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Jozef Kačur

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APPLICATION OF ROTHE'S METHOD TO PERTURBED LINEAR HYPERBOLIC EQUATIONS AND VARIATIONAL INEQUALITIES

JOZEF KAČUR, Bratislava

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Introduction. The main purpose of this paper is to apply Rothe's method to perturbed linear hyperbolic equations and inequalities. This method allows us to use direct variational methods in the case of hyperbolic equations or inequalities and to transfer some results from elliptic equations or inequalities to the corresponding hyperbolic ones.

In Part I we shall be concerned with abstract perturbed linear hyperbolic equations which can be applied to the following equation

$$(1') \qquad \frac{\partial^2 u}{\partial t^2} + \sum_{|i|\cdot|j| \leq k} (-1)^{|i|} D^i(a_{ij}(x)\, D^j u) = f(t, x, u. \nabla u, \ldots, \nabla^k u)$$

for $0 < t < T$ $(T < \infty)$ and $x \in \Omega$, where $\Omega \subset R^N$ is a bounded domain with a Lipschitzian boundary $\partial\Omega$ and $f(t, x, \xi)$ is Lipschitz continuous in t, ξ. Together with $(1')$ the initial conditions

$$(2') \qquad u(x, 0) = U_0(x)\,, \quad \frac{\partial u(x, 0)}{\partial t} = U_1(x)\,, \quad x \in \Omega$$

and the corresponding boundary conditions (e.g. homogeneous Dirichlet boundary conditions) will be considered.

In Part II the corresponding hyperbolic variational inequality (see (6)) is considered.

Let V, H be two Hilbert spaces with the norms $\|\cdot\|$, $|\cdot|$, respectively. We assume that $V \subset H$, V is dense in H and the imbedding $V \hookrightarrow H$ is continuous. We consider a linear, bounded operator A from V into V^* (the dual space to V). By $\langle u, v\rangle$ we denote the duality between $u \in V^*$ and $v \in V$ and by (u, v) we denote the scalar product in H. We assume that the form $\langle Au, v\rangle$ generates (an equivalent) scalar product in V. Let $F(t, u)$ be a Lipschitz continuous operator from $\langle 0, T\rangle \times V$ into H We consider $(1')$, $(2')$ in the abstract form

$$(1) \qquad \frac{\mathrm{d}^2 u(t)}{\mathrm{d}t^2} + A\, u(t) = F(t, u(t)) \text{ for a.e. } \quad t \in (0, T)\,,$$

(2) $$u(0) = U_0 \in V, \quad \frac{\mathrm{d}u(0)}{\mathrm{d}t} = U_1 \in V,$$

where $u : (0, T) \to V$ is an abstract function with $(\mathrm{d}^2u(t)/\mathrm{d}t^2) \in H$. In our particular problem (1'), (2') and in the assertions concerning the regularity of solution we set $V \subset W_2^k(\Omega)$ with $\mathring{W}_2^k(\Omega) \subset V \subset W_2^k(\Omega)$ ($\mathring{W}_2^k$, W_2^k are Sobolev spaces), $H = L_2(\Omega)$, $\langle Au, v\rangle = \sum_{|i|,|j| \leqq k} \int_\Omega a_{ij}(x)\, D^j u\, D^i v\, \mathrm{d}x$, $F(t, u) = f(t, x, u, \nabla u, \ldots, \nabla^k u)$.

We apply Rothe's method (method of lines) to (1), (2) in the following way. Successively, for $i = 1, \ldots, n$, we look for a solution $u_i \in V$ of the linear elliptic equation (corresponding to (1)), i.e., we solve the variational identity

(3) $$\left(\frac{u - 2u_{i-1} + u_{i-2}}{h^2}, v\right) + \langle Au, v\rangle = (F(t_i, u_{i-1}), v)$$

for all $v \in V$, where $u_0 = U_0$, $u_{-1} = U_0 - hU_1$, n is a positive integer, $h = T/n$, $t_i = ih$ and U_0, U_1 are from (2). By means of u_i ($i = 0, 1, \ldots, n$) we construct Rothe's function

(4) $$u_n(t) = u_{i-1} + (t - t_{i-1})\, h^{-1}(u_i - u_{i-1}) \quad \text{for} \quad t_{i-1} \leqq t \leqq t_i,$$

$i = 1, \ldots, n$. We prove the convergence of u_n ($n \to \infty$) to the solution u of (1), (2), i.e., to the solution of the identity

(5) $$\left(\frac{\mathrm{d}^2u(t)}{\mathrm{d}t^2}, v\right) + \langle A\, u(t), v\rangle = (F(t, u(t)), v)$$

for all $v \in V$ and a.e. $t \in (0, T)$.

Analogously we proceed in the case of the corresponding hyperbolic variational inequality. Let K be a nonempty, closed, convex set in V. We look for a solution u of the variational inequality

(6) $$\left(\frac{\mathrm{d}^2u(t)}{\mathrm{d}t^2}, v - \frac{\mathrm{d}u(t)}{\mathrm{d}t}\right) + \left\langle A\, u(t), v - \frac{\mathrm{d}u(t)}{\mathrm{d}t}\right\rangle \geqq \left(F(t, u(t)), v - \frac{\mathrm{d}u(t)}{\mathrm{d}t}\right)$$

for all $v \in K$ and a.e. $t \in (0, T)$, where $\mathrm{d}u(t)/\mathrm{d}t \in K$ and $u(0) = U_0$, $\mathrm{d}u(0)/\mathrm{d}t = U_1$. In this case we apply Rothe's method in the following way. Successively, for $i = 1, \ldots, n$, we look for the solutions $z_i \in K$ of the linear elliptic variational inequalities

(7) $$\left(\frac{z - z_{i-1}}{h}, v - z\right) + h\langle Az, v - z\rangle \geqq \left(F\left(t_i, U_0 + \sum_{j=1}^{i-1} hz_j\right), v - z\right) - h\sum_{j=1}^{i-1}\langle Az_j, v - z\rangle - \langle AU_0, v - z\rangle$$

for all $v \in K$, where $z_0 = U_1$. Then we construct $u_i = U_0 + \sum_{j=1}^{i} hz_j$ and hence Rothe's function u_n (see (4)). Convergence of u_n to the unique solution u of (6) is proved.

