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ON THE INFINITESIMAL GEOMETRIC OBJECTS OF SUBMANIFOLDS

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As a further development of the invariant method of investigation for submanifolds of homogeneous spaces by É. Cartan, some authors (in particular S. P. Finikov and his disciples) have applied a natural modification of this method for finding of infinitesimal invariants and infinitesimal relative invariants of submanifolds. Though they have investigated only concrete situations, we can observe that their evaluations have the following general form. Besides the "variations with respet to the secondary parameters" of some functions, one also finds the corresponding variations of a basis of the principal forms and then proceeds in the same way as in the case of the usual (or "finite") invariants. Since we feel such a "classical" approach to be unsatisfactory from theoretical point of view, we present an intrinsic explanation of this algorithm. Moreover, the present state of the theory of geometric objects of submanifolds, [10], [3], [7], enables us to treat arbitrary infinitesimal geometric objects of submanifolds. We conclude with a detailed discussion of a classical example from our point of view. Since we use a specialization of frames in the example, we also add a general remark to an aspect of this procedure.

Our considerations are in the category C^∞. The standard terminology of the theory of jets is used throughout the paper, cf. [3], [6].

1. Consider the fibre bundle $K^1_m(M)$ of all regular contact m^1-elements on a differentiable manifold M, $m < n = \dim M$. Taking into account that every $\xi \in K^1_{m,x}(M)$ is naturally identified with an m-dimensional subspace $\tau(\xi)$ of $T_x(M)$, we define

$$(1) \qquad VK^1_m(M) = \{(v, \xi) \in T(M) \oplus K^1_m(M);\ v \in \tau(\xi)\},$$

where $\oplus$ means the fibre product over M. The space $VK^1_m(M)$ can be considered either as a vector bundle over $K^1_m(M)$ of fibre dimension m or as an associated fibre bundle of the symbol $(M, \gamma^n_m, L^1_n, H^1(M))$, where $\gamma^n_m = VK^1_{m,0}(\mathbf{R}^n)$ (i.e. the fibre of $VK^1_m(\mathbf{R}^n)$ over $0 \in \mathbf{R}^n$), cf. [1]. Further, let $K^r_m(M)$ be the fibre bundle of all contact m^r-elements on M and let $\rho^s_r : K^r_m(M) \to K^s_m(M)$, $s \leqq r$, be the jet projection. Then we set

$$(2) \qquad VK^r_m(M) = \{(v, \xi) \in T(M) \oplus K^r_m(M);\ v \in \tau(\rho^1_r(\xi))\}.$$

Even this space can be considered either as a vector bundle over $K^r_m(M)$ of fibre dimension m or as an associated fibre bundle of the symbol $(M, VK^r_{m,0}(\mathbf{R}^n), L^r_n, H^r(M))$. If S is an m-dimensional submanifold of M, then we have a natural injection $T(S) \subset \subset VK^r_m(M)$, $v \to (v, k^r_{\eta(v)}S)$, where $\eta : T(M) \to M$ is the bundle projection and $k^r_x S$ means the contact m^r-element determined by S at a point $x \in S$.

Remark 1. Quite analogously, one can treat the tangent vectors of higher order. Let $T^r(M)$ be the bundle of all tangent vectors of order r on M. (In particular, we recall the exact sequence

$$0 \to T^{r-1}(M) \to T^r(M) \to S^r T(M) \to 0 \tag{3}$$

established by Pohl, [12].) Let $\xi = XL^r_m \in K^r_{m,x}(M)$, where X is an m^r-velocity on M at x. Since X is an r-jet of $\mathbf{R}^m$ into M with source 0 and target x, it determines a linear mapping $X_* : T^r_0(\mathbf{R}^m) \to T^r_x(M)$. One sees directly that the subspace $X_*(T^r_0(\mathbf{R}^m)) \subset \subset T^r_x(M)$ is well determined by ξ, i.e. it does not depend on the choice of a representative X of ξ; we shall denote this subspace by $\tau_r(\xi)$. Then we define

$$V^s K^r_m(M) = \{(v, \xi) \in T^s(M) \oplus K^r_m(M);\, v \in \tau_s(\rho^s_r(\xi))\}.$$

Obviously, it is $VK^r_m(M) = V^1 K^r_m(M)$. In the differential geometry of submanifolds, the higher order tangent vectors are sometimes investigated in such a way, that the vectors of $T^r(M)$ are transformed by (3) into the elements of $S^r T(M)$. In this special case, the following method of investigation can be applied. However, we do not see any "natural" invariant algorithm for the higher order tangent vectors in the general case.

