Florian-Horia Vasilescu

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SPECTRAL MAPPING THEOREM FOR THE LOCAL SPECTRUM

F.-H. VASILESCU, Bucharest

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1. Introduction. In the sequel X will be a Fréchet space in the sense of [2]; in particular, the topology of X will be defined by a countable family of semi-norms $\{\|x\|_m\}_{m=1}^{\infty}$. We denote by $C(X)$ the set of all linear closed operators acting in X, and by $L(X)$ the set of all continuous linear operators on X. The space $L(X)$ will be endowed with the topology of the uniform convergence on the bounded subsets of X. We denote also by $\mathbb{C}_\infty$ the one-point compactification of the complex field $\mathbb{C}$.

We recall that the *spectrum* $\sigma(T)$ of an operator $T \in C(X)$ is defined as the complement in $\mathbb{C}_\infty$ of the set $\varrho(T)$ of all points $\lambda \in \mathbb{C}_\infty$ which have a neighbourhood V_λ such that $(\mu - T)^{-1} \in L(X)$ for any $\mu \in V_\lambda \cap \mathbb{C}$ and the set

$$\{(\mu - T)^{-1} x;\ \mu \in V_\lambda \cap \mathbb{C}\}$$

is bounded for any $x \in X$ (this definition is equivalent to the original one given in [9], in Fréchet spaces).

Let us fix now an operator $T \in C(X)$. We recall the concept of local spectrum, as defined in [5]. Namely, for a fixed element $x \in X$ let us denote by $\delta_T(x)$ the set of all $\lambda \in \mathbb{C}_\infty$ for which there are an open neighbourhood V_λ and an X-valued analytic function f_x, defined in V_λ, whose values are actually in the domain of definition $\mathscr{D}(T)$ of T, such that $(\mu - T) f_x(\mu) = x$ for $\mu \in V_\lambda \cap \mathbb{C}$. Such a function f_x will be called *T-associated with x (at λ). The set* $\gamma_T(x) = \mathbb{C}_\infty \setminus \delta_T(x)$ is the *local spectrum of T at x* and it is obviously contained in $\sigma(T)$.

Let us denote by A_T the set of all complex-valued functions, analytic in neighbourhoods of $\sigma(T)$. The analytic functional calculus for T [4], [9], [1] is then defined by

$$f(T) = \begin{cases} \dfrac{1}{2\pi i} \displaystyle\int_\Gamma f(\mu)(\mu - T)^{-1} \, d\mu & \infty \notin \sigma(T), \\ f(\infty) + \dfrac{1}{2\pi i} \displaystyle\int_\Gamma f(\mu)(\mu - T)^{-1} \, d\mu & \infty \in \sigma(T), \end{cases}$$

where Γ is a rectifiable contour surrounding $\sigma(T)$ in $\mathbb{C}$ and $f \in A_T$ is arbitrary. The properties of the analytic functional calculus are well-known and we will not repeat all of them here. One of the most important properties of the analytic functional calculus is known as the *spectral mapping theorem* and it asserts that $\sigma(f(T)) = f(\sigma(T))$, for any $f \in A_T$. The main result of this paper is a variant of this formula, valid for the local spectrum. In this way we improve an older result from [7], extending the continuous case in Banach space, completely solved in [3] for the operators having the single valued extension property in Dunford's sense (see again [3] for this notion). We extend also the case of the continuous operators in Banach spaces, which are not supposed to have the single valued extension property, developed in [8]. The present refinement takes advantage of some ideas of [8].

Let us recall one more concept. As is shown in [5], there exists a unique maximal open set $\Omega_T \subset \mathbb{C}_\infty$ with the property that if $U \subset \Omega_T$ is open and $f_0 : U \to \mathscr{D}(T)$ is analytic and $(\mu - T)f_0(\mu) = 0$ for $\mu \in U \cap \mathbb{C}$ then $f_0 = 0$ in U. In other words, the set Ω_T is the maximal open set in $\mathbb{C}_\infty$ in which the operator T has the single valued extension property. Its complement in $\mathbb{C}_\infty$ will be denoted by S_T. The set S_T is contained in $\sigma(T)$ and has a "good behaviour" with respect to the analytic functional calculus [7]. We shall return to this problem in the third section.