For simplicity we set $I = \langle 0, T \rangle$. We shall use the following functional spaces: $W_2^k(\Omega)$ (Sobolev space), $C(I, V)$ (the space of continuous abstract functions from I into V), $L_\infty(I, V)$ (the space of bounded measurable abstract functions from I into V), $L_2(I, V)$ (the space of measurable abstract functions for which $\|u\|^2_{L_2(I,V)} = (\int_I \|u(t)\|_V^2 \, \mathrm{d}t)^{1/2} < \infty$)) and $C^{r,1}(\bar{\Omega})$ (the space of real functions whose r-th derivatives are Lipschitz continuous in $\bar{\Omega}$), whose definitions and fundamental properties can be found in [11].

The problem (1'), (2') (Part I) has been solved in [1] in the case $f = f(t)$, $a_{ij} = a_{ij}(t, x)$ by using Rothe's method. The solution u possesses the properties $u \in C(I, V)$, $\mathrm{d}u/\mathrm{d}t \in L_\infty(I, V) \cap C(I, L_2)$, $\mathrm{d}^2u/\mathrm{d}t^2 \in L_\infty(I, L_2)$ (under stronger assumptions $\mathrm{d}^2u/\mathrm{d}t^2 \in L_\infty(I, V)$, $\mathrm{d}^3u/\mathrm{d}t^3 \in L_\infty(I, L_2)$). In the more special cases of (1'), (2') this method has been used in [6], [7]. Here, we prove more regularity properties of the solution in the x-variable and a stronger convergence of u_n to u. Let $0 \leqq l \leqq k$ be an integer. When $a_{ij} \in C^{r_i,1}(\bar{\Omega})$ ($r_i = \max\{0, |i| + l - k\}$) then our solution is an element of the space $C(I, V) \cap L_\infty(I, W_2^{k+l}(\Omega'))$ for arbitrary $\Omega' \subset \Omega$ with $\bar{\Omega}' \subset \Omega$ (regularity in the interior of Ω) and $\mathrm{d}^2u/\mathrm{d}t^2 \in L_\infty(I, H)$. Thus, in the case $l = k$ our solution satisfies (1') for a.e. $(x, t) \in \Omega \times (0, T)$ in the classical sense. Under stronger assumptions on $F(t, v)$ (see (22)) we can prove $\mathrm{d}^3u/\mathrm{d}t^3 \in L_\infty(I, H)$, $\mathrm{d}u/\mathrm{d}t \in L_\infty(I, V) \cap C(I, H)$ and the convergence $u_n \to u$ in the norm of the space $C(I, V \cap W_2^{k+l}(\Omega'))$ and $\mathrm{d}^2u_n/\mathrm{d}t^2 \to \mathrm{d}^2u/\mathrm{d}t^2$ in $C(I, H)$. The results concerning the hyperbolic variational inequality (6) are analogous to those on the problem (1), (2) and are interesting also from the numerical point of view. Hyperbolic variational inequalities similar to our problem (6) have been solved, e.g., in [12], [13] by the method of penalization. We use some techniques developed in [1]–[5]. All results can be extended to the case $a_{ij} = a_{ij}(t, x)$ (see Remark 4).

I. LINEAR PERTURBED HYPERBOLIC EQUATION

A priori estimates. Throughout the paper the letter C will be used for positive constants. We allow C to have different values in the course of discussion. We assume that the form $\langle Au, v \rangle$ represents (an equivalent) scalar product in V and

$$\langle Au, u \rangle = \|u\|^2 . \tag{8}$$

The Lipschitz continuity of $F(t, u) : I \times V \to H$ is assumed in the form

$$|F(t, u) - F(t', v)| \leqq L(|t - t'| + |t - t'| \, \|v\| + \|u - v\|) \tag{9}$$

for all $t, t' \in I$ and $u, v \in V$.

We denote

$$z_i = \frac{u_i - u_{i-1}}{h}, \quad s_i = \frac{z_i - z_{i-1}}{h} \quad \text{for} \quad i = 0, 1, \ldots, n ,$$

where $u_0 = U_0$, $u_{-1} = u_0 - hU_1$ and $u_{-2} = h^2(-AU_0 + F(0, U_0)) + 2u_{-1} - u_0$. We assume $U_0 \in V$ and $U_1 \in V$.

The existence of the unique solution u_i ($i = 1, \ldots, n$) of (3) is guaranteed by the Lax-Milgram (or Riesz) theorem.

Lemma 1. *There exists $n_0 > 0$ and C such that the estimate $\|u_i\|^2 + |z_i|^2 \leqq C$ holds for all $n > n_0$, $i = 1, \ldots, n$.*

Proof. From (3) for $v = u_i - u_{i-1}$ and from (8) we have

$$\left(\frac{z_i - z_{i-1}}{h}, u_i - u_{i-1}\right) + \langle Au_i, u_i - u_{i-1}\rangle = (F(t_i, u_{i-1}), u_i - u_{i-1})$$

and hence

$$2^{-1}|z_i|^2 + 2^{-1}|z_i - z_{i-1}|^2 - 2^{-1}|z_{i-1}|^2 + 2^{-1}\|u_i\|^2 + 2^{-1}\|u_i - u_{i-1}\|^2 - \\ - 2^{-1}\|u_{i-1}\|^2 \leqq h(F(t_i, u_{i-1}), z_i)\,.$$

From this inequality and (9) we obtain

$$|z_i|^2 (1 - C_1 h) + \|u_i\|^2 \leqq \|z_{i-1}\|^2 + \|u_{i-1}\|^2 (1 + C_2 h) + Ch\,.$$

Successively, from this inequality we deduce

$$(|z_i|^2 + \|u_i\|^2)(1 - C_1 h)^i \leqq (|z_0|^2 + \|u_0\|^2)(1 + C_2 h)^i + C = \\ = (|U_1|^2 + \|U_0\|^2)(1 + C_2 h)^i + C\,,$$

which implies our assertion.