2. To clear some fundamental ideas up, we shall start with an auxiliary consideration. The tangent bundle $T(M)$ of M is an associated fibre bundle of the symbol $(M, \mathbf{R}^n, L^1_n, H^1(M))$, so that one has a relative image map $\mu : H^1(M) \oplus T(M) \to \mathbf{R}^n$, see [5]. This mapping is in the following close relation to the canonical form Θ of $H^1(M)$, [2], [4]. Let $\pi : H^1(M) \to M$ be the bundle projection. Then its differential $\pi_* : T(H^1(M)) \to T(M)$ together with the bundle projection $\nu : T(H^1(M)) \to H^1(M)$ determine a mapping $\lambda : T(H^1(M)) \to H^1(M) \oplus T(M)$, $\lambda(v) = (\nu(v), \pi_*(v))$.

Lemma 1. *The following diagram commutes*

$$\begin{tikzcd} \mathbf{R}^n & T(H^1(M)) \arrow[l, "\Theta"'] \arrow[d, "\lambda"] \\ & H^1(M) \oplus T(M) \arrow[ul, "\mu"] \end{tikzcd} \tag{4}$$

Proof. If $u \in H^1(M)$ and $v \in T_u(H^1(M))$, then $\Theta(v) = u^{-1}(\pi_*(v))$ by definition of Θ. On the other hand, if $w \in T_x(M)$, then $\mu(u, w) = u^{-1}(w)$ by definition of μ. Hence $\mu(\lambda(v)) = \mu(u, \pi_*(v)) = u^{-1}(\pi_*(v)) = \Theta(v)$, QED.

Let $K^1_{n,m} = K^1_{m,0}(\mathbf{R}^n)$ and let $\hat{K}^1_{n,m} \subset K^1_{n,m}$ be the subspace of all elements transversal with respect to the canonical projection $p : \mathbf{R}^n \to \mathbf{R}^m$. On $\hat{K}^1_{n,m}$, there are natural coordinates y^J_p, see [6]. Further, let $\hat{\gamma}^n_m \subset \gamma^n_m$ be the subspace of all pairs (v, ξ), such that $\xi \in \hat{K}^1_{n,m}$. On $\hat{\gamma}^n_m$, we introduce the coordinates y^J_p, y^p by $y^J_p(v, \xi) = y^J_p(\xi)$ and by

$$(5) \qquad y^p(v, \xi) = x^p(p(v)) = y^p(v), \qquad \begin{aligned} &p, q, \ldots = 1, \ldots, m, \\ &J, K, \ldots = m + 1, \ldots, n, \end{aligned}$$

where x^p are the canonical coordinates on $\mathbf{R}^m$. Obviously, v is completely determined by $p(v) \in \mathbf{R}^m$ and by ξ (the coordinates of $v \in \mathbf{R}^n$ are $y^p(v)$, $y^J_p(\xi)\, y^p(v)$).

Consider an m-dimensional submanifold $S \subset M$. According to [6], we denote by $Q^1(S)$ the restriction of $H^1(M)$ over S and we set

$$\hat{Q}^1(S) = \{u \in Q^1(S);\, u^{-1}(k^1_x S) \in \hat{K}^1_{n,m},\, x = \pi(u)\}.$$

Every $u \in Q^1_x(S)$ carries $T_x(S)$ into an m-dimensional subspace $u^{-1}(T_x(S))$ of $\mathbf{R}^n$, so that it plays a role of a frame for $T_x(S)$. If $u \in \hat{Q}^1_x(S)$, then $u^{-1}(T_x(S))$ is transversal with respect to $p : \mathbf{R}^n \to \mathbf{R}^m$. Consequently, we obtain the natural coordinate functions A^p on $\hat{Q}^1(S) \oplus T(S)$ defined by $A^p(u, v) = x^p(p(u^{-1}(v)))$. Let

$$\tilde{\Theta} = (\tilde{\Theta}_i), \qquad i, j \ldots = 1, \ldots, n$$

be the restriction of the canonical form of $H^1(M)$ to $Q^1(S)$. By (4) and (5), we deduce the commutativity of the diagram

