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THE CANTOR-BERNSTEIN THEOREM FOR FUNCTORS

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Abstract:

We call a category $\mathbb{K}$ Cantor-Bernstein category if each two functors $\mathcal{A}, \mathcal{B} : \mathbb{K} \longrightarrow \mathcal{S}$ are equivalent whenever $\mathcal{A}$ is a subfunctor of $\mathcal{B}$ and $\mathcal{B}$ is a subfunctor of $\mathcal{A}$ (where $\mathcal{S}$ is the category of sets and mappings). A full characterization of Cantor-Bernstein categories is given. Related problems are considered.

Key-words:

transformations, functors, Cantor-Bernstein theorem.

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The present paper brings a categorial generalization of a classical theorem of Cantor-Bernstein. We recall that the Cantor-Bernstein theorem says: if there exists an injection from a set A to a set B and an injection from B to A, then there exists a bijection between A and B. We consider an analogous question for functors to the category $\mathcal{S}$ of all sets and mappings in the following way: we call a category $\mathbb{K}$ a Cantor-Bernstein category provided the following holds: if there exists a monotransformation from a functor $\mathcal{A} : \mathbb{K} \longrightarrow \mathcal{S}$ to a functor $\mathcal{B} : \mathbb{K} \longrightarrow \mathcal{S}$ and a monostransformation from $\mathcal{B}$ to $\mathcal{A}$, then $\mathcal{A}$ and $\mathcal{B}$ are naturally equivalent.

So, the Cantor-Bernstein theorem says that the category with exactly one morphism is a Cantor-Bernstein category.

Analogously, we can define a Banach category, using the Banach's generalization of the mentioned Cantor-Bernstein theorem [1], and a Tarski-Knaster category, based on another generalization by Tarski and Knaster [2, 3]. In the present note we prove that all these definitions are equivalent and give a full characterization of the Cantor-Bernstein categories: they coincide precisely with the Brandt categories. (We recall that a category is said to be a Brandt category if each its morphism is an isomorphism.) A further discussion of the question is sketched at the end.

I. Now, we recall here the mentioned Banach's generalization and the generalization by Tarski and Knaster:

<u>Theorem (Banach)</u>. Let A, B be sets, $f : A \to B$, $g : B \longrightarrow A$ be mappings. Then there exist the sets A_f, A_g, B_f, B_g such that

1) $A_f \cup A_g = A$, $A_f \cap A_g = \emptyset$, $B_f \cup B_g = B$, $B_f \cap B_g = \emptyset$,

2) $f(A_f) = B_f$, $g(B_g) = A_g$.

<u>Theorem (Tarski, Knaster)</u>. Let A, B be two arbitrary sets, $A_0 \subset A$, $B_0 \subset B$ and let $f : A_0 \longrightarrow B$ and $g :$ $: B_0 \longrightarrow A$. Then there exist the sets A_f, A_g, B_f, B_g such that

1) $A_f \cup A_g = A$, $B_f \cup B_g = B$, $A_f \cap A_g = \emptyset$, $B_f \cap B_g = \emptyset$,

2) $f^{-1}(B_f) = A_f$, $g^{-1}(A_g) = B_g$.

<u>Convention</u>: If $\mathbb{K}$ is a category then the class of all its objects (or morphisms) is denoted by $\mathbb{K}^{\sigma}$ (or $\mathbb{K}^{m}$, respectively). The identity-morphism on $\sigma \in \mathbb{K}^{\sigma}$ is denoted by 1_{σ} or only 1. We use the symbols $\cup, \cap, \subset$ also for functors from $\mathbb{K}$ to $\mathbb{S}$. So, if $\mathcal{A}, \mathcal{B} : \mathbb{K} \longrightarrow \mathbb{S}$ are functors then $\mathcal{A} \subset \mathcal{B}$ denotes $\mathcal{A}(\sigma) \subset \mathcal{B}(\sigma)$ for every $\sigma \in \mathbb{K}^{\sigma}$ and $\mathcal{A}(\varphi)$ is a domain-range-restriction of $\mathcal{B}(\varphi)$ for every $\varphi \in \mathbb{K}^{m}$. If $f : \mathcal{A} \longrightarrow \mathcal{B}$ is a transformation, $\mathcal{A}' \subset \mathcal{A}$, then by $f(\mathcal{A}')$ the subfunctor of $\mathcal{B}$ is denoted, such that $[f(\mathcal{A}')](\sigma) = f(\mathcal{A}'(\sigma))$.
$C_0 : \mathbb{K} \longrightarrow \mathbb{S}$ denotes the trivial functor, i.e. $C_0(\sigma) = \emptyset$ for all $\sigma \in \mathbb{K}^{\sigma}$.