Lemma 2. *Let $AU_0 \in H$. There exist n_0 and C such that the estimate*

$$\|z_i\|^2 + |s_i|^2 \leqq C \quad \textit{holds for all} \;\; n > n_0\,, \quad i = 1, \ldots, n\,.$$

Proof. Let us consider (3) for $i = j$ and $i = j - 1$, where $v = s_j$. Subtracting these identities we obtain

$$(s_j, s_j) + \langle Au_j - Au_{j-1}, s_j\rangle = (s_{j-1}, s_j) + (F(t_j, u_{j-1}) - F(t_{j-1}, u_{j-2}), s_j)$$

and hence (see (8), (9))

$$|s_j|^2 + \langle Az_j, z_j - z_{j-1}\rangle \leqq \\ \leqq 2^{-1}|s_j|^2 + 2^{-1}|s_{j-1}|^2 + hC(1 + \|u_{j-2}\| + \|z_{j-1}\|)\,|s_j|\,.$$

From this inequality, analogously as in Lemma 1, we obtain

$$\|z_j\|^2 + \|z_j - z_{j-1}\|^2 + |s_j|^2 (1 - C_1 h) \leqq |s_{j-1}|^2 + \|z_{j-1}\|^2 (1 + C_2 h) + C_3 h\,,$$

where the estimate $\|u_i\| \leqq C$ has been used. From this reccurent inequality we deduce

$$\|z_j\|^2 + |s_j|^2 \leqq C(1 + |s_0|^2 + \|z_0\|^2) \leqq C(1 + |AU_0|^2 + \|U_1\|^2)$$

and the proof is complete.

Analogously as in (4) we construct $z_n(t)$, $s_n(t)$ in terms of z_i, s_i, respectively. We also define the step functions $\bar{u}_n(t)$, $\bar{z}_n(t)$, $\bar{s}_n(t)$ as follows:

$$\bar{u}_n(t) = u_i \quad \text{for} \quad t_{i-1} < t \leqq t_i \,, \quad i = 1, \ldots, n \,, \quad \bar{u}_n(0) = u_0$$

and analogously $\bar{z}_n(t)$, $\bar{s}_n(t)$.

Lemmas 1 and 2 imply the a priori estimates

$$\|u_n(t)\| + \|z_n(t)\| + |s_n(t)| \leqq C \,; \tag{10}$$

$$\|\bar{u}(t)\| + \|\bar{z}_n(t)\| + |\bar{s}_n(t)| \leqq C \,; \tag{11}$$

$$\|u_n(t) - \bar{u}_n(t)\| + |z_n(t) - \bar{z}_n(t)| \leqq \frac{C}{n} \tag{12}$$

for all n, $t \in I$ and

$$\|u_n(t) - u_n(t')\| + |z_n(t) - z_n(t')| \leqq C|t - t'| \tag{13}$$

for all n and $t, t' \in I$.

Moreover, by the regularity results on elliptic equations (see [8]) and from (3) we conclude regularity for u_i in the case of (1′), (2′). Indeed, from the identity

$$\langle Au, v\rangle = (s_i, v) + (F(t_i, u_{i-1})), v \equiv (f_{i,n}, v)$$
$$(|f_{i,n}| \leqq C \quad \text{for all } n \,, \quad i = 1, \ldots, n)$$

and the regularity assumption

$$a_{ij} \in C^{r_i,1}(\bar{\Omega}) \quad \text{where} \quad r_i = \max\{0, |i| + l - k\} \,, \quad 0 \leqq l \leqq k \,, \tag{14}$$

we obtain (see [8])

$$\|u_i\|_{W_2^{k+l}(\Omega')} \leqq C(\Omega')\,(\|u_i\| + |f_{i,n}|) \leqq C(\Omega') \tag{15′}$$

for all n, $i = 1, \ldots, n$, where Ω' is an arbitrary subdomain of Ω with $\bar{\Omega}' \subset \Omega$. The estimate (15′) implies

$$\|u_n(t)\|_{W_2^{k+l}(\Omega')} + \|\bar{u}_n(t)\|_{W_2^{k+l}(\Omega')} \leqq C(\Omega') \quad \text{for all} \quad t \in I \,. \tag{15}$$

As a consequence of our notation we have $\mathrm{d}^- u_n(t)/\mathrm{d}t = \bar{z}_n(t)$, $\mathrm{d}^- z_n(t)/\mathrm{d}t = \bar{s}_n(t)$ where $\mathrm{d}^-/\mathrm{d}t$ is the left hand derivative. The variational identity (3) can be rewriten in the form

$$\left(\frac{\mathrm{d}^- z_n(t)}{\mathrm{d}t}, v\right) + \langle A\bar{u}_n(t), v\rangle = \left(F^{(n)}\left(t, \bar{u}_n\left(t - \frac{T}{n}\right)\right), v\right) \tag{16}$$

for all n and $t \in (0, T)$, where $\bar{u}_n(t) = u_0$ for $t \in (-T/n, 0\rangle$, $F^{(n)}(t, v) = F(t_i, v)$ for $t_{i-1} < t \leqq t_i$, $i = 1, \ldots, n$ and $F^{(n)}(t, v) = F(0, v)$ for $t \in (-T/n, 0\rangle$. By passing to the limit $n \to \infty$ in (16) we prove the convergence of $u_n(t)$ and $z_n(t)$.

Lemma 3. *Let $AU_0 \in H$. Then there exists $u \in C(I, V)$ with*

$$\frac{\mathrm{d}u}{\mathrm{d}t} \in C(I, H) \cap L_\infty(I, V), \quad \frac{\mathrm{d}^2 u}{\mathrm{d}t^2} \in L_\infty(I, H)$$

such that

$$u_n \to u \quad \text{in} \quad C(I, V), \quad z_n \to \frac{\mathrm{d}u}{\mathrm{d}t} \quad in \quad C(I, H)$$

and the estimate

$$\left|z_n - \frac{\mathrm{d}u}{\mathrm{d}t}\right|^2 + \left|\frac{\mathrm{d}u_n}{\mathrm{d}t} - \frac{\mathrm{d}u}{\mathrm{d}t}\right|^2 + \|u_n - u\|^2 \leqq \frac{C}{n}$$

takes place uniformly for $t \in I$.

