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# LANDESMAN-LAZER TYPE CONDITION AND NONLINEARITIES WITH LINEAR GROWTH*)

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## 1. INTRODUCTION

Let us consider the *nonlinear boundary value problem* (BVP) *at resonance*

$$(1)\qquad \begin{aligned} &x''(t) + m^2\, x(t) + g(x(t)) = e(t)\,, \quad t \in [0, \pi]\,,\\ &x(0) = x(\pi) = 0\,, \end{aligned}$$

where $e \in C([0, \pi])$, $m \in \mathbb{N}$ and $g\colon \mathbb{R} \to \mathbb{R}$ is a bounded continuous function on $\mathbb{R}$ such that the following limits are finite:

$$(2)\qquad \lim_{x\to\infty} g(x) = g(\infty) \quad\text{and}\quad \lim_{x\to-\infty} g(x) = g(-\infty)\,.$$

Then BVP (1) has at least one solution provided that

$$(3)\qquad \begin{aligned} &g(-\infty) \int_0^\pi (\sin mt)^+ \,\mathrm{d}t - g(\infty) \int_0^\pi (\sin mt)^- \,\mathrm{d}t <\\ &< \int_0^\pi e(t) \sin mt \,\mathrm{d}t <\\ &< g(\infty) \int_0^\pi (\sin mt)^+ \,\mathrm{d}t - g(-\infty) \int_0^\pi (\sin mt)^- \,\mathrm{d}t \end{aligned}$$

(where $(\sin mt)^+$ and $(\sin mt)^-$ denote the positive and negative parts of $\sin mt$, respectively).

The proof of this assertion may be found e.g. in Fučík [53, Th. 6.4]. It is based on the *Ljapunov-Schmidt reduction* combined with *Schauder's fixed point theorem.* It is worth mentioning that if we suppose $g(-\infty) < g(x) < g(+\infty)$ for any $x \in \mathbb{R}$, the condition (3) is also necessary for the solvability of BVP (1) (see [53]).

The first results in this direction were obtained by Lazer and Leach [73], Landesman and Lazer [72], and Williams [119]. Since 1970 many people have been working on nonlinear BVPs of the type (1) and the results [72], [73], [119] were generalized in various directions. The reader is refered to Fučík [53] for the survey of the results and for an exhaustive list of bibliography up to 1979. For this reason we mention only the first works on this topic and concentrate our attention to results published in 1980 and later.

*) Dedicated to the 10th anniversary of the death of Professor Svatopluk Fučík.

In papers on the problems under consideration which followed after [72], [73], [119] not only *Dirichlet* or *Neumann BVPs* for (1) were considered but also *nonlinear periodic BVPs* for the *equations* or *systems of Liénard type*, *Dirichlet* or *Neumann BVPs for partial differential equations.*

The study of the resonance problems of the type (1) is closely related to the behaviour of the nonlinear term $g$, namely, in connection with the Landesman-Lazer type condition (3). It is clear that if $g(x) = 0$ then BVP is solvable for any $e$ satisfying

$$(4) \qquad \int_0^\pi e(t) \sin mt \, dt = 0 .$$

However, the condition (3) gives no information about the solvability of (1) in the case $g(\infty) = g(-\infty)$. Nevertheless, some further information about the asymptotic behaviour of $g$ guarantees that (4) is sufficient for the solvability of BVP (1) (see e.g. Fučík [53, Chapter 23], Cañada [13], Drábek [34, 35, 36], deFigueiredo and W. M. Ni [50], Hofer [60], Iannacci, Nkashama and Ward [64]).

If the function $g$ has no limits at $\pm\infty$ then the condition (3) can be generalized in the sense that we write $\liminf_{x\to+\infty} g(x)$ and $\limsup_{x\to-\infty} g(x)$ instead of $g(+\infty)$ and $g(-\infty)$ in (3) (see Fučík [53, 11.4]). But the set of the right hand sides $e$ satisfying (3) may be empty, for instance in the case $g(x) = \sin x$. This situation occurs if we consider *periodic problems for pendulum-like equations* or for *systems* of such equations (see Fučík [53, Chapters 19–22], Caristi [19], Dancer [27], Ding [29], Drábek and Invernizzi [40], Fournier and Mawhin [52], Kannan and Ortega [67, 68], Mawhin [83, 84], Mawhin and Willem [87], Ortega [96], Ward [112, 116]) or *Liénard equations* and *systems* (see Fučík [53, Chapter 22], Caristi [20], Conti [25], Habets and Nkashama [58], Mawhin and Ward [86], Nkashama [90, 91], Zanolin [120]), and other types of problems (see Cañada and Martines-Amores [14, 15], Caristi [18], Ding [30], deFigueiredo [48], Petryshyn and Yu [97, 98], Kent Nagle and Singhofer [70]). In these cases, usually some restrictions are imposed on the norm of the forcing term $e$ in order to get the existence result. However, in some special cases when variational approach can be used such restrictions are not necessary (see Mawhin and Willem [87], Lupo and Solimini [82], Solimini [109], Ward [116]).