2. Main result. In this section we shall prove the following

2.1. Theorem. *Consider $T \in C(X)$ and take $f \in A_T$ which is non-constant in any connected component of its domain of definition. Then for every $x \in X$ we have $f(\gamma_T(x)) = \gamma_{f(T)}(x)$.*

In order to prove Theorem 2.1 we need some supplementary results.

2.2. Proposition. *Consider $T \in C(X)$ and take $x \in X$ such that $\gamma_T(x) \not\ni \infty$. Then $x \in \mathscr{D}(T^k)$ for any $k \geqq 1$ and*

$$\sup\{|z|;\, z \in \gamma_T(x)\} \leqq \sup_m \lim_{k\to\infty} \|T^k x\|_m^{1/k} < \infty .$$

The proof of this assertion can be found in [6].

2.3. Lemma. *Assume that $T \in C(X)$ has the properties $\varrho(T) \neq \emptyset$ and $\sigma(T) \ni \infty$. If $f \in A_T$ is such that $\lim_{z\to\infty} z^k f(z) = 0$ $(0 \leqq k < m)$ then for any polynomial P of degree at most m the function $g = Pf \in A_T$, $g(T) = P(T)f(T)$ and $g(T)x = f(T)P(T)x$ for any $x \in \mathscr{D}(P(T))$.*

The proof of this result is similar to that in [4] so that we omit it.

Proof of Theorem 2.1. If there exists at least one $f \in A_T$ which is non-constant in any connected component of its domain of definition then we must have $\varrho(T) \neq \emptyset$; otherwise, the only functions in A_T would be constants.

Let us consider such an $f \in A_T$ and fix $x \in X$. We show first the inclusion $f(\gamma_T(x)) \subset \subset \gamma_{f(T)}(x)$. Take $\lambda_0 \in f^{-1}(\delta_{f(T)}(x))$. Since $\mu_0 = f(\lambda_0) \in \delta_{f(T)}(x)$, we can take an analytic function h which is $f(T)$-associated with x at μ_0. Let us write $f(\lambda) - f(z) = (\lambda - z)\, g_\lambda(z)$, and note that for any fixed λ the function g_λ is analytic in a neighbourhood of $\sigma(T)$. By Lemma 2.3 we have $(\lambda - T)\, g_\lambda(T)\, h(f(\lambda)) = x$ and the mapping

$$g_\lambda(T)\, h(f(\lambda)) = \frac{1}{2\pi i} \int_\Gamma g_\lambda(z)\,(z - T)^{-1}\, h(f(\lambda))\, dz\,,$$

where Γ is a rectifiable contour surrounding $\sigma(T)$, is analytic in a neighbourhood of λ_0. We therefore have $f^{-1}(\delta_{f(T)}(x)) \subset \delta_T(x)$, whence $f(\gamma_T(x)) \subset \gamma_{f(T)}(x)$.

Conversely, take $\mu_0 \in \gamma_{f(T)}(x)$. If $\sigma(T) \ni \infty$, we suppose that $\mu_0 \neq f(\infty)$. Consider the equation $\mu_0 - f(\lambda) = 0$. Since the solutions of this equation have no cluster point in the domain of definition of f, we may suppose, diminishing this domain of definition if necessary, that the above equation has only the distinct roots $\lambda_1, \ldots, \lambda_n$. Let us assume that all these roots are in $\delta_T(x)$. Take g_j which are T-associated with x at λ_j; more precisely, we may suppose that g_j are defined in neighbourhoods of some open and mutually disjoint sets $\Delta_j \ni \lambda_j$, whose boundaries are rectifiable contours $(j = 1, \ldots, n)$.