Proof. Let us subtract (16) for $n = r$ and $n = s$, where $v = \bar{z}_r(t) - \bar{z}_s(t)$ (in the sequel we omit t). Then we obtain

$$\left(\frac{\mathrm{d}^-}{\mathrm{d}t}(z_r - z_s),\, z_r - z_s\right) + \left\langle A(u_r - u_s),\, \frac{\mathrm{d}^-}{\mathrm{d}t}(u_r - u_s)\right\rangle = \left(F^{(r)}\left(t, \bar{u}_r\left(t - \frac{T}{r}\right)\right) -\right.$$

$$\left. - F^{(s)}\left(t, \bar{u}_s\left(t - \frac{T}{s}\right)\right),\, \bar{z}_r - \bar{z}_s\right) + \left(\frac{\mathrm{d}^-}{\mathrm{d}t}(z_r - z_s),\, z_r - \bar{z}_r + (\bar{z}_s - z_s)\right) +$$

$$+ \left\langle A(u_r - \bar{u}_r + \bar{u}_s - u_s),\, \frac{\mathrm{d}^-}{\mathrm{d}t}(u_r - u_s)\right\rangle.$$

We integrate this inequality over $(0, t)$ and use the a priori estimates (10)–(13) and (9). Successively we obtain

$$|z_r(t) - z_s(t)|^2 + \|u_r(t) - u_s(t)\|^2 \leqq C_1\left(\frac{1}{r} + \frac{1}{s}\right) + C_2 \int_0^t \|u_r(\tau) - u_s(\tau)\| \,.$$

$$. \,|z_r(\tau) - z_s(\tau)|\, \mathrm{d}\tau$$

and hence

$$|z_r(t) - z_s(t)|^2 + \|u_r(t) - u_s(t)\|^2 \leqq C_1\left(\frac{1}{r} + \frac{1}{s}\right) \mathrm{e}^{C_2 T} \tag{17}$$

because of Gronwall's lemma. Thus, there exist $z \in C(I, H)$ and $u \in C(I, V)$ such that $u_n \to u$ in $C(I, V)$ and $z_n \to z$ in $C(I, H)$. Passing to the limit for $n \to \infty$ in (13) we have

$$|z(t) - z(t')|^2 + \|u(t) - u(t')\|^2 \leqq C|t - t'|, \tag{13'}$$

which implies (see [9], [10]) $\mathrm{d}z/\mathrm{d}t \in L_\infty(I, H)$ and $\mathrm{d}u/\mathrm{d}t \in L_\infty(I, V)$. Passing to the limit for $n \to \infty$ in the identity

$$(u_n(t), v) = \int_0^t \left(\frac{\mathrm{d}u_n(t)}{\mathrm{d}t}, v\right) \mathrm{d}t + (U_0, v) = \int_0^t (\bar{z}_n(\tau), v) \,\mathrm{d}\tau + (U_0, v)$$

we easily find out $\mathrm{d}u(t)/\mathrm{d}t = z(t)$. Thus, $\mathrm{d}^2u/\mathrm{d}t^2 = \mathrm{d}z/\mathrm{d}t \in L_\infty(I, H)$. Passing to the limit for $s \to \infty$ in (17) and using (12) we obtain the estimate in Lemma 3 and the proof is complete.

Existence of solution. Theorem 1. *Suppose* (8), (9) *and* $AU_0 \in H$. *Then there exists a unique solution of* (1), (2) *with the properties*

$$u \in C(I, V), \ \frac{\mathrm{d}u}{\mathrm{d}t} \in L_\infty(I, V) \cap C(I, H), \ \frac{\mathrm{d}^2u}{\mathrm{d}t^2} \in L_\infty(I, H) \quad \textit{and} \quad Au \in L_\infty(I, H).$$

The estimate

$$\left|\frac{\mathrm{d}u_n(t)}{\mathrm{d}t} - \frac{\mathrm{d}u}{\mathrm{d}t}\right|^2 + \|u_n(t) - u(t)\|^2 \leqq \frac{C}{n} \tag{18}$$

takes place for all $t \in I$ *where* $u_n(t)$ *is from* (4). *In the case of* (1′), (2′), *when* (14) *is satisfied then* $u \in L_\infty(I, W^{k+l}(\Omega'))$ *where* $\Omega' \subset \Omega$ *with* $\bar{\Omega}' \subset \Omega$.

Proof. Integrating (16) over $(0, t)$ we obtain

$$(z_n(t), v) - (U_1, v) + \int_0^t \langle A\bar{u}_n(\tau), v\rangle \,\mathrm{d}\tau = \int_0^t \left(F^{(n)}\left(\tau, \bar{u}_n\left(\tau - \frac{T}{n}\right)\right), v\right) \mathrm{d}\tau. \tag{19}$$

Lemma 3 implies

$$\int_0^t \langle A\,\bar{u}_n(\tau), v), v\rangle \,\mathrm{d}\tau \to \int_0^t \langle A\,u(\tau), v\rangle \,\mathrm{d}\tau.$$

In the case of (1′), (2′), from (15) we have $u \in L_\infty(I, W_2^{k+l}(\Omega'))$.

Lemma 3 and (12) imply

$$\bar{u}_n\left(t - \frac{T}{n}\right) \to u(t)$$

in V for all $t \in (0, T)$. Hence we have $(F^{(n)}(t, \bar{u}_n(t - T/n)), v) \to (F(t, u(t)), v)$ for all $v \in V$ and $t \in (0, T)$. Since $\langle A\,\bar{u}(t), v\rangle$, $(F^{(n)}(t, \bar{u}_n(t - T/n), v)$ are uniformly bounded (with respect to t), by passing to the limit for $n \to \infty$ in (19) we have

$$(z(t), v) - (U_1, v) + \int_0^t \langle A\,u(\tau), v\rangle \,\mathrm{d}\tau = \int_0^t (F(\tau, u(\tau)), v) \,\mathrm{d}\tau \tag{20}$$

and hence, differentiating (20) with respect to t we obtain (1), (2) since $\langle A\,u(t), v\rangle$ and $(F(t, u(t)), v)$ are continuous functions with respect to t. From (20) we have

$\mathrm{d}u(0)/\mathrm{d}t = z(0) = U_1$ and from Lemma 3, $u(0) = U_0$. From (1), (2) we deduce $AU \in L_\infty(I, H)$. The uniqueness of u is obtained in the following way. If U_1, u_2 are two such solutions of (1), (2) then $u = u_1 - u_2$ satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}\left|\frac{\mathrm{d}u}{\mathrm{d}t}\right|^2 + \frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|^2 = 2\left(F(t, u_1(t)) - F_2(t, u_2(t)), \frac{\mathrm{d}u(t)}{\mathrm{d}t}\right)$$

and hence

$$\frac{\mathrm{d}}{\mathrm{d}t}\left|\frac{\mathrm{d}u(t)}{\mathrm{d}t}\right|^2 + \frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|^2 \leqq C\|u(t)\|\left|\frac{\mathrm{d}u(t)}{\mathrm{d}t}\right|. \tag{21}$$

Integrating (21) over $(0, t)$ we obtain

$$\left|\frac{\mathrm{d}u(t)}{\mathrm{d}t}\right|^2 + \|u(t)\|^2 \leqq C\int_0^t\left(\left|\frac{\mathrm{d}u(\tau)}{\mathrm{d}\tau}\right|^2 + \|u(\tau)\|^2\right)\mathrm{d}\tau.$$

Hence and from Gronwall's lemma we conclude $\|u(t)\| = 0$. The estimate (17) is proved in Lemma 3. Thus, proof of Theorem 1 is complete.

