigenvector $y(\varepsilon)$ of $C(\varepsilon)$ corresponding to $\gamma_2(\varepsilon)$ and such that $(y(\varepsilon), y(\varepsilon)) = 1$, $(y(\varepsilon), e) = 0$. Since $y(\varepsilon)$ is that vector for which the minimum in (8) is attained and $y(0)$ is clearly the vector $y = (y_i)$ (determined up to the sign) where

$$y_i = \ \ |N_2|^{1/2}\, |N_1|^{-1/2}\, n^{-1/2} \quad \text{if} \quad i \in N_1 ,$$
$$y_i = -|N_1|^{1/2}\, |N_2|^{-1/2}\, n^{-1/2} \quad \text{if} \quad i \in N_2 ,$$

there exists a subinterval $I_2 = (0, \xi)$ with $0 < \xi < \eta$ in which the sign of $y_i(\varepsilon)$ is positive for $i \in N_1$ and negative for $i \in N_2$. Thus the alternating edges are exactly those in C. The proof is complete.

(3,9) *Let $G = (N, E)$ be a graph with the vertex set $N = \{1, \ldots, n\}$. Let each edge (i, k) of G be valuated by a positive number c_{ik}. If $y = (y_i)$ is a characteristic valuation and $a_C(G)$ the algebraic connectivity of G then*

$$a_C(G)\, y_i = \sum_{k,(i,k)\in E} c_{ik}(y_i - y_k) \quad \text{for all} \quad i \in N\,, \tag{9}$$

and also for any subset $M \subset N$

$$a_C(G) \sum_{i\in M} y_i = \sum_{\substack{(i,k)\in E\\ i\in M, k\notin M}} c_{ik}(y_i - y_k)\,. \tag{10}$$

Proof. (9) follows immediately from $A_C(G)\, y - a_C(G)\, y = 0$. If we sum in (9) over all $i \in M$, we obtain (10) since the terms on the right-hand side for $i \in M$, $k \in M$ cancel.

(3,10) **Corollary.** *Let G be a connected valuated graph with the vertex set $N = \{1, \ldots, n\}$, let $y = (y_i)$ be its characteristic valuation. If $y_i > 0$ then there exists an index j such that $(i, j) \in E$ and $y_j < y_i$.*

Proof. Follows immediately from (9) since $a_C(G) > 0$.

We shall investigate now the properties of the characteristic valuation on blocks of G.

(3,11) **Theorem.** *Let $G = (N, E)$ be a connected graph, y its characteristic valuation. Let k be a point of articulation of G, let $G_0, G_1, \ldots, G_r$ be all components of the graph obtained from G by removing the vertex k and all adjacent edges. Then:*

(i) *If $y_k > 0$ then exactly one of the components G_i contains a vertex negatively valuated in y. For all vertices s in the remaining components $y_s > y_k$.*

(ii) *If $y_k = 0$ and there is a component G_i containing both positively and negatively valuated vertices then there is exactly one such component, all remaining being zero valuated.*

(iii) *If $y_k = 0$ and none component contains both positively and negatively valuated vertices then each component G_i contains either only positively valuated, or negatively valuated, or only zero valuated vertices.*

Proof. Let first $y_k > 0$. Since $\sum_{i\in N} y_i = 0$, there is a vertex in G with negative value; this must be in $G_0 \cup \ldots \cup G_r$, thus in at least one G_i, say G_0. To complete the proof of (i), it suffices to show that for all vertices t in $G_1 \cup \ldots \cup G_r$, $y_t > y_k$ since then

$y_t > 0$ as well. Suppose first that $y_t < y_k$ for some vertex t in $G_1 \cup \ldots \cup G_r$. Then there exists an $\varepsilon > 0$ for which $y_k - \varepsilon > 0$ as well as $y_k - \varepsilon \geqq y_t$. By (3,4), the graph $\tilde{G}$ induced by G on $M = \{s \in N \mid y_s \leqq y_k - \varepsilon\}$ is connected. Since $k \notin M$, $\tilde{G}$ is contained in $G_0 \cup \ldots \cup G_r$; as it contains at least one vertex in G_0 (with a negative value), $\tilde{G} \subset G_0$. However, $t \in G_1 \cup \ldots \cup G_r$ belongs to M, a contradiction. If $y_s = y_k$ for some vertex s in $G_1 \cup \ldots \cup G_r$ then by (3,10), there is a vertex t in G for which $(s, t) \in E$ and $y_t < y_s$. Since $t \neq k$, $t \in G_1 \cup \ldots \cup G_r$, a contradiction by the previous argument.

Let us consider now the case that $y_k = 0$. For notational convenience, we shall assume that $k = n$ and that

$$A(G) = A = \begin{bmatrix} A_0 & 0 & \ldots & 0 & c_0 \\ 0 & A_1 & \ldots & 0 & c_1 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & A_r & c_r \\ c_0^T & c_1^T & \ldots & c_r^T & \gamma \end{bmatrix}$$

where A_i corresponds to vertices in G_i, $i = 0, \ldots, r$.

