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DUAL FINITE ELEMENT ANALYSIS FOR AN INEQUALITY OF THE 2nd ORDER

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In practice we often meet problems when the cogradient of the unknown solution is more important than the solution itself. Using the so called dual variational formulation, one can approximate directly the components of the cogradient. The dual finite element analysis for the case of elliptic equations is given in [3]. Involving Lagrange multipliers in the dual formulation, we obtain the so called dual hybrid formulation, which is studied in [11].

In the present paper, the dual finite element analysis for an elliptic inequality of the 2nd order with an interior obstacle is given. Using piecewise linear equilibrium elements, the rate of convergence of Ritz approximations is established, provided the exact solution is smooth enough. The primal analysis of this problem is given in [2]. The dual analysis for unilateral boundary value problems is given in [12], [13].

1. SETTING OF THE PROBLEM

Let $Q \subset R_n$ be a bounded domain with a Lipschitz boundary Γ. By $H^k(Q)$ ($k \geqq 0$ integer) we denote the classical Sobolev spaces with the following notation:

$$(1.1) \qquad \|v\|_{k,Q} = \left(\int_Q \sum_{|\alpha| \leqq k} |D^\alpha v|^2 \,\mathrm{d}x\right)^{1/2},$$

$$(1.2) \qquad |v|_{m,Q} = \left(\int_Q \sum_{|\alpha| = m} |D^\alpha v|^2 \,\mathrm{d}x\right)^{1/2}.$$

In the case $k = 0$ we set $H^0(Q) = L^2(Q)$ and we write simply $\|v\|_{0,Q} = \|v\|_Q$. By $H_0^k(Q)$ we denote the completion of $\mathscr{D}(Q)$ under the norm (1.1). $\boldsymbol{H}^k(Q)$ denotes the Cartesian product of $H^k(Q)$ with the usual norm $\|\boldsymbol{v}\|_{k,Q}$ and seminorms $|\boldsymbol{v}|_{m,Q}$.

We shall consider the following obstacle problem:

$$(\mathscr{P}) \qquad \textit{find } u \in \mathscr{U} \textit{ such that}$$

$$\mathscr{J}(u) = \min_{v \in \mathscr{U}} \mathscr{J}(v)$$

with

$$\mathscr{U} = \{v \in H_0^1(Q) : v \geqq \varphi \text{ a. e. in } Q\}\,,$$

$$\mathscr{J}(v) = \int_Q |\text{grad}\, v|^2 \,\mathrm{d}x - 2 \int_Q fv \,\mathrm{d}x\,,$$

where $f \in L^2(Q)$ and $\varphi \in H_0^1(Q)$ are given functions. $\mathscr{U}$ is a closed convex subset of $H_0^1(Q)$. Let us recall the following existence and uniqueness result (cf. [8], [10]).

Theorem 1.1. *There is a unique solution u of $(\mathscr{P})$ and this solution is characterized by the relation*

$$a(u, v - u) \geqq \int_Q f(v - u)\,\mathrm{d}x \quad \forall v \in \mathscr{U}\,, \tag{1.3}$$

where

$$a(u, v) = \int_Q \text{grad}\, u \,.\, \text{grad}\, v \,\mathrm{d}x\,.$$

If u is smooth enough, then using Green's formula we deduce from (1.3):

$$-\Delta u = f \quad \text{in} \quad Q_0 \subset Q\,,$$
$$-\Delta u \geqq f \quad \text{in} \quad Q_+ \subset Q\,,$$

where

$$Q_0 = \{x \in Q : u(x) > \varphi(x)\}\,,$$
$$Q_+ = \{x \in Q : u(x) = \varphi(x)\}\,.$$

As $\varphi \in H_0^1(Q)$, we can write $\mathscr{U} = \varphi + \mathscr{U}_0$, where

$$\mathscr{U}_0 = \{w \in H_0^1(Q) : w \geqq 0 \text{ a. e. in } Q\}\,.$$

Let

$$u = \varphi + w^*\,, \quad w^* \in \mathscr{U}_0\,. \tag{1.4}$$

Then we have

Lemma 1.1. *It holds*

$$\langle -\Delta u - f, w^* \rangle = 0\,, \tag{1.5}$$

where $\langle , \rangle$ denotes the duality pairing between $H^{-1}(Q) = (H_0^1(Q))'$ and $H_0^1(Q)$.

Proof. Inserting $v = \varphi$ and $v = \varphi + 2w^*$ into (1.3) and using (1.4) we obtain (1.5).

Next we derive a dual variational formulation to $(\mathscr{P})$. To this end we introduce the following Lagrangian $\mathscr{L}$:

$$\mathscr{L}(\mathscr{N}, v, \lambda) = \int_Q \mathscr{N}_i \mathscr{N}_i \, \mathrm{d}x - 2 \int_Q f v \, \mathrm{d}x + 2 \int_Q \lambda_i \left(\frac{\partial v}{\partial x_i} - \mathscr{N}_i \right) \mathrm{d}x \text{ }^{1)}$$

where $(\mathscr{N}, v, \lambda) \in W = L^2(Q) \times \mathscr{U} \times L^2(Q)$, $\mathscr{N} = (\mathscr{N}_1, \ldots, \mathscr{N}_n)$, $\lambda = (\lambda_1, \ldots, \lambda_n)$. It is easy to verify that

$$\mathscr{J}(u) = \inf_{v \in \mathscr{U}} \mathscr{J}(v) = \inf_{(\mathscr{N}, v)} \sup_{\lambda} \mathscr{L}(\mathscr{N}, v, \lambda) , \tag{1.6}$$

where $(\mathscr{N}, v) \in L^2(Q) \times \mathscr{U}$, $\lambda \in L^2(Q)$.

Theorem 1.2. *There is a unique saddle-point* $(\mathscr{N}^*, v^*, \lambda^*)$ *of* $\mathscr{L}$ *on* W *and*

$$(\mathscr{N}^*, v^*, \lambda^*) = (\operatorname{grad} u, u, \operatorname{grad} u) , \tag{1.7}$$

where u *is the solution of* $(\mathscr{P})$.