A large number of generalizations of the result due to Landesman and Lazer is connected with the *removal of the boundedness* of $g$. Particularly, one can prove the same result as in the case of *bounded nonlinearity* of $g$ if $g$ has a sublinear growth (see Fučík [53, Chapter 14] and Section 3). The situation becomes more complicated if we deal with nonlinearities $g$ having linear growth at $\pm\infty$. In this case simple examples show that in order to get the existence result we have to assume in addition to (3), that the growth of $g$ at $\pm\infty$ is not too fast (see Fučík [53,Chapter 15], Ahmad [1, 2], Ahmad and Lazer [3], Arias [4], d'Aujourd'hui [6, 7],Berestycki and deFigueiredo [11], Castro [17], Cesari and Kannan [21], Ding [31, 32], Drábek [37, 38, 39], Fernandes, Omari and Zanolin [44], deFigueiredo [46], Gupta [57], Iannacci and Nkashama [61, 62], Iannacci, Nkashama, Omari and Zanolin [63], Iannacci, Nkashama and Ward [64], Kannan, Lakshmikantham and Nieto [66], Omari and Zanolin [93, 94, 95], Ruf [102], Sanches [106], Ward [117], Willem [118]).

Roughly speaking, if the nonlinearity is of linear growth, we have to know that

$$\text{(5)} \qquad \lim_{x \to +\infty} x^{-1} g(x) \quad \text{and} \quad \lim_{x \to -\infty} x^{-1} g(x)$$

do not interract (in some sense) with the spectrum of the linear part of the equation (see Section 4).

The values (5) play an important role in the problem of *multiplicity of the solutions.* It appears that the number of solutions of BVP (1) is closely related to the number of eigenvalues of the linear part which lie between the numbers (5) (see e.g. d'Aujourd'hui [5, 8], Chiappinelli, Mawhin and Nugari [24], Costa, deFigueiredo and Consalves [26], Dancer [28], Fabry, Mawhin and Nkashama [43], Gallouët and Kavian [54, 55], Hart, Lazer and McKenna [59], Kent Nagle and Singhofer [71], Lazer and McKenna [74–81], McKenna, Redlinger and Walter [88], Ruf [101], Ruf and Srikanth [105], Schmitt [107], Solimini [108]).

Let us note that the situation is different in the case when the nonlinear term has *superlinear growth* at $+\infty$ or at $-\infty$. In this case the interval with end points $\lim_{x \to +\infty} x^{-1} g(x)$ and $\lim_{x \to -\infty} x^{-1} g(x)$ may contain infinitely many eigenvalues of the linear part and one can get interesting existence and multiplicity results (see Fučík [53, Parts IX and X], Bahri and Berestycki [9, 10], Brézis [12], Cañada and Ortega [16], Chang [22, 23], Drábek [33], deFigueiredo [47], deFigueiredo and Solimini [49], Fortunato and Jannelli [51], Gallouët and Morel [56], Kannan and Ortega [69], Mawhin [85], Milojevič [89], Omari, Villari and Zanolin [92], Ramaswamy [99], Ruf [100], Ruf and Solimini [103], Ruf and Srikanth [104], Triebel [111], Ward [113–115]).

From now on we shall suppose that the nonlinear function $g$ grows at most linearly at $\pm\infty$. The purpose of this paper is to show that if the linear growth of $g$ at $\pm\infty$ is in some sense related to the spectrum of the linear part: $x \mapsto x'' + \lambda x$ then we get the existence result similar to the result of Landesman and Lazer [72].

The paper is organized as follows. In Section 2 we state the main hypotheses on the nonlinear function $g = g(t, x)$ and prove some auxiliary assertions. Section 3 deals with bounded and sublinear nonlinearities. We shall present the method which is (in some sense) simpler and more general than that used in Fučík [53, Chapters 13 and 14]. It is shown, in Section 4, that essentially the same method can be used to prove the existence result for BVPs with linearly growing nonlinearities. Concerning nonlinearities which do not interract with the spectrum of the linear problem

$$\text{(6)} \qquad \begin{aligned} &x''(t) + \lambda\, x(t) = 0\,, \quad t \in [0, \pi]\,, \\ &x(0) = x(\pi) = 0 \end{aligned}$$

we get the same results as Iannacci and Nkashama [61, 62]. Nonetheless, the investigation of the generalized eigenvalue problem

$$(7) \qquad x''(t) + \mu\, x^+(t) - \nu\, x^-(t) = 0\,, \quad t \in [0, \pi]\,,$$
$$x(0) = x(\pi) = 0\,,$$

allows us to consider more general nonlinearities $g$. Particularly, $g$ may be a jumping nonlinearity in the sense of Fučík [53, Part XI].

