Czechoslovak Mathematical Journal

Bedřich Pondělíček

On varieties of regular $*$-semigroups, II

Czechoslovak Mathematical Journal, Vol. 41 (1991), No. 3, 512–517

Persistent URL: http://dml.cz/dmlcz/102485

Czechoslovak Mathematical Journal, 41 (116) 1991, Praha

ON VARIETIES OF REGULAR $*$-SEMIGROUPS, II

BEDŘICH PONDĚLÍČEK, Praha

(Received August 18, 1990)

Following I. Chajda [1] by a *diagonal* on an algebra S we shall mean a reflexive and compatible binary relation on S. The set $\operatorname{Ref}(S)$ of all diagonals on S forms a complete lattice with respect to set inclusion. By a *quasiorder* on S is meant a transitive diagonal on S. Analogously, the set $\operatorname{Qua}(S)$ of all quasiorders forms a complete lattice with respect to set inclusion but $\operatorname{Qua}(S)$ is no sublattice of $\operatorname{Ref}(S)$ in a general case, see [2].

The aim of this paper is to describe all varieties of regular $*$-semigroups whose lattices of all diagonals (quasiorders) are modular, distributive and boolean.

Recall that a *regular $*$-semigroup* (see [3]) is an algebra $(S, \cdot, *)$ where $(S, \cdot)$ is a semigroup and $*$ is a unary operation on S satisfying the following:

(1) $$(x^*)^* = x\,, \quad x = xx^*x \quad \text{and} \quad (xy)^* = y^*x^*\,.$$

By $\mathscr{W}(i = j)$ we denote the variety of all regular $*$-semigroups satisfying the identity $i = j$. Terminology and notation not defined here may be found in [4] and [5].

Let S be a regular $*$-semigroup. For $M, N \subseteqq S \times S$ we put

$$\begin{aligned} MN &= \{(ab, cd);\ (a, c) \in M,\ (b, d) \in N\}\,, \\ M^* &= \{(a^*, c^*);\ (a, c) \in M\}\,, \\ \overline{M} &= \{(c, a);\ (a, c) \in M\}\,. \end{aligned}$$

If $M = \{(a, c)\}$ or $N = \{(b, d)\}$, then we simply write $M = (a, c)$ or $N = (b, d)$, respectively. By a *diagonal* A on S we shall mean a reflexive regular $*$-subsemigroup of the direct product $S \times S$, i.e.

(2) $$\mathrm{id}_s \subseteqq A\,, \quad AA \subseteqq A \quad \text{and} \quad A^* \subseteqq A\,.$$

By $\operatorname{Ref}(S)$ we denote the lattice of all diagonals on S with respect to set inclusion. Denote by $\vee$ or $\wedge$ the join or meet in $\operatorname{Ref}(S)$, respectively. The meet evidently coincides with the set intersection. For $M \subseteqq S \times S$ we denote by $\mathrm{R}(M)$ the least diagonal on S containing M. It is easy to show the following:

(3) $(x, y) \in \mathrm{R}(M)$ if and only if
$x = x_1x_2 \ldots x_m$ and $y = y_1y_2 \ldots y_m$
where either $(x_i, y_i) \in M$ or

$(x_i^*, y_i^*) \in M$ or $x_i = y_i$ for
$i = 1, 2, \ldots, m$.

$$A \vee B = \mathrm{R}(A \cup B) \quad \text{for} \quad A, B \in \mathrm{Ref}(S)\,. \tag{4}$$

It is very easy to show that the mapping $A \to \bar{A}$ on $\mathrm{Ref}(S)$ is an involution lattice automorphism on $\mathrm{Ref}(S)$, i.e.

$$\bar{\bar{A}} = A\,, \quad \overline{A \vee B} = \bar{A} \vee \bar{B} \quad \text{and} \quad \overline{A \wedge B} = \bar{A} \vee \bar{B}\,. \tag{5}$$

for any $A, B \in \mathrm{Ref}(S)$. Evidently $A = \bar{A}$ if and only if A is a *tolerance* (symmetric diagonal) on S. It follows from (5) that the set $\mathrm{Tol}(S)$ of all tolerances on S is a sublattice of $\mathrm{Ref}(S)$.