Proof. Let $(\mathscr{N}^*, v^*, \lambda^*) \in W$ be a saddle-point of $\mathscr{L}$ on W. Then

$$\delta_{\mathscr{N}} \mathscr{L}(\mathscr{N}^*, v^*, \lambda^*) = 0 \Leftrightarrow \mathscr{N}^* = \lambda^* , \tag{1.8}$$

$$\delta_{\lambda} \mathscr{L}(\mathscr{N}^*, v^*, \lambda^*) = 0 \Leftrightarrow \mathscr{N}^* = \operatorname{grad} v^* , \tag{1.9}$$

$$\delta_{v} \mathscr{L}(\mathscr{N}^*, v^*, \lambda^*)(v - v^*) \geqq 0 \quad \forall v \in \mathscr{U} \Leftrightarrow \tag{1.10}$$

$$\Leftrightarrow \int_Q \lambda_i^* \left(\frac{\partial v}{\partial x_i} - \frac{\partial v^*}{\partial x_i} \right) \mathrm{d}x \geqq \int_Q f (v - v^*) \, \mathrm{d}x \quad \forall v \in \mathscr{U} ,$$

where $\delta_{\mathscr{N}} \mathscr{L}$ denotes the partial differentiation of $\mathscr{L}$ with respect to $\mathscr{N}$ (and analogously $\delta_{\lambda} \mathscr{L}$, $\delta_{v} \mathscr{L}$). Taking into account (1.8), (1.9) we deduce from (1.10) that $v^* = u$ is a solution of $(\mathscr{P})$. Hence we conclude that there is at most one saddle-point of $\mathscr{L}$ on W. Conversely, to prove that (1.7) is a saddle-point of $\mathscr{L}$ on W, we must verify:

$$\mathscr{L}(\mathscr{N}^*, v^*, \lambda) \leqq \mathscr{L}(\mathscr{N}^*, v^*, \lambda^*) \quad \forall \lambda \in L^2(Q) , \tag{1.11}$$

$$\mathscr{L}(\mathscr{N}^*, v^*, \lambda^*) \leqq \mathscr{L}(\mathscr{N}, v, \lambda^*) , \quad (\mathscr{N}, v) \in L^2(Q) \times \mathscr{U} . \tag{1.12}$$

It is easy to see that (1.11) is satisfied even with the sign of equality. Let us prove (1.12). We have

$$\int_Q \left(\mathscr{N}_i - \frac{\partial u}{\partial x_i} \right) \left(\mathscr{N}_i - \frac{\partial u}{\partial x_i} \right) \mathrm{d}x \geqq 2 \int_Q \frac{\partial u}{\partial x_i} \left(\frac{\partial u}{\partial x_i} - \frac{\partial v}{\partial x_i} \right) \mathrm{d}x + \tag{1.13}$$

$$+ 2 \int_Q f (v - u) \, \mathrm{d}x$$

for $\forall \mathscr{N} \in L^2(Q)$, $\forall v \in \mathscr{U}$ by virtue of (1.3). A direct calculation shows that (1.12) and (1.13) are equivalent. Theorem is proved.

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1) In what follows a repeated index implies always summation over the range $1, 2, \ldots, n$.

Using the well-known properties of saddle-points, we can write in (1.6):

$$\mathscr{J}(u) = \inf_{(\mathscr{N}, v)} \sup_{\lambda} \mathscr{L}(\mathscr{N}, v, \lambda) = \sup_{\lambda} \inf_{(\mathscr{N}, v)} \mathscr{L}(\mathscr{N}, v, \lambda), \tag{1.14}$$

$$(\mathscr{N}, v) \in \boldsymbol{L}^2(Q) \times \mathscr{U}, \quad \lambda \in \boldsymbol{L}^2(Q).$$

Let $\bar{\lambda} = (\bar{\lambda}_1, \ldots, \bar{\lambda}_n) \in \boldsymbol{L}^2(Q)$ be fixed and let us set

$$\bar{\mathscr{L}}(\mathscr{N}, v) = \mathscr{L}(\mathscr{N}, v, \bar{\lambda}) = \mathscr{L}_1(\mathscr{N}) + \mathscr{L}_2(v),$$

where

$$\mathscr{L}_1(\mathscr{N}) = \int_Q \mathscr{N}_i \mathscr{N}_i \, \mathrm{d}x - 2 \int_Q \bar{\lambda}_i \mathscr{N}_i \, \mathrm{d}x,$$

$$\mathscr{L}_2(v) = 2 \int_Q \bar{\lambda}_i \frac{\partial v}{\partial x_i} \, \mathrm{d}x - 2 \int_Q f v \, \mathrm{d}x.$$

Let $(\bar{\mathscr{N}}, \bar{v}) \in \boldsymbol{L}^2(Q) \times \mathscr{U}$ be such that

$$\bar{\mathscr{L}}(\bar{\mathscr{N}}, \bar{v}) = \inf_{(\mathscr{N}, v)} \mathscr{L}(\mathscr{N}, v, \bar{\lambda}) = \inf_{\mathscr{N}} \mathscr{L}_1(\mathscr{N}) + \inf_{v} \mathscr{L}_2(v),$$

$\mathscr{N} \in \boldsymbol{L}^2(Q)$, $v \in \mathscr{U}$. Then

$$\delta_{\mathscr{N}} \mathscr{L}_1(\bar{\mathscr{N}}) = 0 \Leftrightarrow \bar{\mathscr{N}} = \bar{\lambda}$$

and

$$\mathscr{L}_1(\bar{\mathscr{N}}) = -\int_Q \bar{\lambda}_i \bar{\lambda}_i \, \mathrm{d}x.$$

On the other hand,

$$\inf_{v \in \mathscr{U}} \mathscr{L}_2(v) = \inf_{w \in \mathscr{U}_0} \mathscr{L}_2(\varphi + w) =$$

$$= \inf_{w \in \mathscr{U}_0} \left\{ 2 \int_Q \bar{\lambda}_i \left(\frac{\partial \varphi}{\partial x_i} + \frac{\partial w}{\partial x_i} \right) \mathrm{d}x - 2 \int_Q f(\varphi + w) \, \mathrm{d}x \right\} =$$

$$= \begin{cases} 2 \displaystyle\int_Q \bar{\lambda}_i \frac{\partial \varphi}{\partial x_i} \, \mathrm{d}x - 2 \int_Q f \varphi \, \mathrm{d}x & \text{if } \bar{\lambda} \in \mathscr{N}_f^-(Q) \\ -\infty & \text{if } \lambda \notin \mathscr{N}_f^-(Q), \end{cases}$$

where

$$\mathscr{N}_f^-(Q) = \left\{ \lambda \in \boldsymbol{L}^2(Q) : \int_Q \lambda_i \frac{\partial w}{\partial x_i} \, \mathrm{d}x \geqq \int_Q f w \, \mathrm{d}x \quad \forall w \in \mathscr{U}_0 \right\}.$$

Hence

$$\sup_{\lambda} \inf_{(\mathscr{N}, v)} \mathscr{L}(\mathscr{N}, v, \lambda) = \sup_{\lambda \in \mathscr{N}^-{}_f(Q)} \left\{ -\int_Q \lambda_i \lambda_i \, \mathrm{d}x + 2 \int_Q \lambda_i \frac{\partial \varphi}{\partial x_i} \, \mathrm{d}x \right\} -$$

$$- 2 \int_Q f \varphi \, \mathrm{d}x = - \inf_{\lambda \in \mathscr{N}_f{}^-(Q)} \mathscr{S}(\lambda) - 2 \int_Q f \varphi \, \mathrm{d}x,$$

where

$$\mathcal{S}(\boldsymbol{\lambda}) = \int_Q \lambda_i \lambda_i \, \mathrm{d}x - 2 \int_Q \lambda_i \frac{\partial \varphi}{\partial x_i} \, \mathrm{d}x \, .$$

Let us define the following variational problem:

$(\mathcal{P}^*)$ *find* $\boldsymbol{\lambda}^* \in \mathcal{N}_f^-(Q)$ *such that*

$$\mathcal{S}(\boldsymbol{\lambda}^*) = \inf_{\boldsymbol{\lambda} \in \mathcal{N}_f^-(Q)} \mathcal{S}(\boldsymbol{\lambda})$$

$(\mathcal{P}^*)$ will be called *the dual problem to* $(\mathcal{P})$.