$$(6) \qquad \begin{tikzcd} \mathbf{R} & T(\hat{Q}^1(S)) \arrow[l, "\tilde{\Theta}^p"'] \arrow[d, "\lambda_S"] \\ & \hat{Q}^1(S) \oplus T(S) \arrow[ul, "A^p"] \end{tikzcd}$$

where $\lambda_S = \lambda \mid T(\hat{Q}^1(S))$. Hence the differential forms $\tilde{\Theta}^p : T(\hat{Q}^1(S)) \to \mathbf{R}$ can be considered as a special kind of some "coordinate functions" for $T(S)$ in the following sense. If $v \in T_x(S)$, $u \in \hat{Q}^1_x(S)$ and $w \in T_u(\hat{Q}^1(S))$ is a vector such that $\pi_*(w) = v$, then it is $\hat{\Theta}^p(w) = A^p(u, v)$. In addition, if $a^J_p : \hat{Q}^1(S) \to \mathbf{R}$ are the coordinate functions of the fundamental field of the first order of S, then the coordinates of the vector $u^{-1}(v) \in \mathbf{R}^n$ are $\tilde{\Theta}^p(w)$, $a^J_p(u)\, \tilde{\Theta}^p(w)$.

We shall now deduce the equations of the fundamental distribution on $L^1_n \times \gamma^n_m$ with respect to the coordinates y^J_p, y^p. Though it is not complicated to use direct evaluations, we shall proceed in an indirect way. For the sake of simplicity, we shall assume that S is homotopically trivial. Take a vector field $\zeta : S \to T(S)$ and denote by $a^p : \hat{Q}^1(S) \to \mathbf{R}$ its coordinate functions, i.e. $a^p(u) = A^p(u, \zeta(\pi(u)))$. Consider a fundamental vector field Y on $\hat{Q}^1(S) \subset Q^1(S)$ and choose a vector field Z on $\hat{Q}^1(S)$ such that Z is π-related with ζ and satisfies $[Y, Z] = 0$. (Such a vector field can be constructed easily be means of a trivialization of principal fibre bundle $Q^1(S)$.) According to (6), it holds $a^p = \tilde{\Theta}^p(Z)$. To evaluate $da^p(Y) = Y\tilde{\Theta}^p(Z)$, we use, on the one hand, the well known formula for exterior derivative

$$\text{(7)} \qquad \mathrm{d}\tilde{\Theta}^p(Y, Z) = \frac{1}{2}\{Y\tilde{\Theta}^p(Z) - Z\tilde{\Theta}^p(Y) - \tilde{\Theta}^p([Y, Z])\}$$

and, on the other hand, the structure equations of Θ

$$\text{(8)} \qquad \mathrm{d}\Theta^i = \Theta^j \wedge \Theta_j'^i,$$

where $(\Theta^i, \Theta_j'^i)$ is an admissible extension of Θ, [4]. Using the relation $\tilde{\Theta}^J = a_p^J\tilde{\Theta}^p$, [6], we obtain

$$\text{(9)} \qquad \mathrm{d}\tilde{\Theta}^p = \tilde{\Theta}^q \wedge (\tilde{\Theta}_q'^p + a_q^J\tilde{\Theta}_J'^p).$$

As Y is a vertical vector field, it is $\tilde{\Theta}^p(Y) = 0$. Hence (7) is simplified to

$$\text{(10)} \qquad \mathrm{d}\tilde{\Theta}^p(Y, Z) = \frac{1}{2}\, Y\tilde{\Theta}^p(Z)$$

and (9) implies

$$\text{(11)} \qquad \mathrm{d}\tilde{\Theta}^p(Y, Z) = -\frac{1}{2}\tilde{\Theta}^q(Z)\,[\tilde{\Theta}_q'^p(Y) + a_q^J\tilde{\Theta}_J'^p(Y)].$$

Comparing (10) and (11), we obtain finally

$$\text{(12)} \qquad da^p(Y) + a^q\Theta_q'^p(Y) + a^q a_q^J\tilde{\Theta}_J'^p(Y) = 0.$$