A transitive diagonal on S is said to be a *quasiorder* on S. The set $\mathrm{Qua}(S)$ of all quasiorders on S forms a lattice with respect to set inclusion but $\mathrm{Qua}(S)$ need not be a sublattice of $\mathrm{Ref}(S)$. It is clear that $\mathrm{Con}(S) = \mathrm{Tol}(S) \cap \mathrm{Qua}(S)$ is the lattice of all *congruences* on S. We have the following diagram:

$$\begin{tikzcd} & \mathrm{Ref}(S) & \\ \mathrm{Tol}(S) \arrow[ur, "\text{sublattice}"] & & \mathrm{Qua}(S) \arrow[ul, "\text{subset}"'] \\ & \mathrm{Con}(S) \arrow[ul, "\text{subset}"] \arrow[ur, "\text{sublattice}"'] & \end{tikzcd} \tag{6}$$

Let us note that in [6] there are described all varieties of regular $*$-semigroup whose tolerance (congruence) lattices are modular, distributive and boolean. In fact this paper is a continuation of [6].

Theorem 1. *The following conditions for a variety $\mathcal{V}$ of regular $*$-semigroups are equivalent*:

1. $\mathcal{V} \subseteq \mathcal{W}(xx^* = yy^*)$.
2. $\mathrm{Con}(S) = \mathrm{Ref}(S)$ *for all* $S \in \mathcal{V}$.
3. $\mathrm{Con}(S)$ *is a sublattice of* Ref. (S) *for all* $S \in \mathcal{V}$.
4. $\mathrm{Con}(S)$ *is a sublattice of* $\mathrm{Tol}(S)$ *for all* $S \in \mathcal{V}$.
5. $\mathrm{Qua}(S)$ *is a sublattice of* $\mathrm{Ref}(S)$ *for all* $S \in \mathcal{V}$.
6. $\mathrm{Qua}(S)$ *is a sublattice of* $\mathrm{Tol}(S)$ *for all* $S \in \mathcal{V}$.
7. *The lattice* $\mathrm{Qua}(S)$ *is modular for all* $S \in \mathcal{V}$.
8. *The lattice* $\mathrm{Con}(S)$ *is modular for all* $S \in \mathcal{V}$.

Proof. $1 \Rightarrow 2$. Suppose that $S \in \mathcal{W}(xx^* = yy^*)$. We shall show that $\mathrm{Ref}(S) \subseteq \mathrm{Tol}(S)$. Let $A \in \mathrm{Ref}(S)$. If $(x, y) \in A$, then by (1) and (2) we have $(y, x) = (yy^*y, xx^*x) = (xx^*y, xy^*y) \in A$. It follows from Theorem 1 of [6] that $\mathrm{Tol}(S) = \mathrm{Con}(S)$ and so $\mathrm{Ref}(S) \subseteq \mathrm{Con}(S)$. According to (6) we have $\mathrm{Con}(S) = \mathrm{Ref}(S)$.

$2 \Rightarrow 3$, 4, 5 and 6. It follows from (6).

4 or 5 or $6 \Rightarrow 3$. It is clear.

$3 \Rightarrow 1$. According to (6), we obtain $3 \Rightarrow 4$ and by Theorem 1 of [6] we have $4 \Rightarrow 1$.

$1 \Rightarrow 7$. Suppose that $S \in \mathscr{W}(xx^* = yy^*)$. It follows from $1 \Rightarrow 2$ and (6) that $\operatorname{Qua}(S) = \operatorname{Con}(S)$ and so by Theorem 5 of [6] the lattice $\operatorname{Qua}(S)$ is modular.

$7 \Rightarrow 8$. See (6).

$8 \Rightarrow 1$. Apply Theorem 5 of [6].

Theorem 2. *The following conditions for a variety $\mathscr{V}$ of regular $*$-semigroups are equivalent:*

1. $\mathscr{V} \subseteq \mathscr{W}(xyy^*x^* = xx^*)$.
2. *The lattice* $\operatorname{Ref}(S)$ *is modular for all* $S \in \mathscr{V}$.
3. *The lattice* $\operatorname{Tol}(S)$ *is modular for all* $S \in \mathscr{V}$.

Proof. $1 \Rightarrow 2$. Suppose that $S \in \mathscr{W}(xyy^*x^* = xx^*)$. It is easy to show that

$$(7) \qquad x\,e\,x^* = xx^*$$

for every $x \in S$ and every projection e of S (i.e. $e = e^2 = e^*$).