Theorem 1.3. *There is a unique solution* $\boldsymbol{\lambda}^*$ *of* $(\mathcal{P}^*)$ *and*

$$\boldsymbol{\lambda}^* = \operatorname{grad} u \, , \tag{1.15}$$

where u *is the solution of* $(\mathcal{P})$.

Proof of the existence and uniqueness is standard. The derivation of $(\mathcal{P}^*)$ justifies (1.15).

Remark. It is easy to see that

$$\boldsymbol{\lambda} \in \mathcal{N}_f^-(Q) \Leftrightarrow \operatorname{div} \boldsymbol{\lambda} + f \leqq 0 \quad \text{in} \quad Q$$

in the sense of distributions. Let $\boldsymbol{\lambda}^0 \in \boldsymbol{L}^2(Q)$ be such that

$$\operatorname{div} \boldsymbol{\lambda}^0 = -f \quad \text{in} \quad Q \, .$$

Then $\mathcal{N}_f^-(Q) = \boldsymbol{\lambda}^0 + \mathcal{N}_0^-(Q)$, where

$$\mathcal{N}_0^-(Q) = \left\{ \boldsymbol{\chi} \in \boldsymbol{L}^2(Q) : \int_Q \chi_i \frac{\partial w}{\partial x_i} \, \mathrm{d}x \geqq 0 \quad \forall w \in \mathcal{U}_0 \right\},$$

or equivalently

$$\boldsymbol{\chi} \in \mathcal{N}_0^-(Q) \Leftrightarrow \operatorname{div} \boldsymbol{\chi} \leqq 0 \quad \text{in} \quad Q$$

in the sense of distributions.

Problem $(\mathcal{P}^*)$ can be formulated equivalently as follows:

$$\begin{cases} \textit{find } \boldsymbol{\lambda}^* \in \mathcal{N}_f^-(Q) \textit{ such that} \\ b(\boldsymbol{\lambda}^*, \boldsymbol{\lambda} - \boldsymbol{\lambda}^*) \geqq \mathcal{F}(\boldsymbol{\lambda} - \boldsymbol{\lambda}^*) \quad \forall \boldsymbol{\lambda} \in \mathcal{N}_f^-(Q) , \end{cases}$$

where

$$b(\boldsymbol{\lambda}, \boldsymbol{\mu}) = \int_Q \lambda_i \mu_i \, \mathrm{d}x \, , \quad \mathcal{F}(\boldsymbol{\lambda}) = \int_Q \lambda_i \frac{\partial \varphi}{\partial x_i} \, \mathrm{d}x \, .$$

Approximation of $(\mathcal{P}^*)$

In order to define the Ritz-Galerkin approximations, we introduce a system $\{\mathcal{N}_{0h}^-(Q)\}$, $h \in (0, 1)$ of "finite dimensional approximations" of $\mathcal{N}_0^-(Q)$. Let us suppose that $\mathscr{N}^-_{0h}(Q) \subset \mathscr{N}^-_0(Q)$ for all $h \in (0, 1)$ and let us set

$$\mathscr{N}^-_{fh}(Q) = \boldsymbol{\lambda}^0 + \mathscr{N}^-_{0h}(Q) \subset \mathscr{N}^-_f(Q) \quad \forall h \in (0, 1)\,.$$

We define the following procedure:

$(\mathscr{P}^*_h)$ *find* $\boldsymbol{\lambda}^*_h \in \mathscr{N}^-_{fh}(Q)$ *such that*

$$\mathscr{S}(\boldsymbol{\lambda}^*_h) = \min_{\boldsymbol{\lambda}_h \in \mathscr{N}_{fh}^-(Q)} \mathscr{S}(\boldsymbol{\lambda}_h)$$

or equivalently

$$\begin{cases} \text{find } \boldsymbol{\lambda}^*_h \in \mathscr{N}^-_{fh}(Q) \text{ such that} \\ b(\boldsymbol{\lambda}^*_h, \boldsymbol{\lambda}_h - \boldsymbol{\lambda}^*_h) \geqq \mathscr{F}(\boldsymbol{\lambda}_h - \boldsymbol{\lambda}^*_h) \quad \forall \boldsymbol{\lambda}_h \in \mathscr{N}^-_{fh}(Q)\,. \end{cases}$$

Lemma 1.2. *To every* $h \in (0, 1)$ *there exists precisely one solution* $\boldsymbol{\lambda}^*_h$ of $(\mathscr{P}^*_h)$. *Moreover, it holds*:

$$\|\boldsymbol{\lambda}^* - \boldsymbol{\lambda}^*_h\|^2_{0,Q} \leqq \{\mathscr{F}(\boldsymbol{\lambda}^* - \boldsymbol{\lambda}_h) + b(\boldsymbol{\lambda}^*_h - \boldsymbol{\lambda}^*, \boldsymbol{\lambda}_h - \boldsymbol{\lambda}^*) + b(\boldsymbol{\lambda}^*, \boldsymbol{\lambda}_h - \boldsymbol{\lambda}^*)\} \quad \forall \boldsymbol{\lambda}_h \in \mathscr{N}^-_{fh}(Q)\,. \tag{1.16}$$

Proof. $\mathscr{S}(\boldsymbol{\lambda}_h)$ $(\boldsymbol{\lambda}_h \in \mathscr{N}^-_{fh}(Q))$ is a quadratic function generated by a symmetric, positive-definite matrix, $\mathscr{N}^-_{fh}(Q)$ is a closed convex subset of $\mathscr{N}^-_f(Q)$. Hence the existence and the uniqueness of $\boldsymbol{\lambda}^*_h$ follows. For the proof of (1.16) see e.g. [2], [5].

