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ON SET TIGHTNESS and T-TIGHTNESS

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Abstract: The main purpose of this note is to study the behaviour of the set tightness and the T-tightness under maps and products. A particular result is the following: if X is a compact space and Y a Hausdorff space then $t_s(X\times Y)\le t(X)t_s(Y)$ and $T(X\times Y)\le T(X)(T(Y)$. Finally a little bit refined version of two results in Juhász's first book concerning the depth of a topological space is given.

Key words: Set tightness, T-tightness, weak Lindelöf number.

Classification: Primary 54A25
Secondary 54D20

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0. Introduction. Recently two new cardinal functions, closely related to the tightness have been introduced: the set tightness $t_s(X)$, by Arhangel'skii, Isler and Tironi (see [1]), and the T -tightness T(X), by Juhász (see [8]). In this paper we study the behaviour of these two cardinal functions under some topological operations, in particular the product operation. In the last section we will make some remarks on the depth of a topological space. The author wishes to thank the referee for his useful comments.

1. Some preliminaries. For notation and definitions not explicitly mentioned here we refer to [6] and [7]. m,ρ will denote cardinal numbers and α,α_0 ordinal numbers. A cardinal number is an initial ordinal. For any set S , $|S|$ denotes its cardinality and exp(S) (respectively $\exp_m(S)$) the set of all subsets (respectively the set of all subsets of size at most m) of S. For any family $\gamma\subset\exp(S)$ we briefly write $\bigcup\gamma$ (respectively $\bigcap\gamma$) for the union (respectively the intersection) of all members of γ.

Every topological space is assumed to be T_1 and every map continuous and surjective. Compact means compact Hausdorff. If A is a subset of a topological space X we denote its closure by cl(A) or sometimes $cl_X(A)$. If $\gamma \subset \exp(X)$, $cl(\gamma)$ denotes the family $\{cl(B) \mid B \in \gamma\}$.

We recall the following

Definition 1.1. If A is a subset of a topological space X, the tightness of A with respect to X is the cardinal number $t(A,X)=\min\{m \mid \forall C \subset X$ such that $A \cap cl(C) \neq \emptyset$ there is $C_0 \in \exp_m(C)$ with the property that $A \cap cl(C_0) \neq \emptyset\}$.

If $A=\{x\}$ we briefly write $t(x,X)$ instead of $t(\{x\},X)$.

The tightness of X is defined as $t(X)=\sup_{x \in X} t(x,X)$.

Definition 1.2 (see [8]). For any topological space X the T-tightness of X, denoted by T(X), is the smallest cardinal number m such that whenever $\{F_\alpha\}_{\alpha \in \wp}$ is an increasing sequence of closed subsets of X and $cf(\wp) > m$ also $\bigcup_{\alpha \in \wp} F_\alpha$ is closed.

It is clear that $T(X) \leq t(X)$.

In [8] there is proved the following

Proposition 1.3. a) If X is a compact space then $t(X)=T(X)$; b) if for a space X, t(X) is a successor cardinal then $t(X)=T(X)$.

Definition 1.4. Let X be a topological space, the set tightness at a point $x \in X$, denoted by $t_s(x,X)$, is the smallest cardinal number m such that whenever $x \in cl(C) \setminus C$, where $C \subset X$, then there is a family $\gamma \in \exp_m(\exp(C))$ such that $x \notin \bigcup cl(\gamma)$ but $x \in cl(\bigcup \gamma)$. The set tightness of X is defined as $t_s(X)=\sup_{x \in X} t_s(x,X)$.

It is clear that $t_s(x,X) \leq t(x,X)$ and $t_s(X) \leq t(X)$.

The next two propositions are two typical results concerning the set tightness.

Proposition 1.5 (see [1 , prop. 2.2]). If X is a Hausdorff space then $t_s(X) \leq s(X)$, where s(X) is the spread of X.

Proposition 1.6 (see [3, thm. 5]). If X is a regular space, then $t_s(X) \leq F(X)$, where $F(X)=\sup\{m \mid$ there exists in X a free sequence of length m .

Remark 1. The notion of set tightness was first introduced by Arhangel'skii, Isler and Tironi in [1]. They called it quasi-character and studied several properties of this cardinal function, particularly in the realm of pseudo-radial spaces.