II. The definition of the Cantor-Bernstein category was given in the introduction.

Analogously, we say that a category $\mathbb{K}$ is a <u>Banach category</u> (or a <u>Tarski-Knaster category</u>) if the following is fulfilled: if $\mathcal{A}, \mathcal{B} : \mathbb{K} \longrightarrow \mathbb{S}$ are functors, $f : \mathcal{A} \longrightarrow \mathcal{B}$, $g : \mathcal{B} \longrightarrow \mathcal{A}$ transformations (or $f : \mathcal{A}_0 \longrightarrow \mathcal{B}$, $g : \mathcal{B}_0 \longrightarrow \mathcal{A}$ are transformations, where $\mathcal{A}_0 \subset \mathcal{A}$, $\mathcal{B}_0 \subset \mathcal{B}$), then there exists exactly one quadruple $(\mathcal{A}_f, \mathcal{A}_g, \mathcal{B}_f, \mathcal{B}_g)$ of functors such that

1) $\mathcal{A}_f \cup \mathcal{A}_g = \mathcal{A}$, $\mathcal{A}_f \cap \mathcal{A}_g = C_0$; $\mathcal{B}_f \cup \mathcal{B}_g = \mathcal{B}$, $\mathcal{B}_f \cap \mathcal{B}_g = C_0$,

2) $f(\mathcal{A}_f) = \mathcal{B}_f$, $g(\mathcal{B}_g) = \mathcal{A}_g$.

(or

2') $f^{-1}(\mathcal{B}_f) = \mathcal{A}_f$, $g^{-1}(\mathcal{A}_g) = \mathcal{B}_g$).

3) If $(\mathcal{A}'_f, \mathcal{A}'_g, \mathcal{B}'_f, \mathcal{B}'_g)$ also satisfies 1), 2) (or 1), 2'), respectively), then $\mathcal{A}_f \supset \mathcal{A}'_f$.

<u>Theorem</u>. The following properties of a category $\mathbb{K}$ are equivalent:

(i) $\mathbb{K}$ is a Cantor-Bernstein category;

(ii) $\mathbb{K}$ is a Banach category;

(iii) $\mathbb{K}$ is a Tarski-Knaster category;

(iv) $\mathbb{K}$ is a Brandt category.

The implications (ii) $\Longrightarrow$ (i) and (iii) $\Longrightarrow$ (i) are evident, the other implications will be proved in the next section III.

III. <u>Lemma 1</u>. Every Brandt category is a Banach category as well as a Tarski-Knaster category.

Proof: it is only a modification of that of the Banach theorem or the Tarski-Knaster theorem.

Let $\mathbb{K}$ be a category, $\mathcal{A}, \mathcal{B} : \mathbb{K} \longrightarrow \mathcal{S}$ functors, $f : \mathcal{A} \longrightarrow \mathcal{B}$, $g : \mathcal{B} \longrightarrow \mathcal{A}$ transformations (or $f : \mathcal{A}_0 \longrightarrow \mathcal{B}$, $g : \mathcal{B}_0 \longrightarrow \mathcal{A}$ where $\mathcal{A}_0 \subset \mathcal{A}$, $\mathcal{B}_0 \subset \mathcal{B}$). Let $f = \{ f_\sigma ; \sigma \in \mathbb{K}^\sigma \}$, $g = \{ g^\sigma ; \sigma \in \mathbb{K}^\sigma \}$. Denote $\mathcal{A}(\sigma) = A_\sigma$, $\mathcal{B}(\sigma) = B_\sigma$. Put $A^0_\sigma = A_\sigma$; $B^0_\sigma = B_\sigma - f_\sigma(A^0_\sigma)$;