2. CONSTRUCTION OF $\mathscr{N}^-_{fh}(Q)$

Let us suppose that Q is a bounded polygonal domain. For the sake of simplicity we restrict ourselves to the plane case only. We introduce the following notation:

$$\boldsymbol{C}(\bar{Q}) = [C(\bar{Q})]^2$$

with the norm $\|\boldsymbol{v}\|_{\boldsymbol{C}(\bar{Q})} = \max\limits_{\substack{i=1,2\\ x \in \bar{Q}}} |v_i(x)|$, $\boldsymbol{v} = (v_1, v_2)$ and $\boldsymbol{C}^2(\bar{Q}) = [C^2(\bar{Q})]^2$ with the seminorm

$$|\boldsymbol{v}|_{\boldsymbol{C}^2(\bar{Q})} = \max_{\substack{i,j,k=1,2\\ x \in \bar{Q}}} \left|\frac{\partial^2 v_i}{\partial x_j\, \partial x_k}(x)\right|\,.\ ^{1})$$

Let K be a non-degenerate triangle with vertices a_1, a_2, a_3 and let us set $a_4 = a_1$. Let $P_1(K)$ denote the set of linear polynomials on K and $\boldsymbol{P}_1(K) = [P_1(K)]^2$. We say that $\lambda^{(i)}_1, \lambda^{(i)}_2$ are basic linear functions of the side $a_i a_{i+1}$ if

$$\lambda^{(i)}_j \text{ are linear on } a_i a_{i+1}\,,$$
$$\lambda^{(i)}_1(a_i) = 1\,, \quad \lambda^{(i)}_1(a_{i+1}) = 0\,,$$
$$\lambda^{(i)}_2(a_i) = 0\,, \quad \lambda^{(i)}_2(a_{i+1}) = 1\,.$$

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^{1}) $C^k(\bar{Q})$ ($k \geqq 0$ integer) denotes the usual Banach space of continuous functions on Q, derivatives of which up to the order k are continuous on Q and continuously extensible on $\bar{Q}$.

For $\mathbf{v} \in \mathbf{H}^1(K)$ we define the outward flux by the relation

$$(2.1) \qquad T_i\mathbf{v} = \mathbf{v}|_{a_ia_{i+1}} \, . \, \mathbf{n}^{(i)} \, ,$$

where $\mathbf{v}|_{a_ia_{i+1}}$ are traces of $\mathbf{v}$ on a_ia_{i+1} and $\mathbf{n}^{(i)}$ is the outward unit normal to ∂K. From the trace theorem (see [4]) it follows that $T_i \in L(\mathbf{H}^1(K), L^2(a_ia_{i+1}))$ [2].

For the scalar product in $L^2(a_ia_{i+1})$ we use the notation

$$[u, v]_i = \int_{a_ia_{i+1}} uv \, \mathrm{d}s \, .$$

First we make some observations, the proofs of which are given in [3].

Lemma 2.1. *Let* $\gamma_i, \delta_i \in R_1$ $(i = 1, 2, 3)$ *be given. Then there exists a unique* $\mathbf{v} \in \mathbf{P}_1(K)$ *such that*

$$T_i\mathbf{v}(a_i) = \gamma_i \, , \quad T_i\mathbf{v}(a_{i+1}) = \delta_i \, .$$

Theorem 2.1. *Let* $\mathbf{v} \in \mathbf{H}^1(K)$. *Then the equations*

$$(\mathrm{j}) \qquad [T_i\mathbf{v}, \lambda_k^{(i)}]_i = \alpha_i[\lambda_1^{(i)}, \lambda_k^{(i)}]_i + \beta_i[\lambda_2^{(i)}, \lambda_k^{(i)}]_i \, , \quad k = 1, 2 \, ;$$

$$(\mathrm{jj}) \qquad \Pi\mathbf{v}(a_i) \, . \, \mathbf{n}^{(i)} = \alpha_i \, , \quad \Pi\mathbf{v}(a_{i+1}) \, . \, \mathbf{n}^{(i)} = \beta_i$$

define a mapping $\Pi \in L(\mathbf{H}^1(K), \mathbf{P}_1(K)) \cap L(\mathbf{C}(K), \mathbf{P}_1(K))$.

Let

$$\mathcal{N}_0(K) = \{\mathbf{v} \in \mathbf{H}^1(K) : \operatorname{div} \mathbf{v} = 0 \quad \text{in} \quad K\} \, ,$$
$$\mathcal{M}(K) \; = \{\mathbf{v} \in \mathbf{P}_1(K) : \operatorname{div} \mathbf{v} = 0 \quad \text{in} \quad K\} \, .$$

Theorem 2.2. *Let* Π *be defined by means of the relations* (j), (jj). *Then*

$$\Pi \in L(\mathcal{N}_0(K), \mathcal{M}(K)) \, ;$$
$$\Pi\mathbf{v} = \mathbf{v} \quad \forall \mathbf{v} \in \mathbf{P}_1(K) \, .$$

Theorem 2.3. *Let* $\mathbf{v} \in \mathbf{C}^2(K)$. *Then*

$$\|\mathbf{v} - \Pi\mathbf{v}\|_{C(K)} \leqq 4\left(1 + \frac{6\sqrt{2}}{\sin \alpha}\right) h^2 |\mathbf{v}|_{C^2(K)} \, ,$$

where $h = \operatorname{diam} K$ *and* α *is the minimal interior angle of* K.

Theorem 2.4. *Let* $\mathbf{v} \in \mathbf{H}^j(K)$, $j = 1, 2$. *Then*

$$\|\mathbf{v} - \Pi\mathbf{v}\|_{0,K} \leqq (ch^j/\sin \alpha) \, |\mathbf{v}|_{j,K} \, ,$$

where h, α *have the same meaning as in Theorem* 2.3 *and* $c > 0$ *is an absolute constant.*

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[2] $L(X, Y)$ denotes here the space of linear bounded mappings of X into Y.

Now we extend the above mentioned results. We shall prove that the mapping Π preserves the negativity (or positivity) of divergence. Let us define

$$\mathcal{N}_0^-(K) = \{\mathbf{v} \in \mathbf{H}^1(K) : \operatorname{div} \mathbf{v} \leqq 0 \quad \text{in} \quad K\}\,,$$
$$\mathcal{M}^-(K) = \{\mathbf{v} \in \mathbf{P}_1(K) : \operatorname{div} \mathbf{v} \leqq 0 \quad \text{in} \quad K\}\,.$$

Lemma 2.2.

$$\mathbf{v} \in \mathcal{M}^-(K) \Leftrightarrow \mathbf{v} \in \mathbf{P}_1(K) \quad \text{and} \quad \int_{\partial K} \mathbf{v} \,.\, \mathbf{n} \,\mathrm{d}s \leqq 0\,.$$

Proof. Let

$$\mathbf{v} = (\gamma_1 + \gamma_2 x_1 + \gamma_3 x_2\,, \quad \delta_1 + \delta_2 x_1 + \delta_3 x_2)\,.$$

Using Green's formula we obtain:

$$\int_K \operatorname{div} \mathbf{v} \,\mathrm{d}x = \int_{\partial K} \mathbf{v} \,.\, \mathbf{n} \,\mathrm{d}s\,.$$

If

$$\int_{\partial K} \mathbf{v} \,.\, \mathbf{n} \,\mathrm{d}s \leqq 0$$

then

$$\int_K \operatorname{div} \mathbf{v} \,\mathrm{d}x = (\gamma_2 + \delta_3) \operatorname{mes} K \leqq 0\,.$$

Hence $\gamma_2 + \delta_3 \leqq 0$ which implies $\mathbf{v} \in \mathcal{M}^-(K)$.