We investigate first the case $\sigma(T) \ni \infty$. Then the set where f is not defined is compact in $\mathbb{C}$ and contained in $\varrho(T)$. Let Δ_0 be an open neighbourhood of this set, disjoint with every Δ_j and such that its boundary is a rectifiable contour. Then the set $\Delta = \bigcup_{j=0}^{n} \Delta_j$ has the property that its boundary Γ is a rectifiable contour. Note also that in virtue of continuity and compactness, there exists an open neighbourhood V of μ_0 such that the equation $w - f(\lambda)$ has roots only in Δ, for any $w \in V$. If we denote by g the function which is equal to g_j in Δ_j and to $(z - T)^{-1}x$ in Δ_0 (the set Δ_0 may be supposed to lie in $\varrho(T)$), we can define the function

$$h(w) = (w - f(\infty))^{-1}\, x + \frac{1}{2\pi i} \int_\Gamma \frac{g(\xi)}{w - f(\xi)}\, d\xi\,,$$

which is analytic in V. We shall show that $(w - f(T))\, h(w) = x$ in V. Let Γ_1 be another contour surrounding $\sigma(T)$. We take Γ_1 such that the open set whose boundary is Γ_1 contains the open set whose boundary is Γ, including Γ itself, and which is still in the domain of definition of f. Note the relation

$$(\xi - \eta)\,(\eta - T)^{-1}\, g(\xi) = (\eta - T)^{-1}\, x - g(\xi)\,,$$

from which we derive the equalities

$$\frac{1}{2\pi i} \int_{\Gamma_1} (w - f(\eta)) \left(\frac{1}{2\pi i} \int_\Gamma \frac{(\eta - T)^{-1}\, g(\xi)}{w - f(\xi)}\, d\xi \right) d\eta =$$

$$
\begin{aligned}
&= \frac{1}{2\pi i}\int_{\Gamma_1} (w - f(\eta))\left(\frac{1}{2\pi i}\int_{\Gamma} \frac{d\xi}{(\xi - \eta)(w - f(\xi))}\right)(\eta - T)^{-1} x \, d\eta - \\
&\quad - \frac{1}{2\pi i}\int_{\Gamma_1} (w - f(\eta))\left(\frac{1}{2\pi i}\int_{\Gamma} \frac{g(\xi)}{(\xi - \eta)(w - f(\xi))} d\xi\right) d\eta = \\
&\quad - \frac{(w - f(\infty))^{-1}}{2\pi i}\int_{\Gamma_1} (w - f(\eta))(\eta - T)^{-1} x \, d\eta - \\
&\quad - \frac{1}{2\pi i}\int_{\Gamma} \frac{g(\xi)}{w - f(\xi)}\left(\frac{1}{2\pi i}\int_{\Gamma_1} \frac{w - f(\eta)}{\xi - \eta} d\eta\right) d\xi = \\
&\quad - \frac{(w - f(\infty))^{-1}}{2\pi i}\int_{\Gamma_1} (w - f(\eta))(\eta - T)^{-1} x \, d\eta - \\
&\quad - \frac{w - f(\infty)}{2\pi i}\int_{\Gamma} \frac{g(\xi)}{w - f(\xi)} d\xi ,
\end{aligned}
$$

since we have

$$\frac{1}{2\pi i}\int_{\Gamma} \frac{d\xi}{(\xi - \eta)(w - f(\xi))} = -(w - f(\infty))^{-1} ;$$

indeed, according to the choice of the contours Γ and Γ_1, as $\eta \in \Gamma_1$, the function $(\xi - \eta)^{-1}(w - f(\xi))^{-1}$ is analytic in the open set whose boundary is Γ and we may apply the Cauchy formula at infinity. We have used also the equalities

$$\frac{1}{2\pi i}\int_{\Gamma_1} \frac{w - f(\eta)}{\xi - \eta} d\eta = f(\xi) - f(\infty)$$

and

$$\int_{\Gamma} g(\xi) \, d\xi = 0 .$$

Since we have

$$(w - f(T)) x = (w - f(\infty)) x + \frac{1}{2\pi i}\int_{\Gamma_1} (w - f(\eta))(\eta - T)^{-1} x \, d\eta ,$$

by the above calculation we can write

$$
\begin{aligned}
(w - f(T)) h(w) &= x + \frac{w - f(\infty)}{2\pi i}\int_{\Gamma_1} \frac{g(\xi)}{w - f(\xi)} d\xi + \\
&+ \frac{(w - f(\infty))^{-1}}{2\pi i}\int_{\Gamma_1} (w - f(\eta))(\eta - T)^{-1} x \, d\eta + \\
&+ \frac{1}{2\pi i}\int_{\Gamma_1} (w - f(\eta))\left(\frac{1}{2\pi i}\int_{\Gamma} \frac{(\eta - T)^{-1} g(\xi)}{w - f(\xi)} d\xi\right) d\eta = x ,
\end{aligned}
$$