$$A^i_\sigma = A_\sigma - \bigcup_{j<i} g_\sigma(B^j_\sigma); \; B^i_\sigma = B_\sigma - f_\sigma(A^i_\sigma) ; \; \sigma \in \mathbb{K}^\sigma .$$

(or $B^0_\sigma = B_\sigma$; $A^0_\sigma = A_\sigma - f^{-1}_\sigma(B^0_\sigma)$;

$$B^i_\sigma = B_\sigma - \bigcup_{j<i} g^{-1}_\sigma(A^j_\sigma); \; A^i_\sigma = A_\sigma - f^{-1}_\sigma(B^i_\sigma); \; \sigma \in \mathbb{K}^\sigma).$$

If $\varphi \in \mathbb{K}^m$, $\varphi : \sigma \to \sigma'$ then $\mathcal{A}(\varphi) : \mathcal{A}_\sigma \to \mathcal{A}_{\sigma'}$; $\mathcal{B}(\varphi) : B_\sigma \to B_{\sigma'}$ are bijections commuting with f_σ , $f_{\sigma'}$ and g_σ , $g_{\sigma'}$. By transfinite induction, this implies $[\mathcal{B}(\varphi)](\mathcal{B}^i_\sigma) = B^i_{\sigma'}$; $[\mathcal{A}(\varphi)](A^i_\sigma) = A^i_{\sigma'}$ for all i . So, we may define:

$$\mathcal{A}_f(\sigma) = \bigcap_i A^i_\sigma,\ \mathcal{A}_g(\sigma) = A_\sigma - \mathcal{A}_f(\sigma),\ \mathcal{B}_g(\sigma) = \bigcup_i B^i_\sigma,\ \mathcal{B}_f(\sigma) = B_\sigma - \mathcal{B}_g(\sigma)$$

(or $\mathcal{A}_g(\sigma) = \bigcup_i A^i_\sigma,\ \mathcal{A}_f(\sigma) = A_\sigma - A_g(\sigma),\ \mathcal{B}_f(\sigma) = \bigcap_i B^i_\sigma,\ \mathcal{B}_g(\sigma) = B_\sigma - \mathcal{B}_f(\sigma)$

respectively).

<u>Convention</u>. Let $\nu \cdot \mu = 1$, where μ, ν are morphisms of a category $\mathbb{K}$ (or transformations, or functors, respectively). Then ν is called a retraction, μ a coretraction.

<u>Lemma 2</u>. Let $\mathbb{K}$ be a category that is not a Brandt category. Then there exists a morphism which is not a coretraction.

Proof: Let $\mu \in \mathbb{K}^m$ be a morphism that is not an isomorphism. If μ is a coretraction then $\nu \cdot \mu = 1$ for some $\nu \in \mathbb{K}^m$. Then ν is not a coretraction.

<u>Construction</u>. Let $\mathbb{K}$ be a category which is not a Brandt category, $\mu \in \mathbb{K}^m$, $\mu : a \to b$ be not a coretraction. Put $\mathcal{F}^n = \bigcup_{i=1}^{n} (\mathbb{K}(a,-) \times \{i\})$ where by $\mathbb{K}(a,-)$ we denote the covariant homfunctor from a . Define a factorfunctor $\mathcal{G}^n$ of $\mathcal{F}^n$ by the following equality:

$(\varphi, i) = (\varphi', i') \Longleftrightarrow$ either $\varphi = \varphi', i = i'$ or $\varphi = \varphi'$ and φ is not a coretraction.