In [6], we have deduced the equations of the fundamental distribution on $L_n^1 \subseteq K_{n,m}^1$ in the form

$$\text{(13)} \qquad \mathrm{d}y_p^J - y_q^J\pi_p^q - y_q^J y_p^K\pi_K^q + y_p^K\pi_K^J + \pi_p^J = 0,$$

where π_i^j is the natural basis of the Maurer–Cartan forms of L_n^1. Applying Lemma 2 of [5] to (12), we prove

Proposition 1. *The equations of the fundamental distribution on* $L_n^1 \subseteq \gamma_m^n$ *are* (13) *and*

$$\text{(14)} \qquad \mathrm{d}y^p + y^q\pi_q^p + y^q y_q^J\pi_J^p = 0.$$

Remark 2. Our previous consideration gives a precise explanation of the "classical" manipulation with two "exchangeable symbols of differentiation d, δ, where δ means the differentiation with respect to the secondary parameters" according to É. Cartan, see e.g. [11].

Assume now that M is a homogeneous space with fundamental group G. Fix a point $c \in M$ and denote by H its stability group. Then G has a natural structure of a principal fibre bundle over M with structure group H and $VK_m^1(M)$ can be considered as an associated fibre bundle of the symbol $(M, VK^1_{m,c}(M), H, G)$. Fix a local coordinate system $\varkappa$ on M at c, so that $VK^1_{m,c}(M)$ is identified with γ^n_m. Let

$$\text{(15)} \qquad \begin{aligned} \mathrm{d}\omega^i &= \frac{1}{2} c^i_{jk}\omega^j \wedge \omega^k + c^i_{j\lambda}\omega^j \wedge \omega^\lambda, && \lambda, \mu, \ldots = n+1, \ldots, \dim G, \\ \mathrm{d}\omega^\lambda &= \frac{1}{2} c^\lambda_{\alpha\beta}\omega^\alpha \wedge \omega^\beta, && \alpha, \beta, \ldots = 1, \ldots, \dim G, \end{aligned}$$

be the corresponding structure equations. By means of the homomorphism of H into L^1_n investigated in [6], we find directly the equations of the fundamental distribution on $H \times \gamma^n_m$ in the form

$$\text{(16)} \qquad \mathrm{d}y^J_p - (y^J_q c^q_{p\lambda} + y^J_q y^K_p c^q_{K\lambda} - y^K_p c^J_{K\lambda} - c^J_{p\lambda})\,\pi^\lambda = 0,$$

$$\text{(17)} \qquad \mathrm{d}y^p + y^q c^p_{q\lambda}\pi^\lambda + y^q y^J_q c^p_{J\lambda}\pi^\lambda = 0,$$

where π^λ is the restriction of ω^λ to H. From the practical point of view it is remarkable that these equations can be deduced directly from (15) as follows.

Consider an m-dimensional submanifold $S \subset M$, denote by $Q(S)$ the restrition of principal fibre bundle $G(M, H)$ over S and introduce $\hat{Q}(S) \subset Q(S)$ in the same way as in [6]. Then every $u \in Q_x(S)$ plays a role of a frame for $T_x(S)$, since it caries $T_x(S)$ into an m-dimensional subspace $u^{-1}(T_x(S))$ of $T_c(M)$. If $u \in \hat{Q}(S)$, then this subspace is transversal with respect to $p : \mathbf{R}^n \to \mathbf{R}^m$, provided the local identification of M and $\mathbf{R}^n$ determined by $\varkappa$ is applied. Hence we can define some coordinate functions A^p on $\hat{Q}(S) \oplus T(S)$ by $A^p(u, v) = x^p(p(u^{-1}(v)))$. We also have a mapping $\lambda : T(G) \to G \oplus T(M)$, $v \to (\nu(v), \pi_*(v))$, where $\nu : T(G) \to G$ and $\pi : G \to M$ are the bundle projections. Let $\lambda_S = \lambda \mid T(\hat{Q}(S))$ and let $\tilde{\omega}^\alpha$ be the restriction of ω^α to $\hat{Q}(S)$. According to (6) and [6], we obtain a commutative diagram

$$\text{(18)} \qquad \begin{tikzcd} \mathbf{R} & T(\hat{Q}(S)) \arrow[l, "\tilde{\omega}^p"'] \arrow[d, "\lambda_S"] \\ & \hat{Q}(S) \oplus T(S) \arrow[ul, "A^p"] \end{tikzcd}$$

Taking into account the relations $\tilde{\omega}^J = a^J_p \tilde{\omega}^p$, where $a^J_p : \hat{Q}(S) \to \mathbf{R}$ are the coordinate functions of the fundamental field of the first order of S, we can deduce (17) analogously to (7) and (11), provided the structure equations (15) are used instead of (8).