## 2. PRELIMINARIES

Let $g\colon [0, \pi] \times \mathbb{R} \to \mathbb{R}$ be Carathéodory's function, i.e. $g(., x)$ is measurable for all $x \in \mathbb{R}$ and $g(t, .)$ is continuous for a.a. $t \in [0, \pi]$. Let $e \in L_1(0, \pi)$ and $m \geqq 1$, $m \in \mathbb{N}$. Consider the BVP

$$(8) \qquad x''(t) + m^2\, x(t) + g(t, x(t)) = e(t)\,, \quad t \in [0, \pi]\,,$$
$$x(0) = x(\pi) = 0\,.$$

By the solution of BVP (8) we shall always mean a function $x \in C^1([0, \pi])$ such that $x'$ is absolutely continuous, $x$ satisfies the boundary conditions and the equation (8) holds a.e. in $[0, \pi]$.

We shall suppose that $g$ satisfies the growth restriction

$$(9) \qquad |g(t, x)| \leqq p(t) + c|x|$$

for a.a. $t \in [0, \pi]$, for all $x \in \mathbb{R}$, with some $p \in L_1(0, \pi)$ and $c > 0$. Moreover, we shall assume that there are functions $a, A \in L_1(0, \pi)$ and constants $r, R \in \mathbb{R}$, $r < 0 < R$ with

$$(10) \qquad g(t, x) \geqq A(t) \quad \text{for a.a. } t \in [0, \pi] \quad \text{and all} \quad x \geqq R\,,$$

$$(11) \qquad g(t, x) \leqq a(t) \quad \text{for a.a. } t \in [0, \pi] \quad \text{and all} \quad x \leqq r\,.$$

For nonlinearities $g$ satisfying „dual" one-sided restrictions (i.e. the inequalities (10), (11) are reversed in some sense) see Remark 11. Denote

$$(12) \qquad g_+(t) = \liminf_{x \to +\infty} g(t, x)\,, \quad g_-(t) = \limsup_{x \to -\infty} g(t, x)\,.$$

Note that either $g_+(t) = +\infty$ or $g_-(t) = -\infty$ is possible for some $t \in [0, \pi]$.

Remark 1. It follows from (9), (10), (11) that the function $g$ can be decomposed as

$$(13) \qquad g(t, x) = \gamma(t, x)\, x + h(t, x)\,,$$

where $0 \leqq \gamma(t, x) \leqq q_1(t)$, $|h(t, x)| \leqq q_2(t)$, for all $x \in \mathbb{R}$, a.a. $t \in [0, \pi]$, with some $q_1, q_2 \in L_1(0, \pi)$ (see e.g. Iannacci, Nkashama [62] for details).

Remark 2. It follows from (10), (11) that

$$(14) \qquad \liminf_{x \to \pm\infty} x^{-1}\, g(t, x) \geqq 0$$

for a.a. $t \in [0, \pi]$.

Remark 3. Let us suppose that there are constants $k_1, k_2 > 0$ such that the inequalities

$$0 \leqq \limsup_{x \to +\infty} x^{-1} g(t, x) \leqq k_1 , \quad 0 \leqq \limsup_{x \to -\infty} x^{-1} g(t, x) \leqq k_2$$

hold for a.a. $t \in [0, \pi]$. Then

$$\limsup_{x \to +\infty} \gamma(t, x) \leqq k_1 , \quad \limsup_{x \to -\infty} \gamma(t, x) \leqq k_2$$

for a.a. $t \in [0, \pi]$.

Denote by $H = W_0^{1,2}(0, \pi)$ the usual Sobolev space on $(0, \pi)$ with the inner product

$$(u, v) = \int_0^\pi u'(t)\, v'(t)\, dt$$

and the norm

$$\|u\| = \sqrt{(u, u)} .$$

Let

$$x(t) = \sum_{k=1}^{\infty} a_k \sin kt$$

be the Fourier series of $x \in H$. Then we shall write

$$x(t) = \bar{x}(t) + x^0(t) + \tilde{x}(t) ,$$

where

$$\begin{aligned} \bar{x}(t) &= \sum_{k=1}^{m-1} a_k \sin kt , \\ x^0(t) &= a_m \sin mt , \\ \tilde{x}(t) &= \sum_{k=m+1}^{\infty} a_k \sin kt . \end{aligned}$$