If α is the algebraic connectivity of G then $A - \alpha I$ is singular and

$$(A - \alpha I)\, y = 0\,.$$

Let

$$y = \begin{bmatrix} y^{(0)} \\ y^{(1)} \\ \vdots \\ y^{(r)} \\ 0 \end{bmatrix}$$

be the conformally partitioned vector of the valuation y; we have thus, $I_0, \ldots, I_r$ being identity matrices of the corresponding size,

$$(A_0 - \alpha I_0)\, y^{(0)} = 0\,,$$

$$(A_1 - \alpha I_1)\, y^{(1)} = 0\,,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots$$

$$(A_r - \alpha I_r)\, y^{(r)} = 0\,.$$

Let us distinguish two cases:

α) Some of the components G_i, say G_0, contains both positively and negatively valuated vertices. Thus $y^{(0)}$ is neither nonnegative, nor nonpositive. By (1,7), it follows that $A_0 - \alpha I_0$ which has all off-diagonal entries nonpositive is not positive semidefinite and thus has at least one negative eigenvalue. Consequently, as

$$\begin{bmatrix} A_0 - \alpha I_0, & & & & \\ & A_1 - \alpha I_1, & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot \\ & & & & A_r - \alpha I_r \end{bmatrix}$$

is a principal submatrix of $A - \alpha I$, and thus has, by (1,2), at most one negative eigenvalue, all matrices $A_i - \alpha I_i$, $i = 1, \ldots, r$, are positive semidefinite and thus belong to $\boldsymbol{K}_0$. Assume $y^{(j)} \neq 0$ for an index j, $1 \leqq j \leqq r$. Since A_j is irreducible, $y^{(j)}$ is by (1,5) either positive or negative. At the same time, $c_j \neq 0$ since A is irreducible. But $c_j \leqq 0$ and thus $(y^{(j)})^T c_j \neq 0$. By Lemma (2,4) applied to the vector

$$\begin{bmatrix} 0 \\ \vdots \\ 0 \\ y^{(j)} \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

$$s(A - \alpha I) = \sum_{i=0}^{r} s(A_i - \alpha I_i) + (1,1)$$

which means that $A - \alpha I$ has at least two negative eigenvalues (the other from $A_0 - \alpha I_0$), a contradiction. Thus $y^{(1)} = 0, \ldots, y^{(r)} = 0$ as asserted. The proof of (ii) is complete.

β) None of the components $G_0, G_1, \ldots, G_r$ contains both positively and negatively valuated vertices. Let a component (and there is such since $y \neq 0$), say G_0 contain a vertex with a non-zero valuation. Thus $y^{(0)} \neq 0$ and either $y^{(0)} \geqq 0$, or $y^{(0)} \leqq 0$. If some coordinate of $y^{(0)}$ were zero then by (1,5), $A_0 - \alpha I_0$ would not belong to $\boldsymbol{K}_0$ and it would follow as in α that $y^{(1)} = 0, \ldots, y^{(r)} = 0$, a contradiction to $\sum_{i \in N} y_i = 0$, $y \neq 0$. Thus either $y^{(0)} > 0$, or $y^{(0)} < 0$. The proof of (iii) is complete.

To shorten the formulation of the following theorem, we shall say that a path in a graph G is pure iff it is simple and does not contain more than two points of articulation of each block of G.

(3,12) **Theorem.** *Let G be a connected graph, y its characteristic valuation. Then exactly one of the following two cases occurs:*

Case A. *There is a single block B_0 in G which contains both positively and negatively valuated vertices. Each other block has either vertices with positive valuation only, or vertices with negative valuation only, or vertices with zero valuation only. Every pure path P starting in B_0 and containing just one vertex k in B_0 has the property that the values at the points of articulation contained in P form either an increasing, or decreasing, or a zero sequence along this path according to whether $y_k > 0$, $y_k < 0$ or $y_k = 0$; in the last case all vertices in P have value zero.*

Case B. *No block of G contains both positively and negatively valuated vertices. There exists a single vertex z which has value zero and has a neighbour with a non-zero valuation. This vertex is a point of articulation. Each block contains (with the exception of z) either vertices with positive valuation only, or vertices with negative valuation only, or vertices with zero valuation only. Every pure path P starting in z has the property that the values at its points of articulation either increase, and then all values in vertices of P are (with the exception of z) positive, or decrease, and then all values (up to that of z) are negative, or all values in vertices of P are equal to zero. Every path containing both positively and negatively valuated vertices passes through z.*

Proof. The cases A and B clearly exclude each other. That both can occur, show the examples of graphs with the matrix

$$\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

and characteristic valuation $(1, -1)^T$ (case A) and with the matrix

$$\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

and valuation $(1, 0, -1)^T$ (Case B).