We now introduce the following:

Definition 1.7. Let A be a subset of a topological space X. The set tightness of A with respect to X, denoted by $t_s(A,X)$, is the smallest cardinal number m such that for any set $C\subset X$ satisfying $A\cap C=\emptyset$ and $A\cap cl(C)\neq\emptyset$ there exists a family $\gamma\in \exp_m(\exp(C))$ with the property that $A\cap(\cup cl\,\gamma)=\emptyset$ but $A\cap cl(\cup\gamma)\neq\emptyset$.

If A is open we put $t_s(A,X)=1$. It is clear that in the other cases $t_s(A,X)\geq\aleph_0$.

2. On set tightness

Theorem 2.1. Let A_1, A_2 be two subsets of a topological space X. If $A_1\subset A_2$ and for any set $F\subset A_2\setminus A_1$ that is closed in A_2, there exist two disjoint open sets in X containing respectively A_1 and F, then $t_s(A_1,X)\leq t(A_1,A_2)t_s(A_2,X)$.

Proof. If A_1 is open then the theorem is trivial. Thus we can assume that A_1 is not open. Let $m=t(A_1,A_2)t_s(A_2,X)$. We need to show that for any set $C\subset X$ satisfying $A_1\cap C=\emptyset$ and $A_1\cap cl(C)\neq\emptyset$ there exists $\gamma\in\exp_m(\exp(C))$ such that

$$(*)\qquad A_1\cap(\cup cl\,\gamma)=\emptyset \text{ but } A_1\cap cl(\cup\gamma)\neq\emptyset .$$

Let us fix C and observe that since $A_1\cap cl(C)\neq\emptyset$ either $A_1\cap cl(A_2\cap C)\neq\emptyset$ or $A_1\cap cl(C\setminus A_2)\neq\emptyset$. In the first case from $t(A_1,A_2)\leq m$ and $A_1\cap cl_{A_2}(A_2\cap C)=A_1\cap cl_X(A_2\cap C)\neq\emptyset$ there exists $C_0\in\exp_m(A_2\cap C)$ such that $A_1\cap cl(C_0)\neq\emptyset$. It is obvious that the family γ of all singletons of C_0 has property $(*)$.

Consider now the second case and let $C'=C\setminus A_2$.

We have $A_1\cap cl(C')\neq\emptyset$ and $A_2\cap C'=\emptyset$. Since $t_s(A_2,X)\leq m$ there exists at least a family $\gamma'\in\exp_m(\exp(C'))$ such that $A_2\cap(\cup cl(\gamma'))=\emptyset$ but $A_2\cap cl(\cup\gamma')\neq\emptyset$. Let Γ be the set of all such γ' and $Z=\bigcup_{\gamma'\in\Gamma}(A_2\cap cl(\cup\gamma'))$. Suppose first that $A_1\cap cl(Z)\neq\emptyset$. Since $t(A_1,A_2)\leq m$ there exists $Z_0\in\exp_m(Z)$ such that $A_1\cap cl(Z_0)\neq\emptyset$. For every $z\in Z_0$ choose $\gamma_z\in\Gamma$ such that $z\in cl(\cup\gamma_z)$ and put $\gamma=\bigcup_{z\in Z_0}\gamma_z$. It is clear that $\gamma\in\exp_m(\exp(C))$ and $A_2\cap(\cup cl\,\gamma)=\emptyset$, so a fortiori $A_1\cap(\cup cl\,\gamma)=\emptyset$. In order to prove that γ satisfies property $(*)$ it remains to show that $A_1\cap cl(\cup\gamma)\neq\emptyset$, but this follows easily because $cl(Z_0)\subset cl(\cup\gamma)$ and $A_1\cap cl(Z_0)\neq\emptyset$. To conclude the proof of the theorem it suffices to show the case $A_1\cap cl(Z)=\emptyset$ cannot occur. On the contrary assume $A_1\cap cl(Z)=\emptyset$