Particularly, $\bar{x} = 0$ if $m = 1$. Put

$$x^{\perp}(t) = x(t) - x^0(t) .$$

**Lemma 1.** *Let us assume that for $n \in \mathbb{N}$ we have $0 \leqq f_n(t)$ for a.a. $t \in [0, \pi]$, and $f_n \rightharpoonup 0$ (weakly) in $L_1(0, \pi)$. Then there exists a constant $\varrho > 0$ such that for all $x \in W^{2,1}(0, \pi) \cap H$ one has*

$$\int_0^\pi [x''(t) + m^2 x(t) + f_n(t)\, x(t)]\, [\bar{x}(t) + x^0(t) - \tilde{x}(t)]\, dt \geqq \varrho \|x^{\perp}(t)\|^2 \tag{15}$$

*for $n$ large enough.*

Proof (cf. [39], [62]). The left hand side of (15) is equal to

$$\begin{aligned} L(x) = \int_0^\pi [-(\bar{x}')^2 + m^2 \bar{x}^2]\, dt + \int_0^\pi f_n(t)\, (\bar{x} + x^0)^2\, dt + \\ + \int_0^\pi [(\tilde{x}')^2 - m^2 \tilde{x}^2 - f_n(t)\, \tilde{x}^2]\, dt . \end{aligned} \tag{16}$$

The second integral in (16) is nonnegative. By the decomposition of $x$ we have

$$\int_0^\pi [-(\bar{x}')^2 + m^2 \bar{x}^2]\, dt \geqq \varrho_1 \|\bar{x}\|^2 \tag{17}$$

with some $\varrho_1 > 0$. Put $\tilde{y} = \|\tilde{x}\|^{-1}\,\tilde{x}$. Then

$$(18)\qquad \int_0^\pi [(\tilde{y}')^2 - m^2\tilde{y}^2 - f_n(t)\,\tilde{y}^2]\,\mathrm{d}t \geqq 2\varrho_2 - \left|\int_0^\pi f_n(t)\,\tilde{y}^2\,\mathrm{d}t\right| \geqq \varrho_2$$

with some $\varrho_2 > 0$ for $n$ large enough because $\|\tilde{y}\| = 1$, the imbedding $H \subset C([0, \pi])$ is completely continuous and $f_n \to 0$ in $L_1(0, \pi)$. Hence it follows from (16), (17) and (18) that

$$L(x) \geqq \varrho_1\|\bar{x}\|^2 + \varrho_2\|\tilde{x}\|^2$$

for all $n$ sufficiently large. Taking $\varrho = \min\{\varrho_1, \varrho_2\}$ we complete the proof.

Let us define operators $J, S, G\colon H \to H$ and an element $e^* \in H$ by

$$\begin{aligned}
(Jx, y) &= \int_0^\pi x'(t)\,y'(t)\,\mathrm{d}t\,,\\
(Sx, y) &= \int_0^\pi x(t)\,y(t)\,\mathrm{d}t\,,\\
(G(x), y) &= \int_0^\pi g(t, x(t))\,y(t)\,\mathrm{d}t\,,\\
(e^*, y) &= -\int_0^\pi e(t)\,y(t)\,\mathrm{d}t
\end{aligned}$$

for all $x, y \in H$.

Remark 4. The operators $S$ and $G$ are completely continuous with respect to the completely continuous imbedding $H \subset C([0, \pi])$. The operator $J$ is the identity on $H$.

We shall say that $x$ is a *weak solution of* (8) if $x \in H$ and

$$(19)\qquad Jx = m^2 Sx + G(x) + e^*\,.$$

Remark 5. The usual regularity argument for ODEs yields immediately (see Fučík [53]) that any weak solution of (8) is also a solution in the sense mentioned above. Moreover, if $e \in C([0, \pi])$ and $g$ is a continuous function then every weak solution of (8) satisfies $x \in C^2([0, \pi])$.

Set

$$\begin{aligned}
C_1 &= \{(\mu, \nu) \in \mathbb{R}^2;\ (\mu - 1)(\nu - 1) = 0\}\,,\\
C_{2k} &= \{(\mu, \nu) \in \mathbb{R}^2;\ k(\mu^{-1/2} + \nu^{-1/2}) = 1\}\,,\\
C_{2k+1} &= \{(\mu, \nu) \in \mathbb{R}^2;\ k(\mu^{-1/2} + \nu^{-1/2}) + \mu^{-1/2} = 1\\
&\qquad \text{or}\quad k(\mu^{-1/2} + \nu^{-1/2}) + \nu^{-1/2} = 1\}
\end{aligned}$$

for $k \geqq 1$, and $C = \bigcup_{m=1}^{\infty} C_m$. It is known that (7) has a nontrivial solution if and only if $(\mu, \nu) \in C$ (see Fučík [53, Chapter 42]). Using the shooting argument it is possible to prove the following