Thus let G be a connected graph, y its characteristic valuation. Let first G contain a block B_0 with positively as well as negatively valuated vertices. If G has the only block B_0, we are finished. If not, let B_1 be a block different from B_0. Then there exists a point of articulation k which is contained in B_0 and separates B_1 from B_0. Let $G_0, G_1, \ldots, G_r$ be all components of the graph obtained from G by removing the vertex k and the edges adjacent to k, G_0 containing the remaining vertices from B_0, G_1 the remaining vertices from B_1. If $y_k > 0$, it follows from (i) of Theorem (3,11) that all vertices in G_1 (and thus in B_1) have positive values. If $y_k = 0$, it follows from (ii) of the same theorem that all vertices in G_1, and thus B_1, have value zero. If $y_k < 0$, the valuation $-y$ is also a characteristic valuation and we may apply (i) of (3,11) to this case and complete the proof of the first part of Case A.

Let now P be a pure path, let k be the only vertex of P in B_0 and $k, k_1, k_2, \ldots, k_s$ be all points of articulation in P ordered along the path P. Let first $y_k > 0$. By Theorem (3,11), $y_{k_1} > y_k$. If we apply the same Theorem to the point of articulation k_j where j satisfies $1 \leqq j < s$, we obtain from (i) that $y_{k_{j+1}} > y_{k_j}$. This proves that the sequence $y_k, y_{k_1}, y_{k_2}, \ldots, y_{k_s}$ increases. If $y_k < 0$, this last result applied to $-y$ yields that this sequence decreases. If $y_k = 0$, it follows from (ii) of Thm. (3,11) that even all vertices of P have value zero. The case A is settled.

Let now no block contain both positively and negatively valuated vertices. Let us prove first:

Proposition. *Let a path P contain a vertex with positive value as well as a vertex with negative value. Then P contains a vertex with zero value such that one its neighbour has a non-zero value.*

Proof. Follows immediately from the fact that in this case there is no edge in G one vertex of which has a positive value and the other a negative value.

Since $y \neq 0$ and $\sum_{i \in N} y_i = 0$, there is such path P in G, and thus such a vertex z with $y_z = 0$ and a neighbour with a non-zero valuation. By (9), z must have neighbours with positive as well as negative valuation. Since these cannot belong to the same block, z is a point of articulation of G. It follows then from the properties in (ii) of Thm. (3,11) that the case (iii) in this theorem occurs. Consequently, no other vertex can have value zero and neighbours with a non-zero value. This, together with the Proposition, proves the last assertion. Let now P be a pure path starting in z. It follows from (iii) in Thm. (3,11) that if this path contains a vertex with a positive (alternatively, negative) value then all vertices in P, except z, have positive (alternatively, negative) values. Since $-y$ is also a characteristic valuation, we can restrict ourselves to the first case only. Let $z, k_1, k_2, \ldots, k_s$ denote all points of articulation in P in the ordering induced by P. Let j satisfy $1 \leqq j < s$. Since $y_{k_j} > 0$, we can apply (i) of Thm. (3,11) to obtain that $y_{k_{j+1}} > y_{k_j}$. The case B is thus also settled.

(3,13) **Corollary.** *In Case* B *of Theorem* (3,12), *the subgraph G_0 of G induced by G on the set of vertices with value zero is connected.*

We shall turn now to the case that G is a tree. In this case. the blocks are identical with edges, every path is pure and Theorem (3,12) together with Corollary (3,13) yield immediately the following:

(3,14) **Theorem.** *Let T be a tree, $y = (y_i)$ its characteristic valuation. Then two cases can occur:*

Case A. *All values y_i are different from zero. Then T contains exactly one edge (p, q) such that $y_p > 0$ and $y_q < 0$. The values in vertices along any path in T which starts in P and does not contain q increase, the values in vertices along any path starting in q and not containing p decrease.*

Case B. *The set* $N_0 = \{i \in N \mid y_i = 0\}$ *is non-void. Then the graph* T_0 *induced by* T *on* N_0 *is connected and there is exactly one vertex* $z \in N_0$ *having at least one neighbour not belonging to* N_0. *The values along any path in* T *starting in* z *are increasing, or decreasing, or zero.*

Remark. Using more subtle properties of special matrices, proved in [4] one can prove the following theorem which shows that for valuated trees the characteristic valuation does not have other properties independent on the valuation of edges:

Let T *be a tree (without valuation of edges) with the set of vertices* N. *Let* α *be a positive number, let* $y_1, \ldots, y_n$ *be real numbers not all equal to zero such that* $\sum_{i \in N} y_i = 0$ *and satisfying the following two conditions:* (i) *If all numbers* y_i *are different from zero then there is exactly one edge* (p, q) *in* T *such that* $y_p > 0$, $y_q < 0$; *for the vertices* p *and* q, *the values on any path starting in* p *and not containing* q *increase, the values on any path starting in* q *and not containing* p *decrease.*

(ii) *If the set* N_0 *of vertices with value zero is non-void then the graph* T_0 *induced by* T *on* N_0 *is connected and there is a single vertex* $z \in N_0$ *which has some neighbour not belonging to* N_0. *The values on any path starting in* z *either increase, or decrease, or are all zero.*

Then there exists a positive valuation of edges of T *such that* α *is the algebraic connectivity of* T *and* y *is a characteristic valuation of* T *determined (up to a non-zero factor) uniquely.*

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