and choose a set U_1 open in X, such that $A_1 \subset U$ and $cl(U) \cap Z = \emptyset$. Since $A_1 \cap cl(C') \neq \emptyset$ we have $A_1 \cap cl(U \cap C') \neq \emptyset$ and so a fortiori $A_2 \cap cl(U \cap C') \neq \emptyset$. By virtue of the inequality $t_s(A_2, X) \leq m$ there exists a family $\tilde{\gamma} \in \exp_m(\exp(U \cap C'))$ such that $A_2 \cap (\bigcup cl\,\gamma) = \emptyset$ but $A_2 \cap cl(\bigcup \gamma) \neq \emptyset$. Since $\exp_m(\exp(U \cap C') \subset \exp_m(\exp(C'))$, $\tilde{\gamma}$ belongs to the set Γ defined above. Therefore we must have $Z \cap cl(\bigcup \tilde{\gamma}) \neq \emptyset$, but all members of $\tilde{\gamma}$ are contained in U and hence $cl(\bigcup \tilde{\gamma}) \subset cl(U)$. This is a contradiction because $cl(U) \cap Z = \emptyset$ and the proof is complete.

Corollary 2.2. If F_1, F_2 are two compact subspaces of a Hausdorff space X such that $F_1 \subset F_2$ then $t_s(F_1, X) \leq t(F_1, F_2) t_s(F_2, X)$.

Corollary 2.3. If F is a closed subset of a regular space X then, for any $x \in X$, $t_s(x, X) \leq t(x, F) t_s(F, X)$.

Now we derive from thm. 2.1 some information on the behaviour of the set tightness under maps and products.

Lemma 2.4. Let X, Y be topological spaces. If the map $f: X \to Y$ is closed then $t_s(f^{-1}(y), X) \leq t_s(y, Y)$, $\forall y \in Y$.

Proof. Let $m = t_s(y, Y)$ and $C \subset X$ satisfying $f^{-1}(y) \cap C = \emptyset$ and $f^{-1}(y) \cap cl(C) \neq \emptyset$. Since $y \in cl(f(C)) \setminus f(C)$ and $t_s(y, Y) \leq m$ there exists $\gamma' \in \exp_m(\exp(f(C))$ such that $y \notin \bigcup(cl\,\gamma')$ but $y \in cl(\bigcup \gamma')$. For every $S' \in \gamma'$ choose a set $S \subset C$ such that $f(S) = S'$ and let γ be the family so obtained. It is clear that $\gamma \in \exp_m(\exp(C))$, $f^{-1}(y) \cap (\bigcup cl\,\gamma) = \emptyset$ and, thanks to the closedness of f, $f^{-1}(y) \cap cl(\bigcup \gamma) \neq \emptyset$.

Theorem 2.5. Let X be a regular space and Y a topological space. If $f: X \to Y$ is a closed map, $t_s(Y) \leq m$ and $t(f^{-1}(y)) \leq m$ for all $y \in Y$ then $t_s(X) \leq m$.

Proof. Let $x \in X$. By virtue of Corollary 2.3 we have $t_s(x, X) \leq t(x, f^{-1}(f(x))) t_s(f^{-1}(f(x)), X)$ and, by Lemma 2.4

$$t_s(x, X) \leq t(x, f^{-1}(f(x))) t_s(f(x), Y) \leq m.$$

If the map f in the above theorem is supposed to be perfect the assumption about the regularity of X can be weakened.

Theorem 2.6. Let X, Y be topological spaces, X Hausdorff. If $f: X \to Y$ is perfect, $t_s(Y) \leq m$ and $t(f^{-1}(y)) \leq m$ for all $y \in Y$ then $t_s(X) \leq m$.

Corollary 2.7. Let X, Y be topological spaces. If X is compact and Y Hausdorff then $t_s(X \times Y) \leq t(X) t_s(Y)$.

Proof. It follows directly from Theorem 2.6.

The above corollary can be improved as follows:

Theorem 2.8. Let X be a completely regular space and Y a Hausdorff space. If for every point $x \in X$ there exist a neighbourhood U of x and a compactification $C(U)$ of U such that $t(C(U)) \leq \leq m$, and moreover, $t_s(Y) \leq m$, then $t_s(X \times Y) \leq m$.

Corollary 2.9. If X is a locally compact space and Y a Hausdorff space then $t_s(X \times Y) \leq t(X) t_s(Y)$.

Theorem 2.10. Let $\{X_\alpha\}_{\alpha \in A}$ be a family of topological spaces. If $|A| \leq m$ and for every finite subset B of A, $t_s(\prod_{\alpha \in B} X_\alpha) \leq m$ then $t_s(\prod_{\alpha \in A} X_\alpha) \leq m$.