which contradicts the choice of μ_0. We therefore must have $\lambda_j \in \gamma_T(x)$ for at least one index j, which implies the inclusion

$$\gamma_{f(T)}(x) \subset f(\gamma_T(x)) \cup \{f(\infty)\}.$$

When $\sigma(T) \not\ni \infty$, we consider the function

$$h(w) = \frac{1}{2\pi i} \int_\Gamma \frac{g(\xi)}{w - f(\xi)} \, d\xi,$$

where Γ is chosen as in the previous case. The point at infinity does not play a role any longer and the relations

$$\frac{1}{2\pi i} \int_\Gamma \frac{d\xi}{(\xi - \eta)(w - f(\xi))} = 0$$

and

$$\frac{1}{2\pi i} \int_\Gamma g(\xi) \, d\xi = \frac{1}{2\pi i} \int_{\Gamma_1} (\xi - T)^{-1} x \, d\xi = x$$

imply the inclusion $\gamma_{f(T)}(x) \subset f(\gamma_T(x))$. Consequently, in the case $\sigma(T) \not\ni \infty$ the theorem is proved.

Let us establish the theorem in its general form. It will be sufficient to prove the equality when $\gamma_T(x) \not\ni \infty$. Let us fix $\lambda_0 \in \varrho(T)$ and define $\varphi(z) = (z - \lambda_0)^{-1}$. Note that $\varphi(\infty) = 0$. We show first that $\gamma_{\varphi(T)}(x) \not\ni 0$. According to Proposition 2.2, there is a constant $L \geqq 0$ such that

$$\|T^k x\|_m \leqq M_m L^k \quad (k, m = 1, 2, 3, \ldots).$$

Set $y_k = (T - \lambda_0)^k x$ and notice that

$$\|y_k\|_m \leqq \sum_{j=0}^{k} \binom{k}{j} |\lambda_0|^j \, \|T^{k-j} x\|_m \leqq M_m (L + |\lambda_0|)^k.$$

This estimate shows that the series

$$\alpha(w) = -\sum_{k=0}^{\infty} w^k y_{k+1}$$

defines an analytic function in a neighbourhood of 0, which satisfies the equality $(w - \varphi(T)) \, \alpha(w) = x$, hence $0 \notin \gamma_{\varphi(T)}(x)$.

Finally, a well-known property of the analytic functional calculus [4] yields $f(T) = (f \circ \varphi^{-1})(\varphi(T))$, therefore we can write

$$\gamma_{f(T)}(x) = f \circ \varphi^{-1}(\gamma_{\varphi(T)}(x)) = (f \circ \varphi^{-1})(\varphi(\gamma_T(x)) = f(\gamma_T(x)),$$

and the proof is complete.

As we shall see in the next section, the hypothesis made in Theorem 2.1 on the function f to be non-constant in any connected component of its domain of definition is essential.

3. Some consequences. The main aim of this section is to give a proof of the following

3.1. Theorem. *Consider $T \in C(X)$ and take $f \in A_T$ which is non-constant in any connected component of its domain of definition. Then we have $S_{f(T)} = f(S_T)$.*

This theorem has been already proved in [7]. We shall give here a different proof, based on Theorem 2.1.

The next result is inspired by [8].

3.2. Lemma. *Consider $T \in C(X)$. The set of all $\lambda \in \mathbb{C}$ such that there exists an $x_\lambda \in \mathscr{D}(T)$, $x_\lambda \neq 0$ with the properties $\gamma_T(x_\lambda) = \emptyset$ and $(\lambda - T)x_\lambda = 0$ is contained in S_T and is dense in S_T.*

Proof. It is easily seen that S_T is the set of all points $\lambda \in \mathbb{C}_\infty$ such that in any neighbourhood V_λ of λ one can find an open set U and a $\mathscr{D}(T)$-valued analytic function $f \neq 0$ such that $(\mu - T)f(\mu) = 0$ for $\mu \in U \cap \mathbb{C}$.