If we now consider the fibre bundle $VK^r_m(M)$ for an arbitrary differentiable manifold M, we obtain the equations of the fundamental distribution on $L^r_n \times VK^r_{m,0}(\mathbf{R}^n)$ immediately as the equations (21) of [6] together with (14). Analogously, if M is a homogeneous space, then $VK^r_m(M)$ should be considered as an associated fibre bundle of the symbol $(M, VK^r_{m,c}(M), H, G)$ and the equations of the fundamental distribution on $H \times VK^r_{m,c}(M)$ consist of the equations of the fundamental distribution on $H \times K^r_{m,c}(M)$, [6], and of (17).

3. We shall present the definition of the infinitesimal geometric objects of submanifolds in the most important case, i.e. for submanifolds of a homogeneous space (the case of a space with a fundamental Lie pseudogroup can be treated quite similarly to [3], [6]).

Definition 1. *An infinitesimal geometric m^r-object ψ on a homogeneous space M is an equivariant mapping of H-space $VK^r_{m,c}(M)$ into another H-space W, $\psi : VK^r_{m,c}(M) \to \to W$. Let ψ_2 be the induced mapping $\psi_2 : VK^r_m(M) \to (M, W, H, G)$, see* [3]. *If $S \subset M$ is an m-dimensional submanifold, then the restriction of ψ_2 to $T(S) \subset VK^r_m(M)$ will be called the value of ψ on S. Moreover, taking into account* (18), *we also introduce the auxiliary form of the value of ψ on S as the mapping $\bar{\psi}_S : T(Q(S)) \to W$ defined by $\bar{\psi}_S(v) = \psi(u^{-1}(\pi_*(v)), u^{-1}(k^r_{\pi(u)}S))$, $u = \nu(v)$.*

Let $y^J_p, \dots, y^J_{p_1 \dots p_r}, y^p$ be the local coordinates on $VK^r_{m,c}(M)$ determined by $\varkappa$, let z^a be some local coordinates on W and let

$$z^a = f^a(y^J_p, \dots, y^J_{p_1,\dots,p_r}, y^p), \tag{19}$$

be the coordinate expression of ψ. (Since we have the equations of the fundamental distribution on $H \times VK^r_{m,c}(M)$, the method of G. F. Laptěv, see [10], p. 301 and Appendix to [6], can be used for local analytic constructions of such equivariant mappings.)

Proposition 2. *Let S be an m-dimesional submanifold of M, let $a^J_p, \dots, a^J_{p_1 \dots p_r} : : \hat{Q}(S) \to \mathbf{R}$ be the coordinate functions of the fundamental field of order r of S, let $\tilde{\omega}^p$ be the restriction of ω^p to $T(\hat{Q}(S))$ and let* (19) *be the coordinate expression of an infinitesimal geometric m^r-object ψ on M. Then the coordinate expression of the auxiliary form $\bar{\psi}_S$ of the value of ψ on S is*

$$z^a = f^a(a^J_p, \dots, a^J_{p_1,\dots,p_r}, \tilde{\omega}^p). \tag{20}$$

Proof. This follows directly from (18) and from our results of [6].

Remark 3. Considering the space $\bar{K}^r_m(M)$ of all semi-holonomic contact m^r-elements on M, [7], we introduce $V\bar{K}^r_m(M)$ quite similarly to (2) and we define a semi-holonomic infinitesimal geometric m^r-object on M as an equivariant mapping ψ of H-space $V\bar{K}^r_{m,c}(M)$ into another H-space W. The values of ψ can be considered on m-dimensional manifolds with connection of type M, [7], or on non-holonomic m-dimensional distributions on M, [9].

