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GRAPHS WHOSE ENERGY DOES NOT EXCEED 3

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INTRODUCTION

Throughout the paper, we consider only finite connected graphs having no loops or multiple edges. The *spectrum* of such a graph G is the set of eigenvalues of its $0-1$ adjacency matrix $A(G)$. The sum of all its positive eigenvalues is denoted by $S(G)$, and called the *energy* of G.

For any real $a \geqq 1$, we consider the class of graphs

$$P(a) = \{G \mid S(G) \leqq a\},$$

and, in this paper, we completely describe the class $P(3)$.

Briefly, any graph $G \in P(3)$ is called – *admissible*, and any other graph – *impossible*.

Let G' be any connected (induced) subgraph of a graph G, which is denoted by $G' \subseteqq G$. Since by the known interlacing theorem [1, p. 19] $S(G') \leqq S(G)$, we have that any connected subgraph of an admissible graph is admissible, too. It implies that the method of forbidden subgraphs can be consistently applied.

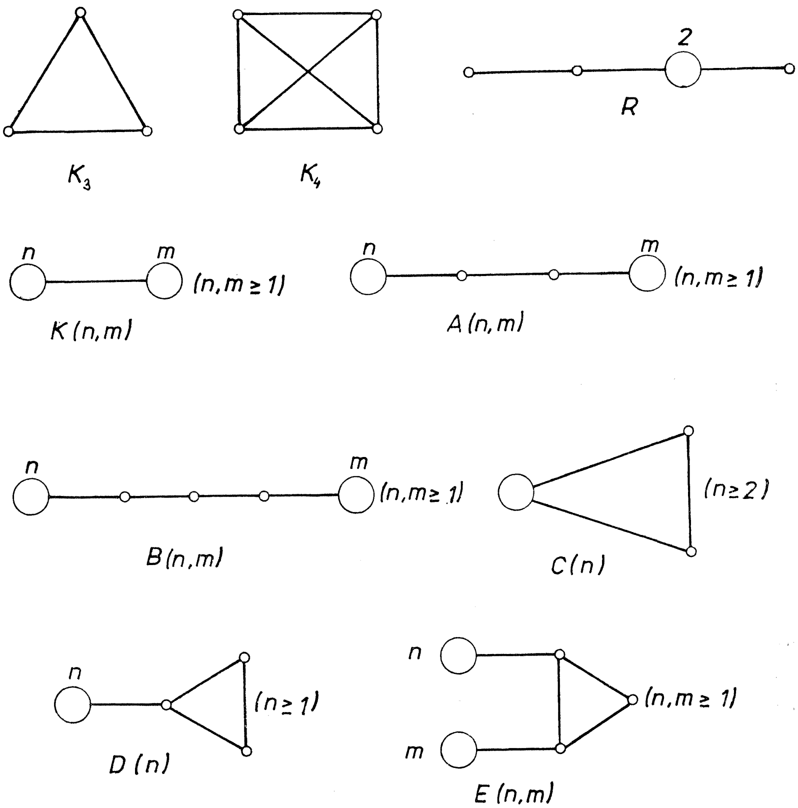
Throughout the paper, K_n, P_n, C_n will be the complete graph, the path and the cycle with n vertices, respectively, while $K_{n,m}$ is the complete bipartite graph with $n + m$ vertices.

In this paper, without a special reference, we often use the lists of spectra of all connected graphs with 2, 3, 4 or 5 vertices (see [1]), or connected graphs with 6 vertices (112 graphs; an internal publication). Using these lists, for each particular graph with this number of vertices we determine whether it is admissible or not.

RESULTS

Denote by a circle any set of isolated vertices, and by the line between two circles the fact that there are all edges between these circles.

Then, by the direct inspection of spectra of all connected graphs with 2, 3, 4, 5 or 6 vertices, we have that all admissible graphs with at most 6 vertices belong to one of the following classes of graphs:



Now, we determine the exact values of parameters for which the above graphs are admissible.

Lemma 1. *The graph* $K(n, m)$ $(1 \leqq n \leqq m)$ *is admissible exactly in the following cases*:

1° $n = 1,\ m \leqq 9$;

2° $n = 2,\ m = 2, 3, 4$;

3° $n = m = 3$.

Proof. As is easily seen, this graph is admissible if and only if $nm \leqq 9$ holds, whence the statement is immediate. □

Lemma 2. *The graph* $A(n, m)$ $(1 \leqq n \leqq m)$ *is admissible exactly in the following cases*:

1° $n = 1,\ m = 1, 2, 3$;

2° $n = m = 2$.