Remark 1. We can prove continuous dependence of the solution of (1), (2) on the data U_0, U_1 and $F(t, v)$ proceeding analogously as in the proof of uniqueness (using (21) and Gronwall's lemma).

Assertion. *Let u_i $(i = 1, 2)$ be two solutions of (1), (2) corresponding to $U_{0,i}, U_{1,i}$ and $F_i(t, v)$. If*

$$|F_1(t, v) - F_2(t, v)| \leqq a(t) + b(t)\|v\| \quad \textit{for all} \quad t \in I \quad \textit{and} \quad v \in V$$

then the estimate

$$\left|\frac{\mathrm{d}(u_1(t) - u_2(t))}{\mathrm{d}t}\right|^2 + \|u_1(t) - u_2(t)\|^2 \leqq C_1 \mathrm{e}^{C_2 t}\Big(\|U_{0,1} - U_{0,2}\|^2 + \\ + |U_{1,1} - U_{1,2}|^2 + \max_I \|u_2(t)\|^2 \int_0^t b^2(t)\,\mathrm{d}t + \int_0^t a^2(t)\,\mathrm{d}t\Big)$$

takes place for all $t \in I$.

Indeed, for $u = u_1 - u_2$ we deduce

$$\frac{\mathrm{d}}{\mathrm{d}t}\left|\frac{\mathrm{d}u(t)}{\mathrm{d}t}\right|^2 + \frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|^2 \leqq (a(t) + b(t)\|u_2(t)\| + L\|u(t)\|)\left|\frac{\mathrm{d}u(t)}{\mathrm{d}t}\right| \quad \text{for all} \quad t \in I. \tag{21'}$$

Hence and from Gronwall's lemma we obtain the required result.

Remark 2. From (20) we deduce $(\mathrm{d}/\mathrm{d}t)(z(t), v) \in C(0, T)$. This means that the weak derivative $z_t(t) = (\mathrm{d}u(t)/\mathrm{d}t)_t$ exists and for a.e. $t \in I$ equals the strong derivative

$\mathrm{d}z(t)/\mathrm{d}t = \mathrm{d}^2u(t)/\mathrm{d}t^2$. Thus, (1), (2) is satisfied for all $t \in (0, T)$ with the weak derivative $(\mathrm{d}u(t)/\mathrm{d}t)_t$.

Under stronger regularity assumptions on $F(t, v)$ and U_0, U_1 we prove $\mathrm{d}^3u/\mathrm{d}t^3 \in L_\infty(I, H)$, which together with (14) (for (1'), (2')) enables us to prove $u_n \to u$ in the norm $C(I, W_2^{k+l}(\Omega'))$ and $\mathrm{d}^2u_n/\mathrm{d}t^2 \to \mathrm{d}^2u/\mathrm{d}t^2$ in $C(I, H)$. We suppose

$$\left|\frac{F(t, v_1) - F(t - h, v_2)}{h} - \frac{F(t - h, v_2) - F(t - 2h, v_3)}{h}\right| \leqq \tag{22}$$

$$\leqq h\, C(v)\left(1 + \alpha\left[\|v_1\|, \|v_2\|, \|v_3\|, \left\|\frac{v_1 - v_2}{h}\right\|, \left\|\frac{v_2 - v_3}{h}\right\|\right] + \left\|\frac{v_1 - 2v_2 + v_3}{h^2}\right\|\right)$$

for all $v_1, v_2, v_3 \in V$, $t, (t - h), (t - 2h) \in I$, where $C(v)$ depends on $\|v_1\|, \|v_2\|, \|v_3\|$ and $\alpha[x_1, x_2, x_3, x_4, x_5]$ is a continuous nonnegative function in its variables.

In the case of (1'), (2'), provided the imbedding $W_2^k(\Omega) \to C(\bar{\Omega})$ is continuous and $|f''_{tt}(t, s)| + |f''_{ts}(t, s)| + |f''_{ss}(t, s)| \leqq C_1$ ($C_1 \equiv C_1(C_2)$, $s \in R$) for $|s| \leqq C_2$, then (22) holds.

Let $S_i = (s_i - s_{i-1})/h$ for $i = 0, 1, \ldots, n$, where we define $u_{-3} = 2u_{-2} - u_{-1} + h^2(-Au_{-1} + F(0, U_0))$. As a consequence of this definition we have $|S_0| = |AU_1|$.

Lemma 4. *Let $U_0, U_1 \in V$ and $AU_0, AU_1 \in H$. If (22) is satisfied, then there exist n_0, C such that the estimate*

$$\|s_i\|^2 + |S_i|^2 \leqq C \tag{23}$$

takes place for all $n > n_0$ and $i = 1, \ldots, n$.

Proof. Let us consider (3) for $u = u_i$, $v = S_j$ and $i = j, j - 1, j - 2$. We multiply the equation corresponding to $i = j - 1$ by -2 and then we add all three equations. We obtain

$$\begin{aligned}(s_j - 2s_{j-1} + s_{j-2}, S_j) + \langle A(u_j - 2u_{j-1} + u_{j-2}), S_j\rangle &= \\ = (F(t_j, u_{j-1}) - 2F(t_{j-1}, u_{j-2}) + F(t_{j-2}, u_{j-3}), S_j)&\end{aligned}$$

for $j = 1, \ldots, n$ and hence, owing to (22),

$$\begin{aligned}(S_j - S_{j-1}, S_j) + \langle As_j, s_j - s_{j-1}\rangle \leqq \\ \leqq C(u)\, h(1 + \alpha[\|u_{j-1}\|, \|u_{j-2}\|, \|u_{j-3}\|, \|z_{j-1}\|, \|z_{j-2}\|] + \|s_{j-1}\|)\, |S_j|\,.\end{aligned} \tag{24}$$

Owing to Lemma 1 and $\|z_0\| + \|z_{-1}\| + \|z_{-2}\| \leqq C(|AU_1| + |AU_0| + \|U_1\| + \|U_0\|)$ we have $\|z_j\| \leqq C$ for $j = -2, \ldots, n$ and analogously $\|u_j\| \leqq C$ for $j = -3, \ldots, n$. Thus, from (24) we conclude analogously as in Lemma 1 and Lemma 2

$$(|S_i|^2 + \|s_i\|^2)(1 - C_1h)^i \leqq (|S_0|^2 + \|s_0\|^2) + C_2 \sum_{j=1}^{i} h + C_3\,,$$

which implies the required estimate (23).