Remark 4. As mentioned in the introduction, in the „classical" differential geometry of submanifolds one meets infinitesimal invariants and relative invariants. he property "to be an invar iant" or "to be a relative invariant" is the following general property of equivariant mappings. Let G be a Lie group and let f be an equivariant mapping of a G-space F into $\mathbf{R}$. If G acts on $\mathbf{R}$ by the identity representation, then f is said to be an (absolute) invariant. If G acts on $\mathbf{R}$ by a homomorphism into the group $\mathbf{R} \setminus \{0\}$ of all homothetic transformations of $\mathbf{R}$, then f will be called a relative invariant. Finally, if G acts on $\mathbf{R}$ by a homomorphism into the subgroup $\mathbf{R}^+ \subset \mathbf{R} \setminus \{0\}$, then f will be said to be an oriented relative invariant. Let y^A be some local coordinates on F, let

$$\mathrm{d}y^A + \eta^A_\alpha(y^B)\, \omega^\alpha = 0 \tag{21}$$

be the equations of the fundamental distribution on $G \times F$ and let $f(y^A)$ be the coordinate formula for $f : F \to \mathbf{R}$. Taking account of Appendix of [6], we shall denote by $\mathrm{d}f$ the expression

$$\mathrm{d}f = -(\partial f / \partial y^A)\, \eta^A_\alpha(y^B)\, \omega^\alpha. \tag{22}$$

One sees directly that f is an invariant if and only if

$$\mathrm{d}f = 0, \tag{23}$$

while f is a relative invariant or an oriented relative invariant if and only if

$$\mathrm{d}f + f\omega = 0, \tag{24}$$

where ω is a linear combination of ω^α with constant coefficients. Combining this general remark with the results of § 2 and with Proposition 2, one can treat the infinitesimal invariants and relative invariants of submanifolds.

Remark 5. A practical inconvenience by the standard use of the so-called Cartan's methods consists in the fact that one cannot decide by (24) whether the relative invariant f is oriented or not. An essential difference between both cases is that an oriented relative invariant f determines three invariant subspaces $f^{-1}(\mathbf{R}^+)$, $f^{-1}(\mathbf{R}^-)$, $f^{-1}(0)$ of F, while in the non-oriented case only two invariant subspaces $f^{-1}(0)$ and $f^{-1}(\mathbf{R} \setminus \{0\})$ are determined by f.

Example 1. To illustrate Definition 1 and Proposition 2, we shall discuss a classical example. To simplify our evaluations, we shall apply a specialization of frames, see also Remark 6. Consider a surface S of the 3-dimensional affine space A_3 and fix an affine coordinate system on A_3. Let H be the stability group of the point $c = (0, 0, 0)$ and let

$$\omega^i, \omega^i_j, \qquad i, j, \ldots = 1, 2, 3,$$

be the natural basis of the Maurer–Cartan forms of the fundamental group of A_3. Let $k^1 S$ be the fundamental field of the first order of S, let c_1 be the subspace $x^3 = 0$ of $T_c(A_3)$ considered as an element of $K^1_{2,c}(A_3)$ and let $Q_1(S)$ be the reduction of

$Q(S)$ determined by the pair (k^1S, c_1), [8]. In particular, the differential equations of the structure group $H_1 \subset H$ of $Q_1(S)$ are

$$\omega^i, = 0, \omega_p^3 = 0, \qquad p, q, \ldots = 1, 2. \tag{25}$$

Let $\tilde{\omega}^i$, $\tilde{\omega}_j^i$ be the restrictions of ω^i, ω_j^i to $Q_1(S)$; then it holds

$$\tilde{\omega}^3 = 0. \tag{26}$$

Introduce $\rho : VK_{2,c}^2(A_3) \to K_{2,c}^1(A_3)$, $(v, \xi) \to \rho_2(\xi)$, and set $N = \rho^{-1}(c_1)$. On $VK_{2,c}^2(A_3)$ we have the coordinates $y_p^3 = y_p$, $y_{pq}^3 = y_{pq}$, y^p and N is characterized by $y_p = 0$. Applying the previous method, we find the equations of the fundamental distribution on $H_1 \times N$ in the form

$$\begin{aligned} &dy_{11} - y_{11}(2\pi_1^1 - \pi_3^3) - 2y_{12}\pi_1^2 = 0. \\ &dy_{12} - y_{12}(\pi_1^1 + \pi_2^2 - \pi_3^3) - y_{11}\pi_2^1 - y_{22}\pi_1^2 = 0, \\ &dy_{22} - y_{22}(2\pi_2^2 - \pi_3^3) - 2y_{12}\pi_2^1 = 0, \end{aligned} \tag{27}$$