Take now $\lambda \in \mathbb{C}$ such that there is an $x_\lambda \in X$ with the above stated properties. Since $\gamma_T(x_\lambda) = \emptyset$, we can find in a neighbourhood V_λ of λ an open set U and an analytic $\mathscr{D}(T)$-valued function g such that $(\mu - T)g(\mu) = x$ for $\mu \in U \cap \mathbb{C}$. If we define $f(\mu) = (\lambda - T)g(\mu)$ then $f \neq 0$ in U and $(\mu - T)f(\mu) = 0$, therefore $\lambda \in S_T$.

Let us show that the set of all points $\lambda \in \mathbb{C}$ with the stated properties is dense in S_T. Indeed, if $\lambda \in S_T$ is arbitrary and V_λ is a neighbourhood of λ then there exists a $\mathscr{D}(T)$-valued function f, defined in an open set $U \subset V_\lambda$, f analytic, such that $(\mu - T)f(\mu) = 0$ in U. If we fix $\mu \in U \cap \mathbb{C}$ such that $x_\mu = f(\mu) \neq 0$, then $\gamma_T(x_\mu) = \emptyset$ (see [5], Proposition 2.2), hence $\mu \in V_\lambda$ has the desired properties.

3.3. Lemma. *Let $U \subset \mathbb{C}_\infty$ be an open set, $U \ni \infty$ and $f : U \to \mathbb{C}$ an analytic non-null function. Then for any $K \subset U$, $K \ni \infty$, K closed, the function f has the representation*

$$f(z) = (\lambda_1 - z)\dots(\lambda_n - z)(\lambda_0 - z)^{-q}g(z),$$

where $\lambda_1, \dots, \lambda_n$ lie in K, $q \geqq n$ is an integer, $\lambda_0 \notin U$ and $g(z) \neq 0$ in K.

Proof. If $U = \mathbb{C}_\infty$ then f is constant and the representation is trivial. If $U \neq \mathbb{C}_\infty$, we take $\lambda_0 \notin U$ and consider the transformation $w = (z - \lambda_0)^{-1}$; therefore we shall study the analytic function $g(w) = f(w^{-1} + \lambda_0)$ in a neighbourhood of zero. The set $K_1 = \{w; z \in K\}$ is compact, therefore g has only a finite number of zeros in K_1. We have then

$$g(w) = w^p(w - w_1)\dots(w - w_n)h(w),$$

and $h(w) \neq 0$ in K_1. Hence we obtain the corresponding representation for f, with $q = p + n \geqq n$.

Note that $\lambda_1, \ldots, \lambda_n$ are not supposed to be necessarily distinct.

Proof of Theorem 3.1. We shall use Lemma 3.2. Let $\lambda \in \mathbb{C}$ have the property that there is $x_\lambda \in \mathscr{D}(T)$, $x_\lambda \neq 0$, such that $(\lambda - T) x_\lambda = 0$ and $\gamma_T(x_\lambda) = \emptyset$. Then we have $(f(\lambda) - f(T)) x_\lambda = g(T)(\lambda - T) x_\lambda = 0$, where $g(z) = g_\lambda(z)$ is defined as in the first part of the proof of Theorem 2.1. By Theorem 2.1, $\gamma_{f(T)}(x_\lambda) = \emptyset$, therefore $f(\lambda) \in S_{f(T)}$. By the density obtained in Lemma 3.2, we infer $f(S_T) \subset S_{f(T)}$.