We define $S_n(t)$, $\bar{S}_n(t)$ analogously as $u_n(t)$, $\bar{u}_n(t)$. As a consequence of Lemma 4 we have

$$\|s_n(t)\| + |S_n(t)| \leqq C\,; \tag{10'}$$

$$\|\bar{s}_n(t)\| + |\bar{S}_n(t)| \leqq C\,; \tag{11'}$$

$$|s_n(t) - \bar{s}_n(t)| \leqq \frac{C}{n}\,; \tag{12'}$$

$$|s_n(t) - s_n(t')| \leqq C|t - t'| \tag{13'}$$

for all $t, t' \in I$ and n.

Theorem 2. *Let $U_0, U_1 \in V$, $AU_0 \in H$, $AU_1 \in H$ and let $V \hookrightarrow H$ be compact. Suppose (8), (9), (22), $u(t)$ is a solution of (1), (2) and $u_n(t)$ is from (4). Then $\mathrm{d}^3u/\mathrm{d}t^3 \in L_\infty(I, H)$, $\mathrm{d}^2u/\mathrm{d}t^2 \in C(I, H) \cap L_\infty(I, V)$ and the convergence $s_n \to \mathrm{d}^2u/\mathrm{d}t^2$ in $C(I, H)$ for $n \to \infty$ takes place. Moreover, in the case of (1'), (2'), if (14) is satisfied, then $u \in C(I, V \cap W_2^{k+l}(\Omega'))$ and $u_n \to u$ in $C(I, V \cap W_2^{k+l}(\Omega'))$ for $n \to \infty$, where $\Omega' \subset \Omega$, $\bar{\Omega}' \subset \Omega$.*

Proof. Let $t \in (0, T)$ be fixed. From (10'), the reflexivity of V and the compactness of the imbedding $V \hookrightarrow H$ we conclude that there exists a subsequence of $\{s_n(t)\}$ (indexed by n again) and $w_t \in V$ such that $s_n(t) \rightharpoonup w_t$ in V and $s_n(t) \to w_t$ in H. By the diagonalization method a subsequence of $\{s_n\}$ can be chosen (denoted again by $\{s_n\}$) such that $s_n(t)$ is weakly convergent in V and strongly convergent in H for all rational points of I. On the other hand (13') implies that this subsequence is convergent (locally uniformly) for all $t \in I$. Hence and from the Borel covering theorem we conclude that $s_n(t) \to s(t)$ in H uniformly with respect to $t \in I$. Thus, we have $s_n \to s$ in $C(I, H)$. Hence and from (10') we conclude $s_n(t) \rightharpoonup s(t)$ in V for all $t \in (0, T)$. Thus, owing to (10'), $s \in L_\infty(I, V)$. Passing to the limit for $n \to \infty$ in the identity

$$(z_n(t), v) - (U_1, v) = \int_0^t (\bar{s}_n(\tau), v)\,\mathrm{d}\tau$$

we obtain

$$\frac{\mathrm{d}z(t)}{\mathrm{d}t} = \frac{\mathrm{d}^2u(t)}{\mathrm{d}t^2} = s(t)\,.$$

Passing to the limit for $n \to \infty$ in (13') we have $|s(t) - s(t')| \leqq C|t - t'|$ for all $t, t' \in I$. Hence we have (see [9], [10])

$$\frac{\mathrm{d}s(t)}{\mathrm{d}t} = \frac{\mathrm{d}^3u(t)}{\mathrm{d}t^3} \in L_\infty(I, H)\,.$$

We find out easily that the original sequence (not only a subsequence) $\{s_n\}$ converges to $\mathrm{d}^2u/\mathrm{d}t^2$ in $C(I, H)$. Now, we prove $u_n \to u$ in $C(I, V \cap W_2^{k+l}(\Omega'))$. The element $u(t) - \bar{u}_n(t)$ (t is fixed) is the weak solution of the linear elliptic equation

$$(25) \qquad A(u(t) - u_n(t)) = Q_n(t) \equiv + \left(s_n(t) - \frac{d^2u(t)}{dt^2}\right) + F(t, u(t)) -$$

$$- F^{(n)}\left(t, \bar{u}\left(t - \frac{T}{n}\right)\right) + A(\bar{u}_n(t) - u_n(t)) .$$

From (12) and (22) (particularly from (9)) we conclude

$$|Q_n(t)| \leqq \frac{C_1}{n} + \left|s_n - \frac{d^2u(t)}{dt^2}\right| \to 0 \quad \text{for} \quad n \to \infty$$

(uniformly with respect to $t \in I$). From the regularity results (see [8]) we obtain the estimate

$$(26) \qquad \|u(t) - u_n(t)\|^2_{W_2{}^{k+l}(\Omega')} \leqq C(\|\bar{u}_n(t) - u(t)\| + |Q_n(t)|) \to 0$$

for $n \to \infty$ uniformly with respect to $t \in I$. Since $(d^2u/dt^2) \in C(I, H)$ and $u \in C(I, V)$, the equation

$$A\, u(t) = - \frac{d^2u(t)}{dt^2} + F(t, u(t))$$

yields $u \in C(I, V \cap W_2^{k+l}(\Omega'))$. Thus, the proof is complete.

Remark 3. We have used regularity results in the interior of the domain Ω. If $\partial\Omega \in C^k$ and if we have, e.g., homogeneous Dirichlet boundary value problem (in the more general case see [8]) then the results obtained hold true for Ω (instead of Ω').