Lemma 2.3. *$\mathbf{v} \in \mathcal{M}^-(K) \Leftrightarrow \mathbf{v} \in \mathbf{P}_1(K)$ and $\sum_{i=1}^{3} (\alpha_i + \beta_i)\, l_i \leqq 0$, where $\alpha_i = T_i\mathbf{v}(a_i)$, $\beta_i = T_i\mathbf{v}(a_{i+1})$ and l_i denotes the length of $a_i a_{i+1}$.*

Proof.

$$\int_{\partial K} \mathbf{v} \,.\, \mathbf{n} \,\mathrm{d}s = \sum_{i=1}^{3} \int_{a_i a_{i+1}} T_i\mathbf{v} \,\mathrm{d}s = \sum_{i=1}^{3} \int_{a_i a_{i+1}} (\alpha_i \lambda_1^{(i)} + \beta_i \lambda_2^{(i)}) \,\mathrm{d}s = \tfrac{1}{2} \sum_{i=1}^{3} (\alpha_i + \beta_i)\, l_i\,.$$

The assertion now follows from Lemma 2.2.

Theorem 2.5. *Let Π be definied by the relations (j), (jj). Then*

$$\Pi \in L(\mathcal{N}_0^-(K)\,, \quad \mathcal{M}^-(K))\,.$$

Proof. Adding the equations (j) for $k = 1, 2$ we obtain

$$\int_{a_i a_{i+1}} T_i\mathbf{v} \,\mathrm{d}s = [T_i\mathbf{v}, \lambda_1^{(i)} + \lambda_2^{(i)}]_i = \alpha_i[\lambda_1^{(i)}, 1]_i + \beta_i[\lambda_2^{(i)}, 1]_i = \tfrac{1}{2}(\alpha_i + \beta_i) l_i$$

using that $\lambda_1^{(i)} + \lambda_2^{(i)} = 1$ on $a_i a_{i+1}$ and $[\lambda_k^{(i)}, 1]_i = l_i/2$. If $\mathbf{v} \in \mathcal{N}_0^-(K)$ then

$$0 \geqq \int_{\partial K} \mathbf{v} \cdot \mathbf{n} \, \mathrm{d}s = \sum_{i=1}^{3} \int_{a_i a_{i+1}} T_i \mathbf{v} \, \mathrm{d}s = \tfrac{1}{2} \sum_{i=1}^{3} (\alpha_i + \beta_i) \, l_i \, .$$

The assertion follows from the definition of Π and Lemma 2.3 (continuity of Π has been proved in [3]).

Let $\mathcal{T}_h$, $h \in (0, 1)$ be a triangulation of $\bar{Q}$ satisfying the usual requirements concerning the mutual position of two triangles and max diam $K = h$ $\forall K \in \mathcal{T}_h$. We say that a family $\{\mathcal{T}_h\}$, $h \in (0, 1)$ of triangulations of $\bar{Q}$ is regular, if there exists a constant $\alpha_0 > 0$ independent of h such that all interior angles of the triangles of $\mathcal{T}_h \in \{\mathcal{T}_h\}$ are not less than α_0. Denote by Π_K the mapping defined on $K \in \mathcal{T}_h$ by means of the conditions (j), (jj). Let $K, K' \in \mathcal{T}_h$ be two adjacent triangles with a common side $a_i a_{i+1}$. The function $T_i \mathbf{v}$ defined by (2.1) with respect to the triangle K will be denoted by $T_{i,K} \mathbf{v}$ (analogously for $T_{i,K'} \mathbf{v}$). We say that *the condition* ($\mathcal{R}$) *is satisfied* on the side $a_i a_{i+1}$, if

$$T_{i,K} \mathbf{v} + T_{i,K'} \mathbf{v} = 0 \quad \text{on} \quad a_i a_{i+1} \, .$$

Now we define

$$\mathcal{N}_{0h}^-(Q) = \{\mathbf{v}, \mathbf{v}|_K \in \mathcal{M}^-(K) \;\; \forall K \in \mathcal{T}_h, (\mathcal{R}) \textit{ is satisfied on each common side of any pair } K, K' \textit{ of adjacent triangles of } \mathcal{T}_h\} \, .$$

For $\mathbf{v} \in \mathbf{H}^1(Q)$ we define the mapping r_h by the relation

$$r_h \mathbf{v}|_K = \Pi_K \mathbf{v} \quad \forall K \in \mathcal{T}_h \, .$$

Theorem 2.6. *Let* $\{\mathcal{T}_h\}$, $h \in (0, 1)$ *be a regular family of triangulations of* $\bar{Q}$. *Then*

(2.2) $$r_h \in L(\mathcal{N}_0^-(Q) \cap \mathbf{H}^1(Q) \, , \;\; \mathcal{N}_{0h}^-(Q)) \, ;$$

(2.3) $$\|\mathbf{v} - r_h \mathbf{v}\|_{0,Q} \leqq c h^2 |\mathbf{v}|_{C^2(\bar{Q})} \quad \forall \mathbf{v} \in \mathbf{C}^2(\bar{Q}) \, ;$$

(2.4) $$\|\mathbf{v} - r_h \mathbf{v}\|_{0,Q} \leqq c h^j |\mathbf{v}|_{j,Q} \quad j = 1, 2, \quad \text{and} \quad \forall \mathbf{v} \in \mathbf{H}^j(Q) \, ,$$

where $c > 0$ *is an absolute constant.*

Proof. The proof of (2.2) follows immediately from Theorem 2.5 and the definition of r_h (see also [3]). (2.3), (2.4) follow immediately from Theorems 2.3, 2.4 respectively.

Remark. *It holds*:

$$\mathcal{N}_{0h}^-(Q) \subset \mathcal{N}_0^-(Q) \quad \forall h \in (0, 1) \, .$$

Proof. Let $\mathbf{v} \in \mathcal{N}^-_{0h}(Q)$, $\varphi \in \mathcal{U}_0$ be arbitrary. Then

$$\langle \operatorname{div} \mathbf{v}, \varphi \rangle = - \int_Q \mathbf{v} \cdot \operatorname{grad} \varphi \, dx = - \sum_{K \in \mathcal{T}_h} \int_K \mathbf{v} \cdot \operatorname{grad} \varphi \, dx =$$

$$= \sum_{K \in \mathcal{T}_h} \int_K \operatorname{div} \mathbf{v} \varphi \, dx - \sum_{K \in \mathcal{T}_h} \int_{\partial K} T_K \mathbf{v} \varphi \, ds \, .$$

The last term vanishes because of the condition $(\mathcal{R})$. Hence

$$\langle \operatorname{div} \mathbf{v}, \varphi \rangle = \sum_{K \in \mathcal{T}_h} \int_K \operatorname{div} \mathbf{v} \varphi \, dx = \int_Q \operatorname{div} \mathbf{v} \varphi \, dx \leqq 0 \quad \forall \varphi \in \mathcal{U}_0 \, .$$

Finally, let us set $\mathcal{N}^-_{fh}(Q) = \boldsymbol{\lambda}^0 + \mathcal{N}^-_{0h}(Q)$.