**Lemma 2.** *Let $f_\pm$ be two mappings in $L_\infty(0, \pi)$. Assume that for $m \geqq 1$ there are two points $(\mu_m, \nu_m) \in C_m$, $(\mu_{m+1}, \nu_{m+1}) \in C_{m+1}$ such that $(\mu_m, \mu_{m+1}) \times (\nu_m, \nu_{m+1}) \subset \subset \mathbb{R}^2 \setminus C$, and for a.a. $t \in [0, \pi]$*

$$\mu_m \leqq f_+(t),\ \nu_m \leqq f_-(t) \quad \textit{and} \quad f_+(t) \leqq \mu_{m+1},\ f_-(t) \leqq \nu_{m+1}$$

*with strict inequalities on the same set $I$ and $J$, respectively, of positive measure* *in* $[0, \pi]$. *Then the Dirichlet BVP*

$$x''(t) + f_+(t)\, x^+(t) - f_-(t)\, x^-(t) = 0 \quad \text{on} \quad [0, \pi]\,,$$
$$x(0) = x(\pi) = 0$$

*has only the trivial solution.*

Proof can be found in Invernizzi [65].

## 3. BOUNDED AND SUBLINEAR NONLINEARITIES

We shall suppose that the nonlinear function $g$ satisfies all hypotheses from Section 2 and, moreover,

$$\lim_{x \to \pm\infty} x^{-1} g(t, x) = 0 \tag{20}$$

uniformly for a.a. $t \in [0, \pi]$.

Remark 6. Every bounded or sublinear function $g$ satisfies (20) (see Fučík [53, Chapters 13, 14]).

**Theorem 1.** *Under the above assumptions BVP* (8) *has at least one solution provided that*

$$\int_0^\pi g_-(t)\,(\sin mt)^+\,\mathrm{d}t - \int_0^\pi g_+(t)\,(\sin mt)^-\,\mathrm{d}t < \int_0^\pi e(t) \sin mt\,\mathrm{d}t < \tag{21}$$
$$< \int_0^\pi g_+(t)\,(\sin mt)^+\,\mathrm{d}t - \int_0^\pi g_-(t)\,(\sin mt)^-\,\mathrm{d}t\,.$$

Proof of Theorem 1. With respect to Remark 5 it is sufficient to prove the existence of a solution of equation (19). Fix $\delta \in (0, 2m + 1)$ and define $\mathscr{H}: [0, 1] \times \times H \to H$ by

$$\mathscr{H}(\tau, x) = Jx - m^2 Sx - (1 - \tau)\, \delta Sx - \tau\, G(x) - \tau e^*$$

for all $x \in H$ and $\tau \in [0, 1]$. We shall prove that there is $\xi > 0$ such that

$$\mathscr{H}(\tau, x) \neq 0 \tag{22}$$

for all $\tau \in [0, 1]$ and $x \in H$, $\|x\| = \xi$. Assume that this is not true. Then there is a sequence $\{\tau_n\} \subset [0, 1]$ and a sequence $\{x_n\} \subset H$ such that $\|x_n\| \to \infty$ and

$$\mathscr{H}(\tau_n, x_n) = 0\,. \tag{23}$$

Setting $y_n = \|x_n\|^{-1} x_n$, equation (23) is equivalent to

$$Jy_n - m^2 Sy_n - (1 - \tau_n)\, \delta Sy_n - \tau_n \|x_n\|^{-1} G(x_n) - \tau_n \|x_n\|^{-1} e^* = 0\,. \tag{24}$$

By the assumption (20) one has

$$\lim_{\|x_n\| \to \infty} \tau_n \|x_n\|^{-1} G(x_n) = 0\,. \tag{25}$$

Complete continuity of $S$, (24) and (25) yield that there is $y \in H$ such that $y_n \to y$

in $H$, $\tau_n \to \tau \in [0, 1]$ (taking a subsequence if necessary) and

$$Jy - m^2Sy - (1 - \tau)\,\delta Sy = 0\,.$$

Hence we should have $\tau = 1$ and either $y(t) = (1/m)\,(2/\pi)^{1/2} \sin mt$ or $y(t) = -(1/m)\,(2/\pi)^{1/2} \sin mt$. Let us suppose that $y(t) = (1/m)\,(2/\pi)^{1/2} \sin mt$. Taking the inner product of (23) with $\sin mt$ and realizing that $0 \leqq \tau_n \leqq 1$ we get

$$-\int_0^\pi g(t, x_n(t)) \sin mt \,\mathrm{d}t + \int_0^\pi e(t) \sin mt \,\mathrm{d}t \geqq 0\,, \quad \text{i.e.}$$

$$\liminf_{n\to\infty} \int_0^\pi g(t, x_n(t)) \sin mt \,\mathrm{d}t \leqq \int_0^\pi e(t) \sin mt \,\mathrm{d}t\,. \tag{26}$$