Proof. Let $X = \prod_{\alpha \in A} X_\alpha$, A^* the set of all finite subsets of A, $X_B = \prod_{\alpha \in B} X_\alpha$ for every $B \in A^*$ and π_B the natural projection from X to X_B. Let C be a non closed subset of X and $x \in cl(C) \setminus C$.

Let $A_x^* = \{B \mid B \in A^*$ and $\pi_B(x) \notin \pi_B(C)\}$. For any $B \in A_x^*$ there exists a family $\gamma'_B \in \exp_m(\exp(\pi_B(C)))$ such that $\pi_B(x) \notin \notin \bigcup cl(\gamma'_B)$ but $\pi_B(x) \in cl(\bigcup \gamma'_B)$. For any $S' \in \gamma'_B$ choose $S \subset C \setminus$ such that $\pi_B(S) = S'$ and let γ_B be the family so obtained. Now, for any $B \in A^* \setminus A_x^*$, choose an element $x_B \in C$ such that $\pi_B(x_B) = = \pi_B(x)$. Since $|A^*| \leq m$ then the family $\gamma = \{\bigcup_{B \in A_x^*} \gamma_B\} \cup \{\{x_B\} \mid B \in \in A^* \setminus A_x^*$ has cardinality at most m, so $\gamma \in \exp_m(\exp(C))$. From our construction it follows easily that $x \notin \bigcup cl(\gamma)$ but $x \in cl(\bigcup \gamma)$ and this completes the proof.

To conclude this section we give some theorems dealing with the set tightness of the image of a space under a closed map.

Definition 2.11. A topological space X is said to be scattered iff it has no dense in itself subspace, i.e., every subspace of X has an isolated point.

Theorem 2.12. Let X be a regular space and Y a topological space. If $f: X \to Y$ is a closed map with scattered fibres then $t_s(Y) \leq t_s(X)$.

Proof. Let $m = t_s(X)$, $C \subset Y$ and $y \in cl(C) \setminus C$. By virtue of the closedness of f, $f^{-1}(y) \cap cl(f^{-1}(C)) \neq \emptyset$. Let x be an isolated point of $f^{-1}(y) \cap cl(f^{-1}(C))$. Clearly $x \in cl(f^{-1}(C)) \setminus f^{-1}(C)$. Since $t_s(X) \leq m$ then there exists a family $\gamma \in \exp_m(\exp(f^{-1}(C)))$ such that $x \notin \bigcup \mathrm{cl}(\gamma)$ but $x \in \mathrm{cl}(\bigcup\gamma)$.

The set $[f^{-1}(y) \cap \mathrm{cl}(\bigcup\gamma)] \setminus \{x\}$, is closed in $f^{-1}(y)$ and hence in X. By the regularity of X there exists a closed neighbourhood U of x such that $U \cap \{[f^{-1}(y) \cap \mathrm{cl}(\bigcup\gamma)] \setminus \{x\}\} = \emptyset$.

Let $\gamma' = \{B \cap U | B \in \gamma\}$. It is clear that $x \in \mathrm{cl}(\bigcup\gamma')$ and $f^{-1}(y) \cap (\bigcup \mathrm{cl}(\gamma')) = \emptyset$. Let $\gamma'' = \{f(B) | B \in \gamma'\}$. Thanks to the closedness we have $y \notin \bigcup \mathrm{cl}(\gamma'')$ but clearly $y \in \mathrm{cl}(\bigcup\gamma'')$ and this concludes the proof since $\gamma'' \in \exp_m(\exp(C))$.

Corollary 2.13. Let X be a regular space and Y a topological space. If $f: X \rightarrow Y$ is a closed map and each of its fibres is a scattered space of countable tightness then $t_s(Y) = t_s(X)$.

Corollary 2.14. Let X be a regular space and Y a topological space. If $f: X \rightarrow Y$ is a closed map with discrete fibres then $t_s(Y) = t_s(X)$.

Corollary 2.15. Let X be a Hausdorff space. If there exists a locally finite closed cover $\mathcal{F}(X)$ such that $t_s(F) \leq m$ for all $F \in \mathcal{F}$ then $t_s(X) \leq m$.

Proof. Let $\oplus\mathcal{F}$ be the topological sum of the spaces belonging to $\mathcal{F}$, and let $f: \oplus\mathcal{F} \rightarrow X$ be the natural map. Now we can proceed as in Theorem 2.12 - just by observing that in this case since the fibres of f are finite, it is sufficient to assume the space X Hausdorff.