Let $A, B \in \operatorname{Ref}(S)$. We shall prove that for every projection e of S we have

$$(8) \qquad AB = A(e, e)\,B\,,$$

$$(9) \qquad (e, e)\,A(e, e) = (e, e)\,\bar{A}(e, e)\,,$$

$$(10) \qquad (e, e)\,AB(e, e) = (e, e)\,BA(e, e)\,,$$

$$(11) \qquad ABAB \subseteq AB\,.$$

Identity (8). Suppose that $(a, c) \in A$ and $(b, d) \in B$. There by (1), (2) and (7) we have $(a, c)(b, d) = (a, c)(bb^*c^*c, bb^*c^*c)(e, e)(c^*c, c^*c)(b, d) \in A(e, e)\,B$. Consequently $AB \subseteq A(e, e)\,B \subseteq AB$.

Identity (9). Assume that $(a, c) \in A$. According to (1), (2) and (7), we obtain $(e, e)(a, c)(e, e) = (e, e)(ce, ce)(c^*, a^*)(ea, ea)(e, e) \in (e, e)\,\bar{A}(e, e)$. Thus we have $(e, e)\,A(e, e) \subseteq (e, e)\,\bar{A}(e, e)$. Analogously we can show that $(e, e)\,\bar{A}(e, e) \subseteq$ $\subseteq (e, e)\,A(e, e)$.

Identity (10). Let $(a, c) \in A$ and $(b, d) \in B$. By (1), (2), (8), (7) and (9) we have $(e, e)(a, c)(b, d)(e, e) = (e, e)(cde, cde)(d^*, b^*)(c^*, a^*)(eab, eab)(e, e) \in (e, e)\,.$ $.\,\bar{B}\,\bar{A}(e, e) = (e, e)\,\bar{B}(e, e)\,\bar{A}(e, e) = (e, e)\,B(e, e)\,A(e, e) = (e, e)\,BA(e, e)$. Thus we obtain $(e, e)\,AB(e, e) \subseteq (e, e)\,BA(e, e)$ and analogously we can get $(e, e)\,BA(e, e) \subseteq$ $\subseteq (e, e)\,AB(e, e)$.

Inclusion (11). It follows from (8), (10) and (2) that $ABAB = A(e, e)\,BA(e, e)\,B \subseteq$ $\subseteq A(e, e)\,AB(e, e)\,B \subseteq AB$.

Suppose that $A, B, C \in \operatorname{Ref}(S)$ and $A \subseteq C$. First we shall show that

$$(12) \qquad AB \cap C \subseteq A(B \cap C)\,,$$

$$(13) \qquad BA \cap C \subseteq (B \cap C)\,A\,,$$

$$(14) \qquad ABA \cap C \subseteq A(B \cap C)\,A\,,$$

$$(15) \qquad BAB \cap C \subseteq (B \cap C)\,A(B \cap C)\,.$$

Inclusion (12). Suppose that $(x, y) \in AB \cap C$. Then by (8) and (2) we have $(x, y) = (a, c)(eb, ed)$, where $(a, c) \in A$, $(eb, ed) \in B$ and e is a projection of S. It follows from (7) and (2) that $(eb, ed) = (ea^*e, ec^*e)(x, y) \in AC \subseteqq C$.

Inclusion (13). This is dual to (12).

Inclusion (14). Assume that $(x, y) \in ABA \cap C$. Then by (8) we have $(x, y) = (ue, ve)(a, c)$, where $(ue, ve) \in AB$, $(a, c) \in A$ and e is a projection of S. According to (7) and (2) we obtain $(ue, ve) = (x, y)(ea^*e, ec^*e) \in CA \subseteqq C$. It follows from (12) that $(ue, ve) \in A(B \cap C)$ and so $(x, y) \in A(B \cap C) A$.