For our next purpose we estimate $\|\operatorname{div} \boldsymbol{\lambda} - \operatorname{div} r_h \boldsymbol{\lambda}\|_{0,K}$.

Theorem 2.7. *Let* $\boldsymbol{\lambda} \in \mathbf{H}^1(K)$ *be such that* $\operatorname{div} \boldsymbol{\lambda} \in H^1(K)$. *Then*

$$\|\operatorname{div} \boldsymbol{\lambda} - \operatorname{div} \Pi_K \boldsymbol{\lambda}\|_{0,K} \leqq ch |\operatorname{div} \boldsymbol{\lambda}|_{1,K} \, , \tag{2.5}$$

where $c > 0$ *depends on the minimal interior angle* α *of* K *only.*

Proof. Let P_0 denote the set of all constants on K. Green's formula and the definition of Π_K yield

$$(v, \operatorname{div} \boldsymbol{\lambda} - \operatorname{div} \Pi_K \boldsymbol{\lambda})_{0,K} = \int_{\partial K} v(\boldsymbol{\lambda} - \Pi_K \boldsymbol{\lambda}) \, \mathbf{n} \, ds = 0 \tag{2.6}$$

for every $v \in P_0$. It means that $\operatorname{div} \Pi_K \boldsymbol{\lambda} \in P_0$ is the orthogonal $L^2(K)$ projection of $\operatorname{div} \boldsymbol{\lambda}$ on P_0. Using the well-known property of orthogonal projections and the approximation property of P_0 in $H^1(K)$ (see [9]) we obtain the assertion.

One can easily extend (2.5) to the whole domain Q.

Theorem 2.8. *Let* $\{\mathcal{T}_h\}$ *be a regular family of triangulations of* $\bar{Q}$. *Then for every* $\boldsymbol{\lambda} \in \mathbf{H}^1(Q)$ *with* $\operatorname{div} \boldsymbol{\lambda} \in H^1(Q)$ *we have*

$$\|\operatorname{div} \boldsymbol{\lambda} - \operatorname{div} r_h \boldsymbol{\lambda}\|_{0,Q} \leqq ch |\operatorname{div} \boldsymbol{\lambda}|_{1,Q} \, , \tag{2.7}$$

where $c > 0$ *is an absolute constant.*

3. APPLICATIONS OF $\mathcal{N}^-_{fh}(Q)$ TO THE DUAL VARIATIONAL FORMULATION

In this section we establish the rate of convergence of the Ritz-Galerkin approximations $\boldsymbol{\lambda}^*_h \in \mathcal{N}^-_{fh}(Q)$ to the exact solution $\boldsymbol{\lambda}^* \in \mathcal{N}^-_f(Q)$ of $(\mathcal{P}^*)$. Let us recall that

$$\lambda^* = \lambda^0 + \chi^* ,$$
$$\lambda_h^* = \lambda^0 + \chi_h^* ,$$

where $\chi^* \in \mathcal{N}_0^-(Q)$, $\chi_h^* \in \mathcal{N}_{0h}^-(Q)$ and $\operatorname{div} \lambda^0 = -f$. In what follows we shall suppose that a family $\{\mathcal{T}_h\}$, $h \in (0, 1)$ of triangulations of $\bar{Q}$ used for the construction of $\mathcal{N}_{0h}^-(Q)$, is regular.

Theorem 3.1. *Let* $\chi^* \in \mathcal{N}_0^-(Q) \cap H^j(Q)$, $j = 1, 2$. *Then*

$$\|\lambda^* - \lambda_h^*\|_{0,Q} = 0(h^{j/2}) , \quad h \to 0+ . \tag{3.1}$$

Proof. Let us set $\lambda_h = \lambda^0 + r_h\chi^* \in \mathcal{N}_{fh}^-(Q)$. Then according to (1.16) we can write

$$\begin{aligned}\|\lambda^* - \lambda_h^*\|_{0,Q}^2 &\leqq \mathcal{F}(\lambda^* - \lambda_h) + b(\lambda_h^* - \lambda^*, \lambda_h - \lambda^*) + \\ &\quad + b(\lambda^*, \lambda_h - \lambda^*) \leqq \\ &\leqq \mathcal{F}(\chi^* - r_h\chi^*) + c\varepsilon\|\lambda_h^* - \lambda^*\|_{0,Q}^2 + \frac{c}{\varepsilon}\|r_h\chi^* - \chi^*\|_{0,Q}^2 + \\ &\quad + b(\lambda^*, r_h\chi^* - \chi^*) \quad \forall\varepsilon > 0 .\end{aligned} \tag{3.2}$$

Using the estimate (2.4) we deduce

$$|\mathcal{F}(\chi^* - r_h\chi^*)| = 0(h^j) , \tag{3.3}$$

$$|b(\lambda^*, r_h\chi^* - \chi^*)| = 0(h^j) , \quad h \to 0+ . \tag{3.4}$$

Taking $\varepsilon > 0$ sufficiently small, (3.2)–(3.4) implies (3.1).

Taking into account (3.1) we see that the optimal rate of convergence has not been obtained. Next we shall try to improve (3.1). We shall suppose that the following conditions are satisfied:

$$(u - \varphi)(-\Delta u - f) = 0 \quad \text{a. e. in} \quad Q , \tag{3.5}$$

$$Q_0 = \bigcup_{t=1}^{p} Q_{0t} , \quad Q_{0r} \cap Q_{0s} = \emptyset \quad \text{for} \quad r \neq s , \tag{3.6}$$

where Q_{0t}, $t = 1, \ldots, p$ are domains with sufficiently smooth parts of boundaries $\Gamma_{0t} \cap Q$.

Let us give another equivalent form of the right hand side of (1.16). Using the definition of $\mathcal{F}$ and the fact that $\lambda^* = \operatorname{grad} u$, we obtain

$$\begin{aligned}\|\lambda^* - \lambda_h^*\|_{0,Q}^2 &\leqq b(\lambda^* - \lambda_h^*, \chi^* - \chi_h) + \int_Q \operatorname{grad}(\varphi - u)(\chi^* - \chi_h)\,\mathrm{d}x = \\ &= b(\lambda^* - \lambda_h^*, \chi^* - \chi_h) + \langle u - \varphi, \operatorname{div}(\chi^* - \chi_h)\rangle \quad \forall \chi_h \in \mathcal{N}_{0h}^-(Q) ,\end{aligned} \tag{3.7}$$

where $\langle,\rangle$ denotes the duality pairing between $H_0^1(Q)$ and $H^{-1}(Q)$.