Conversely, take $\mu \in S_{f(T)}$ and $x_\mu \neq 0$ with $\gamma_{f(T)}(x_\mu) = \emptyset$ and $(\mu - f(T) x_\mu = 0$. We suppose that $\sigma(T) \ni \infty$ and apply Lemma 3.3 to the function $\mu - f$ and to $\sigma(T)$. We therefore have

$$\mu - f(z) = (\lambda_1 - z) \ldots (\lambda_n - z)(\lambda_0 - z)^{-q} g(z),$$

where $g(z) \neq 0$ in a neighbourhood of $\sigma(T)$ and $\lambda_0 \in \varrho(T)$. By Lemma 2.3 we can write

$$(\lambda_1 - T) \ldots (\lambda_n - T)(\lambda_0 - T)^{-q} g(T) x_\mu = 0.$$

Let j be the largest index with the property

$$(\lambda_j - T) \ldots (\lambda_n - T)(\lambda_0 - T)^{-q} g(T) x_\mu = 0.$$

We have then $y = (\lambda_{j+1} - T) \ldots (\lambda_n - T)(\lambda_0 - T)^{-q} g(T) x_\mu \neq 0$ (note that if $j = n$, we have $(\lambda_0 - T)^{-q} g(T) x_\mu \neq 0$ since $g(T)^{-1}$ exists). On the other hand, $h_j(z) = (\lambda_{j+1} - z) \ldots (\lambda_n - z)(\lambda_0 - z) g(z)$ is analytic at infinity by virtue of $q \geqq n \geqq n - j$, therefore $h_j(T) \in L(X)$. As $h_j(T)(z - T) x = (z - T) h_j(T) x$ for any $x \in \mathscr{D}(T)$, we derive that $\gamma_T(h_j(T) x_\mu) = \gamma_T(y) \subset \gamma_T(x_\mu) = \emptyset$, where the last equality is obtained by Theorem 2.1. Summarizing, we have $(\lambda_j - T) y = 0$, $y \neq 0$ and $\gamma_T(y) = \emptyset$, hence $\lambda_j \in S_T$. Then $\mu = f(\lambda_j) \in f(S_T)$, which completes the proof.

The case $\sigma(T) \not\ni \infty$ can be obtained in a similar manner, the decomposition of $\mu - f$ being of the same type, with $q = 0$.

3.4. Remarks. 1° Note that the hypothesis that f be non-constant in any connected component of its domain of definition is essential. Indeed, if $T \in C(X)$ has the property $S_T \neq \emptyset$ and $f(z) = 1$, then $f(T) = 1$ and $S_{f(T)} = \emptyset$, while $f(S_T) = \{1\}$.

Furthermore, we can find an $x \in X$, $x \neq 0$, such that $\gamma_T(x) = \emptyset$ by Lemma 3.2 while $\gamma_1(x) = \{1\}$. Therefore, the properties of f are also essential for Theorem 2.1.

2° By similar techniques, one can improve the results of [7] concerning the calculus with polynomials. Namely, if $T \in C(X)$ has the property $\varrho(T) \neq \emptyset$ then for any non-constant polynomial P we have $\gamma_{P(T)}(x) = P(\gamma_T(x))$ for every $x \in X$. As a consequence, we can also prove the relation $S_{P(T)} = P(S_T)$ which is already proved directly in [7]. We will not develop these ideas here; they will be published elsewhere.

References

[1] *G. R. Allan:* A spectral theory for locally convex algebras, J. London Math. Soc., *15* (1965), 399—421.

[2] *N. Bourbaki:* Espaces vectoriels topologiques, Hermann, Paris, 1955.

[3] *I. Colojoară* and *C. Foiaş:* Theory of generalized spectral operators, Gordon and Breach, New York, 1968.

[4] *N. Dunford* and *J. Schwartz:* Linear operators, Part I, Interscience Publishers, New York, 1958.

[5] *F.-H. Vasilescu:* Residually decomposable operators in Banach spaces, Tôhoku Math. J., *21* (1969), 509—522.

[6] *F.-H. Vasilescu:* Residual properties for closed operators on Fréchet spaces, Illinois J. Math., *15* (1971), 377—386.

[7] *F.-H. Vasilescu:* Analytic functions and some residual spectral properties, Rev. Roum. Math. Pures Appl., *15* (1970), 435—451.

[8] *P. Vrbová:* On local spectral properties of operators in Banach spaces, Czech. Math. J., *23* (1973), 483—492.

[9] *L. Waelbroeck:* Le calcul symbolique dans les algèbres commutatives, J. Math. Pures Appl., *33* (1954), 147—186.

Author's address: National Institute for Scientific and Technical Creation, INCREST, Bucharest, Romania.