Proof. Immediately, this graph is admissible if and only if $\sqrt{n} + \sqrt{m} \leqq 2\sqrt{2}$, whence the statement is obvious. □

Lemma 3. *The graph* $B(n, m)$ $(1 \leqq n \leqq m)$ *is admissble exactly in the following cases*:

1° $n = 1$, $m = 1, 2, 3$;

2° $n = m = 2$.

Proof. As is easily seen, the graph $B(n, m)$ is admissible if and only if $n + m + 2\sqrt{(n + m)} \leqq 9$, whence the statement. □

Lemma 4. *The graph* $C(n)$ $(n \geqq 2)$ *is admissible iff* $n = 2, 3$.

Proof. Indeed, since it is a complete 3-partite graph, it has exactly one positive eigenvalue $r_n = r(C(n))$, and $r_n = (1 + \sqrt{(1 + 8n)})/2 \leqq 3$ iff $n = 2, 3$. □

Lemma 5. *The graph* $D(n)$ $(n \geqq 1)$ *is admissible iff* $n = 1, 2$.

Proof. The graphs $D(1)$, $D(2)$ are admissible while $D(3)$ is not. Hence, all $D(n)$ $(n \geqq 3)$ are impossible. □

Lemma 6. *The graph* $E(n, m)$ $(n, m \geqq 1)$ *is admissible iff* $n = m = 1$.

Proof. Indeed, since $E(1, 1)$ is admissible and $E(1, 2)$ is an impossible graph, we have that $E(n, m)$ for $n \geqq 2$ or $m \geqq 2$ are impossible graphs. □

Now, we prove the main result of the paper.

Theorem 1. *Each admissible graph G is one of the graphs displayed in Figure* 1.

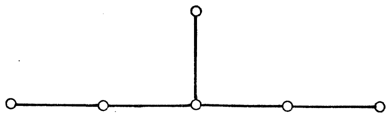
Proof. We distinguish the following three cases:

I. There is no C_3 or C_4 in G as a subgraph.

II. There is a C_3 in G.

III. There is C_4 but no C_3 as an induced subgraph in G.

Case I. Since C_n $(n \geqq 5)$ cannot be a subgraph of an admissible G, we conclude that, in this case, there is no contour in G; thus G is a tree. Further, since there is no P_5 or



in G, we have that G must be one of the graphs $K(1, n)$ $(n \geqq 1)$, $A(n, m)$ $(n, m \geqq 1)$, $B(n, m)$ $(n, m \geqq 1)$.

Case II. Denote the vertices of $L = C_3$ in G by 1, 2, 3. Next, denote by T_i $(i = 1, 2, 3)$ the vertices of G which are (with respect to L) adjacent exactly to the vertex i; the notations T_{12}, T_{13}, T_{23} and T_{123} have a similar meaning. Put

$$T = T_1 + T_2 + T_3 + T_{12} + T_{13} + T_{23} + T_{123} .$$

Next, denote by $\tilde{T}_i$ the vertices of G (non-adjacent to L) which are (with respect to T) adjacent exactly to some vertices of T_i; the notations $\tilde{T}_{ij}$ $(i \neq j)$ and $\tilde{T}_{123}$ have a similar meaning.

Now, we are interested in determining the edge structure of each particular subset between T_i, T_{ij} and T_{123}, as well as the edge structure between these subsets.

For any two subsets A, B we use the notation $A/A = 0$ if A consists of isolated vertices only, $A/A = 1$ if it is complete, $A/B = 0$ or 1 or $\emptyset$ or $*$, if there is no edge between A and B, or there are all such edges, or A and B are not consistent, or we cannot determine this structure, respectively.

All the above information is obtained by choosing arbitrarily two vertices $a \in A$, $b \in B$, then testing the subgraph $123ab$ in the two possible cases: either a, b are adjacent or not.

So, considering the impossible graphs of order 5, we easily obtain the following relations:

$$T_i/T_i = 0 \ (i = 1, 2, 3)\,, \quad T_{ij}/T_{ij} = 0 \ (i \neq j)\,,$$

$$|T_{123}| \leqq 1\,, \quad T_i/T_j = 0 \ (i \neq j)\,, \quad T_i/T_{ij} = \emptyset\,,$$

$$T_i/T_{jk} = \emptyset\,, \quad T_i/T_{123} = \emptyset\,, \quad T_{ij}/T_{ik} = \emptyset\,, \quad T_{ij}/T_{123} = \emptyset\,.$$

Next, testing the graph $123abc$ $(a \in T_1,\ b \in T_2,\ c \in T_3)$, we obtain that the 3-tuple T_1, T_2, T_3 is not consistent in G.