Question 2.16. In the statement of Theorem 2.1 is it possible to replace $t(A_1, A_2)$ with $t_s(A_1, A_2)$? In particular, if X is a compact space and Y a Hausdorff space, is it true that $t_s(X \times Y) \leq t_s(X) t_s(Y)$?

Question 2.17. Let $f: X \rightarrow Y$ be a closed map, is it true that $t_s(Y) \leq t_s(X)$?

3. On T-tightness

Theorem 3.1. Let X, Y be topological spaces. If $f: X \rightarrow Y$ is a quotient map then $T(Y) \leq T(X)$.

The proof is straightforward.

Theorem 3.2. Let X be a regular space and Y a topological space. If $f: X \rightarrow Y$ is a closed map, $T(Y) \leq m$ and $T(f^{-1}(y)) \leq m$ for all $y\in Y$ then $T(X)\leq m$.

Proof. Let $\{F_\alpha\}_{\alpha\in\rho}$ be an increasing sequence of closed subsets of X such that $cf(\rho)>m$. Put $F=\bigcup_{\alpha\in\rho}F_\alpha$ and assume that there exists a point $x\in cl(F)\setminus F$.

Let $F'_\alpha=f^{-1}(f(x))\cap F_\alpha$, $\forall\alpha\in\rho$; the family $\{F'_\alpha\}_{\alpha\in\rho}$ is an increasing sequence of closed subsets of $f^{-1}(f(x))$. Since $T(f^{-1}(f(x)))\leq m$ then the set $\bigcup_{\alpha\in\rho}F'_\alpha=F\cap f^{-1}(f(x))$ is closed in the subspace $f^{-1}(f(x))$ and hence in X.

By regularity of X, there exist two disjoint sets U and V, open in X, such that $x\in U$ and $F\cap f^{-1}(f(x))\subset V$.

Let $F''_\alpha=F_\alpha\setminus V$ $\forall\alpha\in\rho$. It is clear that $x\in cl(\bigcup_{\alpha\in\rho}F''_\alpha)$ and $f^{-1}(f(x))\cap(\bigcup_{\alpha\in\rho}F''_\alpha)=\emptyset$. The family $\{f(F''_\alpha)\}_{\alpha\in\rho}$ is an increasing sequence of closed subsets of Y. Since $T(Y)\leq m$ then the set $\bigcup_{\alpha\in\rho}f(F''_\alpha)=f(\bigcup_{\alpha\in\rho}F''_\alpha)$ must be closed and, by the continuity of f, $f(x)\in cl(f(\bigcup_{\alpha\in\rho}F''_\alpha))=f(\bigcup_{\alpha\in\rho}F''_\alpha)$, but this is impossible because $f^{-1}(f(x))\cap(\bigcup_{\alpha\in\rho}F''_\alpha)=\emptyset$.

This proves that F is closed and so $T(X)\leq m$.

If in the above theorem the map f is supposed to be perfect the assumption about the regularity of X can be weakened.

<u>Theorem 3.3</u>. Let X, Y be topological spaces, X Hausdorff. If $f:X\to Y$ is a perfect map, $T(Y)\leq m$, and $T(f^{-1}(y))\leq m$ $\forall y\in Y$ then $T(X)\leq m$.

<u>Corollary 3.4</u>. Let X, Y be topological spaces. If X is compact and Y Hausdorff then $T(X\times Y)\leq T(X)T(Y)$.

As in Theorem 2.8 we can improve the preceeding result as follows:

<u>Theorem 3.5</u>. Let X be a completely regular space and Y a Hausdorff space. If for every $x\in X$ there exist a neighbourhood U of x and a compactification C(U) of U such that $T(C(U))\leq m$, and moreover $T(Y)\leq m$ then $T(X\times Y)\leq m$.

<u>Corollary 3.6</u>. If X is a locally compact space and Y a Hausdorff space then $T(X\times Y)\leq T(X)T(Y)$.

<u>Theorem 3.7</u>. Let $\{X_\alpha\}_{\alpha\in A}$ be a family of topological spaces. If $|A|\leq m$ and for any finite set $B\subset A$, $T(\prod_{\alpha\in B}X_\alpha)\leq m$ then $T(\prod_{\alpha\in A}X_\alpha)\leq m$.