Suppose for a moment that there is a function $\zeta(t) \in L_1(0, \pi)$ such that

$$g(t, x_n(t)) \sin mt \geqq \zeta(t) \tag{27}$$

for a.a. $t \in [0, \pi]$ and for all $n$ sufficiently large. Then Fatou's lemma and (26) yield

$$\int_0^\pi g_+(t)\,(\sin mt)^+ \,\mathrm{d}t - \int_0^\pi g_-(t)\,(\sin mt)^- \,\mathrm{d}t \leqq \int_0^\pi e(t) \sin mt \,\mathrm{d}t\,,$$

a contradiction with (21). Analogously we proceed in the case $y(t) = -(1/m)\,.\,(2/\pi)^{1/2} \sin mt$. Hence (22) is established and $\mathscr{H}$ is an admissible homotopy of compact perturbations of the identity. Then the homotopy invariance property of the Leray-Schauder degree implies

$$\deg[J - m^2S - G - e^*;\, B_\xi, 0] = \deg[J - (m^2 + \delta)\, S;\, B_\xi, 0]\,, \tag{28}$$

where $B_\xi = \{x \in H;\ \|x\| = \xi\}$. The right hand side of (28) is equal to an odd number by the Borsuk theorem (see Fučík [53, Chapter 20]). Particularly, this means that

$$\deg[J - m^2S - G - e^*;\, B_\xi, 0] \neq 0\,,$$

i.e., by the existence theorem (Fučík [53, Chapter 20]), there exists $x \in B_\xi$ such that

$$Jx = m^2Sx + G(x) + e^*\,. \tag{19}$$

To complete the proof it remains to prove (27). It follows from

$$\|x_n\|^{-1}\, x_n(t) = \|x_n\|^{-1}\,(x_n^\perp(t) + x_n^0(t)) \to (1/m)\,(2/\pi)^{1/2} \sin mt = y(t)$$

that

$$\|x_n\|^{-1}\, \|x_n^\perp\| \to 0 \quad \text{and} \quad \|x_n\|^{-1}\, \|x_n^0\| \to 1\,.$$

Then

$$\|x_n^0\|^{-1}\, \|x_n^\perp\| = \|x_n\|\, \|x_n^0\|^{-1}\, \|x_n\|^{-1}\, \|x_n^\perp\| \to 0\,. \tag{29}$$

The regularity argument for ODEs yields that $x_n \in W^{2,1}(0, \pi)$ for any solution $x_n$ of (23). We obtain from (23) and from Lemma 1 (where we put $f_n(t) = (1 - \tau_n)\,\delta + \tau_n\gamma(t, x_n(t))$ that

$$\begin{aligned} 0 &= \int_0^\pi [x_n'' + m^2x_n + (1 - \tau_n)\,\delta x_n + \tau_n\gamma(t, x_n)\, x_n + \\ &\quad + \tau_n h(t, x_n) - \tau_n e]\,[\bar{x}_n + x_n^0 - \tilde{x}_n]\,\mathrm{d}t \geqq \\ &\geqq \varrho\|x_n^\perp\|^2 - (\|q\|_{L_1} + \|e\|_{L_1})(\|x_n^0\| + \|x_n^\perp\|) \end{aligned}$$

for $n$ sufficiently large. Hence there exists a constant $c_1 > 0$, independent of $n$, such that

$$\|x_n^\perp\|^2 \leqq (c_1/2)\,(\|x_n^0\| + \|x_n^\perp\|)\,. \tag{30}$$

The inequality (30) together with (29) imply that there is $n_0 \in \mathbb{N}$ such that

$$\|x_n^\perp\|^2\,\|x_n^0\|^{-1} \leqq c_1 \tag{31}$$

for any $n \geqq n_0$.

Using (31) we get the estimate

$$\begin{aligned}\gamma(t, x_n(t))\,x_n(t)\sin mt &= (m/2)\,(\sqrt{(\pi/2)})\,\|x_n^0\|^{-1}\,\gamma(t, x_n(t))\,.\\ &\quad\cdot\big[(x_n(t))^2 + (x_n^0(t))^2 - (x_n(t) - x_n^0(t))^2\big] \geqq\\ &\geqq -(m/2)\,(\sqrt{(\pi/2)})\,\gamma(t, x_n(t))\,\|x_n^0\|^{-1}\,(x_n^\perp(t))^2 \geqq\\ &\geqq -c_2\gamma(t, x_n(t))\,\|x_n^0\|^{-1}\,\|x_n^\perp\|^2 \geqq -c_2c_1\gamma(t, x_n(t))\,.\end{aligned} \tag{32}$$

Now, on the basis of Remark 1 and (32) one has

$$\begin{aligned}g(t, x_n(t))\sin mt &= \gamma(t, x_n(t))\,x_n(t)\sin mt + h(t, x_n(t))\sin mt \geqq\\ &\geqq -c_2c_1\gamma(t, x_n(t)) - q_2(t) \geqq -c_2c_1\,q_1(t) - q_2(t) \equiv \zeta(t)\end{aligned}$$

for $n \geqq n_0$, where $\zeta(t) \in L_1(0, \pi)$. Hence (27) is established. This completes the proof of Theorem 1.