Inclusion (15). Suppose that $(x, y) \in BAB \cap C$. Then we have $(x, y) \in (b, d) AB$ where $(b, d) \in B$. Using (1), (2) and (7) we get $(xx^*, yy^*) = (bb^*, dd^*) \in B \cap C$. It follows from (1), (2), (7) and (11) that $(x, y) = (xx^*e, yy^*e)(ex, ey) \in (xx^*e, yy^*e) \,.\, ABAB \subseteqq (xx^*e, yy^*e) AB$, where e is a projection of S. Consequently by (8) we obtain $(x, y) = (xx^*, yy^*)(eu, ev)$, where $(eu, ev) \in AB$. According to (2), we have $(eu, ev) = (ex, ey) \in C$ and so, by (12), we get $(eu, ev) \in A(B \cap C)$. Therefore $(x, y) = (xx^*, yy^*)(eu, ev) \in (B \cap C) A(B \cap C)$.

Finally, it follows from (11), (12), (13), (14), (3) and (4) that $(A \vee B) \wedge C = (A \cup B \cup AB \cup BA \cup ABA \cup BAB) \cap C \subseteqq A \cup (B \cap C) \cup A(B \cap C) \cup (B \cap C) \,.\, A \cup A(B \cap C) A \cup (B \cap C) A(B \cap C) = A \vee (B \wedge C) \subseteqq (A \vee B) \vee C$.

Therefore the lattice $\mathrm{Ref}(S)$ is modular.

$2 \Rightarrow 3$. This follows from (6).

$3 \Rightarrow 1$. See Theorem 4 of [6].

Theorem 3. *The following conditions for a variety $\mathscr{V}$ of regular $*$-semigroups are equivalent*:

1. $\mathscr{V} \subseteqq \mathscr{W}(xyx^* = xx^*)$.
2. *The lattice* $\mathrm{Ref}(S)$ *is boolean for all* $S \in \mathscr{V}$.
3. *The lattice* $\mathrm{Tol}(S)$ *is boolean for all* $S \in \mathscr{V}$.
4. *The lattice* $\mathrm{Ref}(S)$ *is distributive for all* $S \in \mathscr{V}$.
5. *The lattice* $\mathrm{Tol}(S)$ *is distributive for all* $S \in \mathscr{V}$.

Proof. $1 \Rightarrow 2$. Suppose that $S \in \mathscr{W}(xyx^* = xx^*) \subseteqq \mathscr{W}(xyy^*x^* = xx^*)$. We have

$$(16) \qquad exe = e\,, \quad x = xex \quad \text{and} \quad xyz = xez$$

for any $x, y, z \in S$ and each projection e of S. Indeed, it follows from (7) and (1) that $xex = xex^*ex = xx^*x = x$ and $xyz = xexyzez = xez$.

Let $A, B, C \in \mathrm{Ref}(S)$. We shall show that

$$(17) \qquad ABC = AC\,,$$

$$(18) \qquad AB \cap C = (A \cap C)(B \cap C)\,.$$

Identity (17). According to (8), (16) we have $ABC = A(e, e) B(e, e) C = A(e, e) \,.\, C = AC$.

Identity (18). Suppose that $(u, v) \in AB \cap C$. Then by (8) we obtain $(u, v) = (a, c)(e, e)(b, d)$ where $(a, c) \in A$ and $(b, d) \in B$. It follows from (16) and (2) that $(ae, ce) = (aebe, cede) = (ue, ve) \in A \cap C$ and analogously $(eb, ed) = (eu, ev) \in$ $\in B \cap C$. Hence we have $(u, v) = (ae, ce)(eb, ed) \in (A \cap C)(B \cap C)$. Therefore $AB \cap C \subseteqq (A \cap C)(B \cap C) \subseteqq AB \cap C$.

According to (3), (4), (17) and (18), we have $(A \vee B) \wedge C = (A \cup B \cup AB \cup BA) \cap$ $\cap C = (A \cap C) \cup (B \cap C) \cup (A \cap C)(B \cap C) \cup (B \cap C)(A \cap C) = (A \wedge C) \vee$ $\vee (B \wedge C)$.

Therefore the lattice $\mathrm{Ref}(S)$ is distributive.

Now we shall prove that the lattice $\mathrm{Ref}(S)$ is boolean. Let $A \in \mathrm{Ref}(S)$. Choose a projection e of S and put $B = \mathrm{R}((Se \times Se) \setminus A)$.

Let $u, v \in S$. According to (1) and (16), we have $(u, v) = (ue, ve)(u^*e, v^*e)^*$. It is easy to show that $(ue, ve), (u^*e, v^*e) \in A \cup B$. By (3) and (4) we have $(u, v) \in A \vee B$. Therefore $A \vee B = S \times S$.