Remark 4. All the results can be obtained also in the case $a_{ij} = a_{ij}(t, x)$ under the assumptions

$$\frac{\partial^2 a_{ij}}{\partial t^2} \in L_\infty(\Omega) \quad \left(\text{in Theorem 2 we need } \frac{\partial^3 a_{ij}}{\partial t^3} \in L_\infty(\Omega)\right).$$

The proofs are technically more complicated (in this direction see also [1]).

II. HYPERBOLIC VARIATIONAL INEQUALITIES

Applying Rothe's method we reduce the solution of the hyperbolic variational inequality (6) to the solution of the elliptic variational inequalities (7). Existence and uniqueness of the solutions $z_i \in K$ $(i = 1, \ldots, n)$ of the variational inequality (7) is guaranteed, e.g., by [12] (Theorems 8.1, 8.2). By means of z_i $(i = 1, \ldots, n)$ we define $u_i = U_0 + \sum_{j=1}^{i} hz_i$ for $1 \leqq i \leqq n$ and $u_0 = U_0$, $u_{-1} = -hU_1 + u_0$, $u_{-2} = h^2(-AU_0 + F(0, U_0)) + 2u_{-1} - u_0$, where U_0, U_1 are from (2). In this part we shall assume $U_0 \in V$, $U_1 \in K$. Analogously as in Part I we define $u_n(t)$, $\bar{u}_n(t)$, $z_n(t)$, $\bar{z}_n(t)$, $s_n(t)$ and $\bar{s}_n(t)$.

A priori estimates. Lemma 5. *Let $U_1 \in K$, $AU_0 \in H$ and let* (8), (9) *be satisfied. Then there exist n_0 and C such that the estimate*

$$\|u_i\| + \|z_i\| + |s_i| \leqq C \tag{27}$$

holds for all $n > n_0$ and $i = 1, \ldots, n$.

Proof. Let us consider (7) for $i = j$, $z = z_j$, $v = z_{j-1}$ and $i = j - 1$, $z = z_{j-1}$, $v = z_j$. Subtracting these inequalities we obtain

$$|s_j|^2 + h^{-1}\langle A(u_j - u_{j-1}), z_j - z_{j-1}\rangle \leqq$$
$$\leqq \left(F(t_j, u_{j-1}) - F(t_{j-1}, u_{j-2}), \frac{z_j - z_{j-1}}{h}\right) + (s_j, s_{j-1})$$

and hence

$$|s_j|^2 + 2^{-1}\|z_j\|^2 + 2^{-1}\|z_j - z_{j-1}\|^2 - 2^{-1}\|z_{j-1}\|^2 \leqq$$
$$\leqq 2^{-1}|s_j|^2 + 2^{-1}|s_{j-1}|^2 + hL(1 + \|u_{j-1}\| + \|z_{j-1}\|)\,|s_j|\,.$$

From this estimate analogously as in Section I we obtain

$$(|s_j|^2 + \|z_j\|^2)(1 - C_1 h)^j \leqq \|z_0\|^2 + |s_0|^2 + C_2 T \sum_{i=1}^{j-1} h\|z_i\|^2 + C_3 \leqq \tag{28}$$
$$\leqq C(1 + \|U_0\|^2 + \|U_1\|^2 + |AU_0|^2 + \sum_{i=1}^{j-1} h\|z_i\|^2)\,.$$

In particular we have

$$\|z_j\|^2 \leqq C_1 + C_2 \sum_{i=1}^{j-1} h\|z_i\|^2\,,$$

which implies (Gronwall's lemma) $\|z_j\|^2 \leqq C$ for all $n, j = 1, \ldots, n$. Hence and from $u_i = U_0 + h\sum_{j=1}^{i} z_j$ the estimate $\|u_i\| \leqq C$ follows. Thus, (28) implies the required result.

As a consequence of Lemma 4, analogously as in Part I, the a priori estimates

$$\|u_n(t)\| + \|z_n(t)\| + |s_n(t)| \leqq C\,; \tag{29}$$

$$\|\bar{u}_n(t)\| + \|\bar{z}_n(t)\| + |\bar{s}_n(t)| \leqq C\,; \tag{30}$$

$$\|u_n(t) - \bar{u}_n(t)\| + |z_n(t) - \bar{z}_n(t)| \leqq \frac{C}{n}\,; \tag{31}$$

$$\|u_n(t) - u_n(t')\| + |z_n(t) - z_n(t')| \leqq C|t - t'| \tag{32}$$

hold for all n and $t, t' \in I$. We rewrite the inequality into the form

(33)
$$\left(\frac{\mathrm{d}^- z_n(t)}{\mathrm{d}t}, v - \bar{z}_n(t)\right) + \langle A\,\bar{u}_n(t), v - \bar{z}_n(t)\rangle \geqq \left(F^{(n)}\left(t, \bar{u}_n\left(t - \frac{T}{n}\right)\right), v - \bar{z}_n(t)\right)$$

for all $t \in (0, T)$.

Before passing to the limit for $n \to \infty$ in (33) we prove the convergence of $\{u_n(t)\}$ and $\{z_n(t)\}$.

Lemma 6. *Let $U_1 \in K$, $AU_0 \in H$ and let (8), (9) be satisfied. Then there exists $u \in C(I, V)$ with $\mathrm{d}u/\mathrm{d}t \in L_\infty(I, V) \cap C(I, H)$, $\mathrm{d}^2u/\mathrm{d}t^2 \in L_\infty(I, H)$ such that $u_n \to u$ in $C(I, V)$ and $z_n \to \mathrm{d}u/\mathrm{d}t$ in $C(I, H)$. The estimate*

$$\left|\frac{\mathrm{d}^- u_n(t)}{\mathrm{d}t} - \frac{\mathrm{d}u(t)}{\mathrm{d}t}\right|^2 + \|u_n(t) - u(t)\|^2 \leqq \frac{C}{n}$$

takes place for all n and $t \in I$.