$$\begin{aligned} &dy^1 + y^1\pi_1^1 + y^2\pi_2^1 = 0, \\ &dy^2 + y^1\pi_1^2 + y^2\pi_2^2 = 0, \end{aligned} \tag{28}$$

where the π's are the restrictions of the corresponding ω's to H_1. (In practice, one can deduce (28) by the following evaluation, in which the "classical" notation mentioned in Remark 2 is used as a kind of shorthand. On $Q_1(S)$, the structure equations of A_3 give

$$\begin{aligned} d\tilde{\omega}^1 &= \tilde{\omega}^1 \wedge \tilde{\omega}_1^1 + \tilde{\omega}^2 \wedge \tilde{\omega}_2^1, \\ d\tilde{\omega}^2 &= \tilde{\omega}^1 \wedge \tilde{\omega}_1^2 + \tilde{\omega}^2 \wedge \tilde{\omega}_2^2. \end{aligned}$$

Analogously to (7) and (11) one obtains,

$$\begin{aligned} \delta(\omega^1(d)) &= -\tilde{\omega}^1(d)\,\pi_1^1 - \tilde{\omega}^2(d)\,\pi_2^1, \\ \delta(\omega^2(d)) &= -\tilde{\omega}^1(d)\,\pi_1^2 - \tilde{\omega}^2(d)\,\pi_2^2. \end{aligned} \tag{29}$$

By Lemma 2 of [5], (29) implies (28).) Using (27) and (28), we deduce that the mapping

$$f = y_{11}(y^1)^2 + 2y_{12}y^1y^2 + y_{22}(y^2)^2$$

satisfies $df + f\pi_3^3 = 0$. Hence f is a relative invariant. Further, let $a_{pq} : Q_1(S) \to \mathbf{R}$ be the coordinate functions of the fundamental field of the second order of S. By Proposition 2, the restriction to $Q_1(S)$ of the auxiliary form of the value of f on S is

$$a_{11}(\tilde{\omega}^1)^2 + 2a_{12}\tilde{\omega}^1\tilde{\omega}^2 + a_{22}(\tilde{\omega}^2)^2,$$

which is the well known asymptotic form of S.

Remark 6. In Example 1, we have constructed an equivariant mapping f of the H_1-space N, though Definition 1 requires an equivariant mapping of the H-space

$VK^2_{2,c}(A_3)$. But f can be naturally extended to such a mapping as follows. In general, consider a homogeneous space A with fundamental group H and denote by H_1 the stability group of a point $p \in A$. Let A_1 be another H-space and let $\eta : A_1 \to A$ be an equivariant surjection. Set $A_0 = \eta^{-1}(p)$, which is an H_1-space. Consider another H_1-space $\bar{A}_0$ and an H_1-equivariant mapping $\varphi : A_0 \to \bar{A}_0$. Since H has a natural structure of a principal fibre bundle $H(A, H_1)$ over A with structure group H_1, we can construct an associated fibre bundle $\bar{A}_1 = \bar{A}_1(A, \bar{A}_0, H_1, H)$. Every element of $\bar{A}_1$ being an equivalence class $\{(h, y)\}$ with respect to the equivalence relation $(h, y) \sim (hh_1^{-1}, h_1 y)$, $h_1 \in H_1$, we first introduce a left action of H on A_1 by $\bar{h}\{(h, y)\} = \{(\bar{h}h, y)\}$, $\bar{h} \in H$. This definition is correct, since $\bar{h}\{(hh_1, h_1^{-1}y)\} = \{(\bar{h}hh_1, h_1^{-1}y)\} = \{(\bar{h}h, y)\}$. Then we define an H-equivariant mapping $\tilde{\varphi} : A_1 \to \bar{A}_1$ by $\tilde{\varphi}(hy) = h\varphi(y)$, $h \in H$, $y \in A_0$. Even this is a correct definition, since $hy = \bar{h}\bar{y}$, $y, \bar{y} \in A_0$, implies $h^{-1}\bar{h} \in H_1$, so that $\tilde{\varphi}(\bar{h}\bar{y}) = \bar{h}\varphi(\bar{y}) = hh^{-1}\bar{h}\varphi(\bar{y}) = h\varphi(y)$. The H-equivariant mapping $\tilde{\varphi} : A_1 \to \bar{A}_1$ is the above-mentioned natural extension of an H_1-equivariant mapping $\varphi : A_0 \to \bar{A}_0$.

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