Assume that $A \wedge B \neq \mathrm{id}_S$. Then there exist $u, v \in S$ such that $(u, v) \in A \cap B$ and $u \neq v$. According to (3) and (16), we have $(u, v) = (a, c)(e, e)(b, d)$, where (a, c), $(b, d) \in \mathrm{id}_S \cup ((Se \times Se) \setminus A) \cup ((Se \times Se) \setminus A)^*$. If $(a, c) \in (Se \times Se) \setminus A$, then by our assumption we obtain $(a, c) = (ae, ce) = (aeb, ced)(e, e) = (u, v)(e, e) \in A$, which is a contradiction. Thus we have $(a, c) \notin (Se \times Se) \setminus A$. If $(b, d) \in ((Se \times$ $\times Se) \setminus A)^*$, then $(b^*, d^*) \in (Se \times Se) \setminus A$ and so by our assumption we have $(b^*, d^*) = (b^*e, d^*e) = (b^*ea^*, d^*ec^*)(e, e) = (u, v)^*(e, e) \in A$, a contradiction. Therefore $(b, d) \notin ((Se \times Se) \setminus A)^*$.

Consequently we have the following possibilities:

Case 1. $a = c$. Then $b \neq d$ and so $(b, d) \in (Se \times Se) \setminus A$. Hence by our assumption we have $(u, v) = (aebe, aede) = (ae, ae)$, a contradiction.

Case 2. $b = d$. Then $a \neq c$ and so $(a, c) \in ((Se \times Se) \setminus A)^* \subseteqq eS \times eS$. Therefore $(u, v) = (eaeb, eceb) = (eb, eb)$, a contradiction.

Case 3. $a \neq c$ and $b \neq d$. Then $(a, c) \in eS \times eS$ and $(b, d) \in Se \times Se$. Thus we have $u = aeb = e = ced = v$, a contradiction.

Therefore $A \wedge B = \mathrm{id}_S$. Consequently the lattice $\mathrm{Ref}(S)$ is boolean.

$2 \Rightarrow 4$. It is clear.

$4 \Rightarrow 5$. This follows from (6).

$5 \Rightarrow 1$. See Theorem 6 of [6].

$2 \Rightarrow 3$. Suppose that the lattice $\mathrm{Ref}(S)$ is boolean.
According to (6), $\mathrm{Tol}(S)$ is a sublattice of $\mathrm{Ref}(S)$ and so the lattice $\mathrm{Tol}(S)$ is distributive. Let $A \in \mathrm{Tol}(S)$. Then there exists $B \in \mathrm{Ref}(S)$ such that $A \wedge B = \mathrm{id}_s$ and $A \vee B = S \times S$. We have $A = \bar{A}$ and so by (5) we obtain $A \wedge \bar{B} = \mathrm{id}_s$ and $A \vee$ $\vee \bar{B} = S \times S$. Therefore $B = \bar{B} \in \mathrm{Tol}(S)$. Consequently the lattice $\mathrm{Tol}(S)$ is boolean.

$3 \Rightarrow 5$. This follows from (6).

Note. Let be a variety of regular $*$-semigroups. If the lattice $\mathrm{Qua}(S)$ is distributive for all $S \in \mathscr{V}$, then $\mathscr{V}$ is trivial.

This follows from (6) and from Theorem 7 of [6].

References

[1] *Chajda, I.:* Varieties having distributive lattices of quasiorders. Czech. Math. J. *41 (116)* 1991, 85—89.

[2] *Chajda, I.:* Lattices of compatible relations. Arch. Math. (Brno), *13* (1977), 89—96.

[3] *Nordahl, T. E.; Scheiblich, H. E.:* Regular ∗-semigroups. Semigroup Forum *16* (1978), 369—377.

[4] *Clifford, A. H.; Preston, G. B.:* The algebraic theory of semigroups. Vol. I. Am. Math. Soc., 1961.

[5] *Petrich, M.:* Introduction to semigroups. Merill Publishing Company, 1973.

[6] *Pondělíček, B.:* On varieties of regular ∗-semigroups. Czech. Mat. J. *41 (116)* 1991, 110—119.

Author's address: 166 27 Praha 6, Technická 2, Czechoslovakia (FEL ČVUT).