Remark 7. The reader is invited to compare the above proof with the method used in Fučík [53, Chapters 13, 14], Lazer and Leach [73], Landesman and Lazer [72].

## 4. NONLINEARITIES WITH LINEAR GROWTH

In this section we abandon the assumption (20) but suppose that $g = g(t, x)$ satisfies all hypotheses from Section 2. Using the same approach as in the proof of Theorem 1 we prove now a more general result.

**Theorem 2.** *Let the function $g$ satisfy all hypotheses from Section 2. Assume that there is $(\mu_{m+1}, \nu_{m+1}) \in C_{m+1}$ such that $(m^2, \mu_{m+1}) \times (m^2, \nu_{m+1}) \subset \mathbb{R}^2 \setminus C$ if $m$ is even, and*

$$\limsup_{x\to+\infty} x^{-1}\,g(t, x) \leqq \mu_{m+1} - m^2\,, \quad \limsup_{x\to-\infty} x^{-1}\,g(t, x) \leqq \nu_{m+1} - m^2 \tag{33}$$

*with strict inequalities on certain subsets of $[0, \pi]$ of positive measure. Then BVP (8) has at least one solution provided that (21) holds.*

Proof. The idea is the same as in Section 3. Take $0 < \delta < \min\{\mu_{m+1} - m^2, \nu_{m+1} - m^2\}$ and define the homotopy $\mathscr{H}$. In order to prove (22) we proceed again via contradiction, arriving at (24). It follows from (9) and (24) that there are $y \in H$, $\tau \in [0, 1]$ and $g^* \in H$ such that

$$y_n \to y\,, \quad \tau_n\|x_n\|^{-1}\,G(x_n) \to g^* \text{ in } H\,, \quad \tau_n \to \tau \in [0, 1]\,. \tag{34}$$

Since the sequence $\gamma_n(t) := \gamma(t, x_n(t))$ is both bounded in $L_1(0, \pi)$ and equi-integrable, we can also assume

$$(35) \qquad \gamma_n \rightharpoonup f \quad \text{in} \quad L_1(0, \pi)$$

(see Dunford, Schwartz [42]). It follows from (14), (33) and Remark 3 that

$$(36) \qquad \begin{aligned} &f(t) \geqq 0 \quad \text{a.e. on} \quad [0, \pi] \quad \text{and} \\ &f(t) \leqq \mu_{m+1} - m^2 \quad \text{a.e. on} \quad \{t \in [0, \pi];\ y(t) > 0\}\,, \\ &f(t) \leqq \nu_{m+1} - m^2 \quad \text{a.e. on} \quad \{t \in [0, \pi];\ y(t) < 0\}\,, \end{aligned}$$

with strict inequalities on some subsets of $[0, \pi]$ of positive measure. We obtain by the limiting process in (24) and the usual regularity argument for ODEs that $y \in W^{2,1}(0, \pi) \cap H$ and

$$(37) \qquad y''(t) + m^2 y(t) + (1 - \tau)\,\delta\, y(t) + \tau f(t)\, y(t) = 0\,.$$

Set

$$\begin{aligned} f_+(t) &= m^2 + (1 - \tau)\,\delta + \tau f(t) \quad \text{on} \quad \{t \in [0, \pi];\ y(t) > 0\}\,, \\ f_+(t) &= (m^2 + \mu_{m+1})/2 \quad \text{elsewhere}\,, \\ f_-(t) &= m^2 + (1 - \tau)\,\delta + \tau f(t) \quad \text{on} \quad \{t \in [0, \pi];\ y(t) < 0\}\,, \\ f_-(t) &= (m^2 + \nu_{m+1})/2 \quad \text{elsewhere}\,. \end{aligned}$$

Then $f_\pm$ satisfy the assumptions of Lemma 2 with $\mu_m = \nu_m = m^2$, i.e. (37) has only the trivial solution if $\tau = 1$ and $f(t) \equiv 0$ do not hold simultaneously. Hence we should have $\tau = 1$ (i.e. $\tau_n \to 1$), $f(t) \equiv 0$ (i.e. $\gamma_n \rightharpoonup 0$ in $L_1(0, \pi)$) and either $y = (1/m)(2/\pi)^{1/2} \sin mt$ or $y = -(1/m)(2/\pi)^{1/2} \sin mt$. The rest of the proof is the same as that of Theorem 1.