Proof. Let us consider (33) for $n = r$, $v = \bar{z}_s(t)$ and $n = s$, $v = \bar{z}_r(t)$. Subtracting these inequalities we obtain (t is omitted)

$$\left(\frac{\mathrm{d}^-(z_r - z_s)}{\mathrm{d}t}, z_r - z_s\right) + \langle A(\bar{u}_r - \bar{u}_s), \bar{z}_r - \bar{z}_s\rangle \leqq \left(F^{(r)}\left(t, \bar{u}_r\left(t - \frac{T}{r}\right)\right) -\right.$$
$$\left. - F^{(s)}\left(t, \bar{u}_s\left(t - \frac{T}{s}\right)\right), \bar{z}_r - \bar{z}_s\right) + \left(\frac{\mathrm{d}^-(z_r - z_s)}{\mathrm{d}t}, z_r - \bar{z}_r + \bar{z}_s - z_s\right).$$

Hence and from (31), analogously as in Lemma 3, we obtain the required result. The assertion $\mathrm{d}u/\mathrm{d}t \in L_\infty(I, V)$ follows from (32) after passing to the limit for $n \to \infty$. The proof is complete.

Theorem 3. *Suppose (8), (9) and $U_0 \in V$, $U_1 \in K$, $AU_0 \in H$. Then there exists a unique solution of the hyperbolic variational inequality (6) with the following properties:*

$$u \in C(I, V), \quad \frac{\mathrm{d}u}{\mathrm{d}t} \in C(I, H) \cap L_\infty(I, K), \quad \frac{\mathrm{d}^2u}{\mathrm{d}t^2} \in L_\infty(I, H).$$

The estimate

$$\|u_n(t) - u(t)\|^2 + \left|\frac{\mathrm{d}^- u_n(t)}{\mathrm{d}t} - \frac{\mathrm{d}u}{\mathrm{d}t}\right|^2 \leqq \frac{C}{n}$$

takes place for all n and $t \in I$.

Proof. Integrating (33) over (t_1, t_2), $0 < t_1 < t_2 < T$, we have

$$\text{(34)} \qquad \int_{t_1}^{t_2} \left(\frac{\mathrm{d}^- z_n(t)}{\mathrm{d}t}, v - \bar{z}_n(t)\right) \mathrm{d}t + \int_{t_1}^{t_2} \langle A\, \bar{u}_n(t), v - \bar{z}_n(t)\rangle\, \mathrm{d}t \geqq$$
$$\geqq \int_{t_1}^{t_2} \left(F^{(n)}\left(t, \bar{u}_n\left(t - \frac{T}{n}\right)\right), v - \bar{z}_n(t)\right) \mathrm{d}t\,.$$

Owing to (30) there exists a subsequence of $\{\bar{s}_n(t)\}$ and $s \in L_2(I, H)$ such that $\bar{s}_n \rightharpoonup s$ in $L_2(I, H)$. Passing to the limit for $n \to \infty$ in the identity

$$(z_n(t), v) - (U_1, v) = \int_0^t (\bar{s}(\tau), v) \, d\tau$$

and using $z(t) = du(t)/dt$ we easily find out $s(t) = d^2u(t)/dt^2$. Hence and from Lemma 6 we conclude

$$\int_{t_2}^{t_1} \left(\frac{d^- z_n(t)}{dt}, v - \bar{z}_n(t) \right) dt \to \int_{t_1}^{t_2} \left(\frac{d^2 u(t)}{dt^2}, v - \frac{du(t)}{dt} \right) dt$$

for $n \to \infty$ and arbitrary $v \in K$. From (30) and Lemma 6 we conclude that there exists a subsequence of $\{\bar{z}_n\}$ (we use the index n again) such that $\bar{z}_n \rightharpoonup du/dt$ in $L_2(I, V)$ (moreover, w^* – weakly in $L_\infty(I, V)$). Hence and from Lemma 6 we obtain

$$\int_{t_1}^{t_2} \langle A \, \bar{u}(t), v - \bar{z}_n(t) \rangle \, dt \to \int_{t_1}^{t_2} \left\langle A \, u(t), v - \frac{du(t)}{dt} \right\rangle dt \quad \text{for} \quad n \to \infty$$

and for all $v \in K$. Finally, from the estimates (30)–(32), (9) and Lemma 6 we easily obtain

$$\int_{t_1}^{t_2} \left(F^{(n)}\left(t, \bar{u}_n\left(t - \frac{T}{n}\right)\right), v - \bar{z}_n(t) \right) dt \to \int_{t_1}^{t_2} \left(F(t, u(t)), v - \frac{du(t)}{dt} \right) dt$$

for $n \to \infty$. Thus, taking the limit for $n \to \infty$ in (34) we obtain (6) since $t_1, t_2 \in I$ are arbitrary. For fixed $t \in (0, T)$ there exists a subsequence $\{\bar{z}_{n_k}(t)\}$ such that $\bar{z}_{n_k}(t) \rightharpoonup$ $\rightharpoonup w_t$ in V because of (30). On the other hand, $\bar{z}_{n_k}(t) \to du(t)/dt$ in H and hence $w_t =$ $= du(t)/dt$. Thus, the original sequence $\bar{z}_n(t)$ weakly converges to $du(t)/dt$ in V for all $t \in I$. Since $\bar{z}_n(t) \in K$ (for all n and $t \in I$) and K is a closed convex set, $du(t)/dt \in K$ for all $t \in I$. Lemma 6 implies $du(0)/dt = z(0) = U_1$ since $z_n(0) = U_1$. Similarly we have $u(0) = U_0$. The uniqueness of such a solution can be proved from (6) by a standard procedure. When u_1, u_2 are two solutions of (6) then $u = u_1 - u_2$ satisfies the inequality

$$\left(\frac{d^2 u(t)}{dt^2}, \frac{du(t)}{dt} \right) + \left\langle A \, u(t), \frac{du(t)}{dt} \right\rangle \leqq \left(F(t, u_1(t)) - F(t, u_2(t)), \frac{du(t)}{dt} \right).$$

Integrating this inequality and using (8), (9) we have

$$\left| \frac{du(t)}{dt} \right|^2 + \|u(t)\|^2 \leqq C \int_0^t \|u(\tau)\| \left| \frac{du(\tau)}{d\tau} \right| d\tau \, .$$

Since $du(0)/dt = 0$ and $u(0) = 0$, owing to Gronwall's lemma we conclude $u(t) = 0$ which implies uniqueness. The proof of Theorem 3 is complete.

Remark 5. All the results in Section I and Section II hold true if we assume

(8′) there exists $\alpha > 0$ such that $\langle Au, v \rangle + \alpha(u, v)$ is an equivalent scalar product in V instead of (8).

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Author's address: 842 15 Bratislava, Mlýnska dolina, ČSSR (Ústav aplikovanej matematiky a výpočtovej techniky).