Theorem 3.2. *Let* $\chi^* \in H^j(Q)$, $u - \varphi \in H^k(Q_{0t})$ *and* $\partial^{k-1}/\partial \mathbf{n}^{k-1}(u - \varphi) = 0$ *on* $\Gamma_{0t} \cap Q$, $\operatorname{div} \chi^* \in H^m(Q)$, j, $k = 1, 2$, $m = 0, 1$, $t = 1, \ldots, p$. *Let* (3.5), (3.6) *be satisfied. Then*

$$\|\lambda^* - \lambda_h^*\|_{0,Q} \leqq ch^{(k+m)/2} \sqrt{\left(\sum_{t=1}^{p} \|u - \varphi\|_{k,Q_{0t}} |\operatorname{div} \chi^*|_{m,Q_{0t}{}^{2h}}\right)} + 0(h^j)\,, \tag{3.8}$$

where

$$Q_{ot}{}^{\eta h} = \{x \in Q : \operatorname{dist}(x, \Gamma_{0t}) < \eta h, \eta > 0\}\,.$$

Proof. We need to estimate the term

$$\langle u - \varphi, \operatorname{div}(\chi^* - \chi_h)\rangle = \sum_{K \in \mathscr{T}_h} \int_K (u - \varphi) \operatorname{div}(\chi^* - \chi_h)\, \mathrm{d}x$$

with $\chi_h = r_h \chi^*$. Let $K \in \mathscr{T}_h$ be fixed. If $u = \varphi$ a. e. in K, then

$$\int_K (u - \varphi) \operatorname{div}(\chi^* - \chi_h)\, \mathrm{d}x = 0\,. \tag{3.9}$$

If $K \subset Q_{0t}$ for some $t = 1, \ldots, p$ then $\operatorname{div} \chi^* = 0$ a. e. in K so that $\operatorname{div} \chi_h = \operatorname{div} \Pi_K \chi^* = 0$ in K by virtue of Theorem 2.2. So (3.9) holds again. Let G be a system of all $K \in \mathscr{T}_h$ such that $K \cap Q_{0t} \neq \emptyset$ but $K \not\subset Q_{0t}$. Let us set $Q_{0t}^{+h} = Q_{0t}^h \cap Q_{0t}$. Then

$$\left|\int_Q (u - \varphi) \operatorname{div}(\chi^* - \chi_h)\, \mathrm{d}x\right| \leqq \sum_{K \in G} \int_K |u - \varphi|\, |\operatorname{div} \chi^* - \operatorname{div} \chi_h|\, \mathrm{d}x \leqq$$

$$\leqq \sum_{t=1}^{p} \int_{Q_{0t}{}^{+h}} |u - \varphi|\, |\operatorname{div} \chi^* - \operatorname{div} \chi_h|\, \mathrm{d}x\,.$$

If $u - \varphi \in H^k(Q_{0t})$, $\partial^{k-1}/\partial \mathbf{n}^{k-1}(u - \varphi) = 0$ on $\Gamma_{0t} \cap Q$ and $\Gamma_{0t} \cap Q$ is sufficiently smooth, then (cf. [1]):

$$\|u - \varphi\|_{0,Q_{0t}{}^{+h}} \leqq ch^k \|u - \varphi\|_{k,Q_{0t}}\,. \tag{3.10}$$

Using (2.5) we obtain

$$\|\operatorname{div} \chi^* - \operatorname{div} \chi_h\|_{0,Q_{0t}{}^{+h}}^2 \leqq \sum_{K \cap Q_{0t}{}^{+h} \neq \emptyset} \|\operatorname{div} \chi^* - \operatorname{div} \chi_h\|_{0,K}^2 \leqq ch^{2m} |\operatorname{div} \chi^*|_{m,Q_{0t}{}^{2h}}^2\,. \tag{3.11}$$

The term $b(\lambda^* - \lambda_h^*, \chi^* - \chi_h)$ has been estimated in Theorem 3.1. (3.8) now follows from (3.2), (3.10), (3.11).

Up to now, very strong regularity assumption concerning the solution of $(\mathscr{P})$ and $(\mathscr{P}^*)$ have been imposed. In what follows the convergence of λ_h^* to λ^* without the rate of convergence will be proved under the only assumption that $\Delta u \in L^2(Q)$.

In the sequel, let us suppose that Q is a polygonal simply connected domain in R_2. ¹)

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¹) After a slight modification of the following proof, one can easily extend the results to the case of multiply connected domains.

Lemma 3.1. *Let* $\chi \in \mathcal{N}_0^-(Q)$, $\operatorname{div} \chi \in L^2(Q)$ *and let* $\tilde{Q} \supset \bar{Q}$ *be a simply connected domain. Then there exists a function* $\tilde{\chi} \in \boldsymbol{L}^2(\tilde{Q})$ *with the following properties:*

$$\tilde{\chi} = \chi \quad \text{in} \quad Q\,; \tag{3.12}$$

$$\operatorname{div} \tilde{\chi} \in L^2(\tilde{Q})\,; \tag{3.13}$$

$$\operatorname{div} \tilde{\chi} \leqq 0 \quad \text{in} \quad \tilde{Q}\,. \tag{3.14}$$

Proof. Since $\operatorname{div} \chi \in L^2(Q)$, we can define $\chi \,.\, \boldsymbol{n} \in H^{-1/2}(\partial Q)$ by means of Green's formula:

$$\int_Q \chi \,.\, \operatorname{grad} \tilde{\varphi} \,\mathrm{d}x + \int_Q \operatorname{div} \chi \,\tilde{\varphi} \,\mathrm{d}x = \int_{\partial Q} \chi \,.\, \boldsymbol{n} \tilde{\varphi} \,\mathrm{d}s \quad \forall \tilde{\varphi} \in \mathscr{D}(\tilde{Q})\,,$$

where $\int_{\partial Q}$ denotes the duality pairing between $H^{1/2}(\partial Q)$ and $H^{-1/2}(\partial Q)$. Let w be a solution of the boundary value problem

$$-\Delta w = g \quad \text{in} \quad \tilde{Q} - Q$$

$$w = 0 \quad \text{on} \quad \partial \tilde{Q}$$

$$\partial w / \partial \boldsymbol{n}^{(1)} = -\chi \,.\, \boldsymbol{n}^{(2)} \quad \text{on} \quad \partial Q\,,$$

where $g \in L^2(\tilde{Q})$ is a given non-negative function, $\boldsymbol{n}^{(2)}$ is the outward unit normal to ∂Q and $\boldsymbol{n}^{(1)} = -\boldsymbol{n}^{(2)}$. Let us set

$$\tilde{\chi} = \begin{cases} \chi & \text{in} \quad Q \\ \operatorname{grad} w & \text{in} \quad \tilde{Q} - Q\,. \end{cases}$$