Similarly, we obtain that

$$\tilde{T}_i = \emptyset\,, \quad \tilde{T}_{ij} = \emptyset\,, \quad \tilde{T}_{123} = \emptyset\,,$$

which implies that each admissible G, in the case II, consists of $L = C_3$ and possibly of the classes T_i, T_{ij}, T_{123}.

In view of all the above results, excluding the symmetric cases, we have that G, in case II, consists only of one of the following subsets: $L = C_3$, $L + T_1$, $L + T_{12}$, $L + T_{123}$, $L + T_1 + T_2$.

Consequently, G is one of the following graphs: $C_3, K_4, C(n)$ $(n \geqq 2)$, $D(n)$ $(n \geqq 1)$, $E(n, m)$ $(n, m \geqq 1)$.

Case III. Denote the vertices of $L = C_4$ in G by 1, 2, 3, 4.

Then, similarly as in the previous case, we have the subsets T_i, T_{ij}, T_{ijk} and T_{1234} in G.

By assumption, or by considering the forbidden subgraphs, we easily conclude that

$$T_{12} = T_{23} = T_{34} = T_{14} = \emptyset\,, \quad T_{ijk} = \emptyset \quad \text{and} \quad T_{1234} = \emptyset\,,$$

so in T only the subsets T_i $(i = 1, 2, 3, 4)$ and T_{13}, T_{24} remain.

By the impossible graphs of order 6 (and by the assumption), we conclude that $\tilde{T}_i = \emptyset$ and $\tilde{T}_{13} = \tilde{T}_{24} = \emptyset$. Thus, in this case, each admissible G consists only of the subsets L, T_1, T_2, T_3, T_4, T_{13} and T_{24}.

As in case II, we conclude that

$$|T_i| \leqq 1\,, \quad T_{13}/T_{13} = T_{24}/T_{24} = 0\,,$$
$$T_i/T_j = \emptyset\,, \quad T_i/T_{13} = T_i/T_{24} = \emptyset\,, \quad T_{13}/T_{24} = 1\,.$$

Hence, excluding the symmetric cases, we have that G consists of one of the following subsets:

$$L = C_4\,, \quad L + T_1\,, \quad L + T_{13}\,, \quad L + T_{13} + T_{24}\,.$$

Consequently, in the case III, G must be one of the following graphs: $K(2, 2)$, R, $K(2, n)$ $(n \geqq 3)$, $K(n, m)$ $(n, m \geqq 3)$, which completes the proof. □

Note that Theorem 1 and Lemmas 1 – 6 completely describe the class P(3).

Moreover, note that the previous results imply that class $P(a)$ is finite if $a = 3$. In the following theorem, we prove this for any $a \geqq 1$.

Theorem 2. *The class $P(a)$ is finite for any $a \geqq 1$.*

Proof. Choose an arbitrary graph $G \in P(a)$ and its arbitrary (not necessarily induced) subgraph $K(1, n)$. Then, since $a \geqq S(G) \geqq r(G) \geqq \sqrt{n} = r(K(1, n))$, where $r(G)$ is the spectral radius of G (see Theorem 0.9 [1, p. 19] for the last inequality), we conclude that $K(1, n) \in P(a)$, thus all such n are uniformly bounded by $b = a^2$. Hence, the degrees of all vertices in G cannot exceed the constant b.

Next, choose any path P_n. Since, for an arbitrary $q \in N$, its q-th positive eigenvalue tends to 2 as $n \to \infty$, we get that $S(P_n) \to \infty$ as $n \to \infty$. Consequently, for any path $P_n \in P(a)$ we have that all n's are uniformly bounded by a constant $k = f(a)$.

Now assume, contrary to the statement, that the set $P(a)$ is infinite for an $a \geqq 1$. Then it is easily seen that either there is a sequence of complete bipartite graphs $K(1, n_i) \in P(a)$ $(n_1 < n_2 < \ldots)$, or there is a sequence of paths $P(n_i) \in P(a)$ $(n_1 < n_2 < < \ldots)$, and both these cases yield contradictions.

This proves the theorem. □

References

[1] *D. M. Cvetković, M. Doob, H. Sachs:* "Spectra of graphs — Theory and Application", VEB Deutscher Verlag der Wissen., Berlin, 1980: Academic Press, New York, 1980.

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