Put $\mathcal{A} = \bigvee_{n=1}^{\infty} \mathcal{Y}^n$, $\mathcal{B} = \bigvee_{n=2}^{\infty} \mathcal{Y}^n$. So, for any $c \in \mathbb{K}^\sigma$ we may suppose that $\mathcal{A}(c)$ (or $\mathcal{B}(c)$) is the set of all triples (φ, i, n) where $\varphi \in \mathbb{K}(a, c)$, $n = 1, 2, \dots$ (or $n = 2, 3, \dots$, respectively), $i = 1, 2, \dots, n$ and that $(\varphi, i, n) = (\varphi', i', n')$ iff $n = n'$ and either $\varphi = \varphi'$, $i = i'$ or $\varphi = \varphi'$, φ is not a coretraction.

Lemma 3. There exist monotransformations $f : \mathcal{A} \longrightarrow \mathcal{B}$, $g : \mathcal{B} \longrightarrow \mathcal{A}$.

Proof: $g : \mathcal{B} \longrightarrow \mathcal{A}$ can be chosen as an inclusion; $f(\varphi, i, n) = (\varphi, i, n+1)$.

Lemma 4. The functors $\mathcal{A}$, $\mathcal{B}$ are not naturally equivalent.

Proof: Suppose that there exists an isotransformation $h : \mathcal{A} \longrightarrow \mathcal{B}$. Put $h_a(1_a, 1, 1) = (\varphi, i, j)$.

a) Let φ be not a coretraction: Find $(\chi, k, l) \in \mathcal{A}(a)$ with $h_a(\chi, k, l) = (1_a, i, j)$. Then necessarily $l \neq 1$ because $h_a(\psi, 1, 1) = (\psi \circ \varphi, i, j) \neq (1_a, i, j)$ for all $\psi : a \longrightarrow a$. But then $h_a(\varphi \circ \chi, k, l) = (\varphi, i, j)$ which is a contradiction.

b) Let φ be a coretraction: Choose ψ with $\psi \circ \varphi = 1_a$. Then $h_a(\psi, 1, 1) = (1_a, i, j)$, consequently $h_a(\varphi \circ \psi, 1, 1) = (\varphi, i, j)$. Since h_a is a bijection, then necessarily $\varphi \circ \psi = 1_a$, φ is an isomorphism, $\psi = \varphi^{-1}$. Since $j \neq 1$, one can choose $i' \in \{1, \dots, j\}$, $i' \neq i$. The construction of $\mathcal{A}$, $\mathcal{B}$ implies $(\varphi, i', j) \neq$ $\neq (\lambda, i, j)$ for all $\lambda : a \longrightarrow a$. So, if we find

$(\chi, k, l) \in \mathcal{A}_a$ with $h_a(\chi, k, l) = (\varphi, i', j)$,
Then necessarily $l \neq 1$. But
$h_b(\mu \circ \psi \circ \chi, k, l) = (\mu, i', j) = (\mu, i, j) = h_b(\mu \circ \psi, 1, 1)$,
which is a contradiction.

Now, the proof of the theorem in II follows easily from the above lemmas and the construction.

IV. We can proceed analogously, when considering epitransformations, retractions and coretractions instead of monotransformations in the definition of the Cantor-Bernstein category.

Definition. A category $\mathbb{K}$ is called c-category (or r-category, or e-category) if the following is fulfilled: if $\mathcal{A}, \mathcal{B} : \mathbb{K} \longrightarrow \mathbb{S}$ are functors such that there exists a coretraction (or a retraction or an epitransformation, respectively) from $\mathcal{A}$ to $\mathcal{B}$ and another one from $\mathcal{B}$ to $\mathcal{A}$ then $\mathcal{A}$ and $\mathcal{B}$ are naturally equivalent.

While the c-categories as well as the r-categories coincide with the Brandt categories, the e-categories do not. The following theorem may be proved:

Theorem. A category $\mathbb{K}$ is an e-category iff it is a thin Brandt category.
(A category $\mathbb{K}$ is called thin if there exists at most one morphism from σ to σ', $\sigma' \in \mathbb{K}^{\sigma}$ arbitrary.)

The proof is omitted.

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