From the definition of $\tilde{\chi}$, (3.12) follows. Let $\tilde{\varphi} \in \mathscr{D}(\tilde{Q})$ be fixed. Then

$$\langle \operatorname{div} \tilde{\chi}, \tilde{\varphi} \rangle = -\int_Q \tilde{\chi} \,.\, \operatorname{grad} \tilde{\varphi} \,\mathrm{d}x - \int_{\tilde{Q}-Q} \tilde{\chi} \,.\, \operatorname{grad} \tilde{\varphi} \,\mathrm{d}x = -\int_Q \chi \,.\, \operatorname{grad} \tilde{\varphi} \,\mathrm{d}x -$$

$$- \int_{\tilde{Q}-Q} \operatorname{grad} w \,.\, \operatorname{grad} \tilde{\varphi} \,\mathrm{d}x = \int_Q \operatorname{div} \chi \tilde{\varphi} \,\mathrm{d}x - \int_{\partial Q} \chi \,.\, \boldsymbol{n}^{(2)} \tilde{\varphi} \,\mathrm{d}s +$$

$$+ \int_{\tilde{Q}-Q} \Delta w \tilde{\varphi} \,\mathrm{d}x - \int_{\partial Q} \frac{\partial w}{\partial \boldsymbol{n}^{(1)}} \tilde{\varphi} \,\mathrm{d}s = \int_{\tilde{Q}} q \tilde{\varphi} \,\mathrm{d}x\,,$$

where

$$q = \begin{cases} \operatorname{div} \chi \in L^2(Q) \\ \Delta w \in L^2(\tilde{Q} - Q)\,. \end{cases}$$

Hence (3.13), (3.14) follows.

Lemma 3.2. $\mathcal{N}_0^-(Q) \cap [C^\infty(\bar{Q})]^2$ *is dense in* $\mathcal{N}_0^-(\operatorname{div}, Q) = \{\boldsymbol{\lambda} \in \mathcal{N}_0^-(Q)$ *and* $\operatorname{div} \boldsymbol{\lambda} \in L^2(Q)\}$ *in the* $\boldsymbol{L}^2(Q)$*-norm.*

Proof. Let $\chi \in \mathcal{N}_0^-(\mathrm{div}, Q)$ be fixed, let $\tilde{\chi}$ be its extension on $\tilde{Q} \subset \bar{Q}$, given by (3.12)–(3.14). Let us set

$$\tilde{\chi}_h = (\tilde{\chi}_{1h}, \tilde{\chi}_{2h}),$$

where $\tilde{\chi}_{jh} \in C^\infty(\bar{Q})$, $j = 1, 2$ are the regularizations of $\tilde{\chi}_j$ defined by

$$\tilde{\chi}_{jh}(x) = \int_{\tilde{Q}} \tilde{\chi}_j(y)\, \omega(x - y, h)\, \mathrm{d}y\,, \quad h > 0\,, \quad x, y \in \tilde{Q}\,.$$

$\omega(x, h)$ is the usual kernel of the regularization (see [4]). It is known that

$$\|\tilde{\chi}_j - \tilde{\chi}_{jh}\|_{0,\tilde{Q}} \to 0 \quad \text{for} \quad h \to 0+\,.$$

Let $h > 0$ be sufficiently small and $x \in Q$. Then

$$\mathrm{div}\, \tilde{\chi}_h(x) = \frac{\partial}{\partial x_1} \tilde{\chi}_{1h} + \frac{\partial}{\partial x_2} \tilde{\chi}_{2h} = - \int_{\tilde{Q}} \tilde{\chi}(y) \cdot \mathrm{grad}_y\, \omega(x - y, h)\, \mathrm{d}y =$$

$$= \int_{\tilde{Q}} \mathrm{div}\, \tilde{\chi}(y)\, \omega(x - y, h)\, \mathrm{d}y \leqq 0$$

by virtue of the fact that $\omega \geqq 0$. Finally,

$$\|\mathrm{div}\, \tilde{\chi}_h - \mathrm{div}\, \tilde{\chi}\|_{0,Q} \to 0 \quad \text{for} \quad h \to 0+\,.$$

Theorem 3.3. *Let the solution u of $(\mathscr{P})$ be such that $\Delta u \in L^2(Q)$. Then*

$$\|\lambda^* - \lambda_h^*\|_{0,Q} \to 0\,, \quad h \to 0+\,.$$

Proof. $\mathrm{div}\, \lambda^* = \Delta u \in L^2(Q)$ and $\mathrm{div}\, \lambda^0 = -f \in L^2(Q)$ yield $\mathrm{div}\, \chi^* \in L^2(Q)$. In order to prove the convergence of λ_h^* to λ^* (or χ_h^* to χ) it is sufficient to prove that there exists a space of smooth functions dense in $\mathcal{N}_0^-(\mathrm{div}, Q)$ (see [8]). Such an assertion follows from Lemma 3.2.

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References

[1] *I. Babuška:* Approximation by hill-functions II. Institute for fluid Dynamics and Applied mathematics. Technical note BN-708.

[2] *F. Brezzi, W. W. Hager, P. A. Raviart:* Error estimates for the finite element solution of variational inequalities. Part I: Primal Theory (preprint).

[3] *J. Haslinger, I. Hlaváček:* Convergence of finite element method based on the dual variational formulation. Apl. Mat. 21 1976, 43–65.

[4] *J. Nečas:* Les méthodes directes en théorie des équations elliptiques. Academia, Prague 1967.

[5] *J. Haslinger:* Finite element analysis for unilateral problems with obstacles on the boundary. Apl. Mat. 22 1977, 180—189.
[6] *J. Haslinger:* A note on a dual finite element method. CMUC 17, 4 1976, 665—673.
[7] *P. G. Ciarlet, P. A. Raviart:* General Lagrange and Hermite interpolation in R^n with applications to finite element methods. Arch. Rational Mech. Anal. 46 1972, 217—249.
[8] *J. Cea:* Optimisation, théorie et algorithmes. Dunod, Paris 1971.
[9] *G. Strang:* Approximations in the finite element method. Numer. Math. 19, 81—98.
[10] *J. L. Lions:* Quelques Méthodes de résolution des problémes aux limites non linéaires. Dunod, Paris.
[11] *P. A. Raviart:* Hybrid finite element methods for solving 2nd order elliptic equations. Conference on Numer. Analysis, Dublin, 1974.
[12] *I. Hlaváček:* Dual finite element analysis for unilateral boundary value problems. Apl. Mat. 22 1977, 14—51.
[13] *I. Hlaváček:* Dual finite element analysis for elliptic problems with obstacles on the boundary, I. Ap. Mat. 22 (1977), 244—255.

Souhrn

DUÁLNÍ ANALÝZA NEROVNIC II. ŘÁDU METODOU KONEČNÝCH PRVKŮ

JAROSLAV HASLINGER

V práci je studována duální variační formulace k okrajovým eliptickým problémům s nerovnostmi (překážkami) zadanými uvnitř oblasti. K numerickému řešení je navržena metoda konečných prvků. Užitím po částech lineárních rovnovážných prvků, zavedených v [3], se dokazuje řád konvergence Ritzových aproximací za předpokladu jisté hladkosti přesného řešení. V dalším se předpoklady na hladkost řešení zeslabují.

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