**Corollary.** *Let $g$ satisfy all hypotheses from Section 2. Moreover, let*

$$(38) \qquad \limsup_{|x| \to \infty} x^{-1} g(t, x) \leqq 2m + 1$$

*with strict inequality on a subset of $[0, \pi]$ of positive measure. Then BVP (8) has at least one solution provided that (21) holds.*

Proof follows from Theorem 2 where we put $\mu_{m+1} = \nu_{m+1} = (m + 1)^2$.

Remark 8. The reader is invited to compare the method of the proof of Theorem 2 with the results in Fučík [53, Chapter 15].

Essentially the same assertion as our Corollary can be proved for periodic problem (see Iannacci and Nkashama [62]) and for BVPs for partial differential equations (see Iannacci and Nkashama [61]).

Remark 9. Elementary calculation yields that if $(\mu_{m+1}, \nu_{m+1}) \in C_{m+1}$ and $\nu_{m+1}$ (or $\mu_{m+1}$) is close to $m^2$ then $\mu_{m+1}$ (or $\nu_{m+1}$) is greater than $2m + 1$ (the distance between eigenvalues $(m + 1)^2$ and $m^2$ of (6)). Hence the difference between the results of Corollary and Theorem 2 may be understood as follows. While the hypotheses of Corollary are satisfied by a nonlinearity $g$ which may asymptotically "touch" the eigenvalue $(m+1)^2$ on the set of positive measure in $[0, \pi]$, the assumptions of Theorem 2 are satisfied also by $g$ "jumping" over $(m+1)^2$.

Remark 10. The assertion similar to Theorem 2 is proved in Drábek [39] for the periodic problem.

Let us present two examples of nonlinear functions $g$ and discuss the solvability of the corresponding BVP (8).

Example 1. Let $g(t, x) = (2m+1)\,x$ (i.e. the function $g$ does not depend on $t$). Then the function $g$ satisfies the assumptions from Section 2 but we have equality in (38) for all $t \in [0, \pi]$. Then condition (21) is fulfilled with any $e \in L_1(0, \pi)$ but BVP (8) has no solution if we take e.g. $e(t) = \sin(m+1)\,t$. It follows that the assumptions of Theorem 2 concerning the growth of $g$ cannot be weakened if we do not distinguish between the growth at $+\infty$ and $-\infty$.

Example 2. Let us define the function $g$ by

$$g(t, x) = \begin{cases} kx & \text{for } x \geqq 0\,, \quad t \in [0, \pi]\,, \\ 0 & \text{for } x < 0\,, \quad t \in [0, \pi]\,, \end{cases}$$

where $k > 0$ is a fixed real number. Then BVP (8) with $m = 1$ and $g$ defined above has a solution for arbitrary $e \in L_1(0, \pi)$, $\int_0^\pi e(t) \sin t \, dt > 0$. Indeed, we have $g_+(t) = +\infty$ on $[0, \pi]$, $g_-(t) = 0$ on $[0, \pi]$, $g$ satisfies the assumptions of Theorem 2 and (21) is fulfilled with any $e \in L_1(0, \pi)$, $\int_0^\pi e(t) \sin t \, dt > 0$. Note that we can take $(\mu_2, \nu_2) \in C_2$ such that $\mu_2 > k + 1$ because $\nu_2$ may be chosen arbitrarily close to 1. On the other hand the function $g$ does not satisfy (38) if $k \geqq 3$, i.e. this case is not covered by Corollary.

Remark 11. Assume that instead of (10), (11) the function $g$ satisfies

$$g(t, x) \leqq a(t) \quad \text{for a.a.} \quad t \in [0, \pi] \quad \text{and all} \quad x \geqq R\,,$$

$$g(t, x) \geqq A(t) \quad \text{for a.a.} \quad t \in [0, \pi] \quad \text{and all} \quad x \leqq r\,.$$

Then it is possible to prove in the same way in some sense "dual versions" of Theorems 1, 2, where $g_+(t)$ and $g_-(t)$ defined by (12) are replaced by

$$\limsup_{x \to +\infty} g(t, x) \quad \text{and} \quad \liminf_{x \to -\infty} g(t, x)\,,$$

respectively, and the inequalities in (21) are reversed.

Remark 12. The result which is similar to our Theorem 2 was proved in Arias [4]. However, the nonlinear function $g = g(x)$ is supposed to be independent of $t$ and to satisfy more restrictive conditions than those stated in Section 2.

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