Proof. Let $X=\prod_{\alpha\in A} X_\alpha$, A^* the set of all finite subsets of A, $X_B=\prod_{\alpha\in B} X_\alpha$ and π_B the natural projection from X to X_B. Let $\{F_\alpha\}_{\alpha\in\rho}$ be an increasing sequence of closed subsets of X such that $cf(\rho)>m$. Let $x\in cl(F)$, where $F=\bigcup_{\alpha\in\rho} F_\alpha$. For every $B\in A^*$ the family $\{cl(\pi_B(F_\alpha))\}_{\alpha\in\rho}$ is an increasing sequence of closed subsets of X_B and $\pi_B(x)\in cl(\bigcup_{\alpha\in\rho} cl(\pi_B(F_\alpha)))$. Since $T(X_B)\leq m$ then there exists an index α_B such that $\pi_B(x)\in cl(\pi_B(F_{\alpha_B}))$. Since $|A^*|\leq m$ and $cf(\rho)>m$ then there exists an index α_0 such that $\alpha_B\in\alpha_0$ $\forall B\in A^*$. Now we have $\pi_B(x)\in cl(\pi_B(F_{\alpha_0}))$, $\forall B\in A^*$, and this clearly implies $x\in F_{\alpha_0}$. The last assertion shows that $x\in F$ and so F is closed.

4. Depth and T-tightness

Two well known cardinal inequalities involving the depth are the following (see [7, th.2.18 and 2.19]):

a) if X is a connected space then $k(X)\leq[\chi(X)]^+$,

b) if X is a topological space then $k(X)\leq L(X)t(X)$,

where $k(X)$, $\chi(X)$ and $L(X)$ denote respectively the depth, the character and the Lindelöf number of the space X.

The aim of this section is to give refined versions of the previous inequalities.

We recall the following:

Definition 4.1. Let X be a topological space. A family of subsets of X, $\{G_\alpha\}_{\alpha\in\rho}$, is said to be a strongly decreasing sequence of length ρ if $\overline{G}_\beta \subsetneqq G_\alpha$ for any $\alpha\in\beta\in\rho$. The depth of X, denoted by $k(X)$, is the supremum of the cardinal numbers ρ such that in X there exists a strongly decreasing sequence of open sets of length ρ.

Theorem 4.2. If X is a connected space then $k(X)\leq[T(X)]^+$.

Proof. Let $m=T(X)$, We need to show that every strongly decreasing sequence of open sets in X has length at most m^+. Assume the contrary and let ρ be a cardinal number such that $m^+<\rho$. Suppose there exists in X a strongly decreasing sequence of open sets of length ρ, say $\{G_\alpha\}_{\alpha\in\rho}$. The set $H=\bigcap_{\alpha\in m^+} G_\alpha=\bigcap_{\alpha\in m^+}\overline{G}_\alpha$ is closed and, moreover, it is nonempty because $G_{m^+} \subset H$. Observe that the family $\{X \setminus G_\alpha\}_{\alpha \in m^+}$ is an increasing chain of closed subsets of X. Since $T(X)=m$ and $cf(m^+)=m^+$, the set $\bigcup_{\alpha \in m^+}(X \setminus G_\alpha)=X \setminus H$ is closed. As the space is connected, this leads to a contradiction and so the theorem is proved.

Remark 2. The "long line" (see [7, example 6.4]) is an example of a connected space in which $k(X)=[T(X)]^+$.

Definition 4.3. A family $\{G_s\}_{s\in S}$ of subsets of a topological space X is said to be a weak cover for X provided that $X=\overline{\bigcup_{s\in S} G_s}$. The weak Lindelöf number of X, denoted by wL(X), is the smallest cardinal number m such that every open cover of X has a weak subcover of cardinality at most m.

It is clear that $wL(X) \leq L(X)$ for any space X. In fact, spaces can be found for which the gap between the weak Lindelöf number and the Lindelöf number is arbitrarily large (see examples in [2]).

Theorem 4.4. If X is any topological space then $k(X) \leq wL(X)T(X)$.

Proof. Let $m=wL(X)T(X)$ and assume there exists in X a strongly decreasing sequence of open sets of length m^+, say $\{G_\alpha\}_{\alpha \in m^+}$. Let $H=\bigcap_{\alpha \in m^+} G_\alpha = \bigcap_{\alpha \in m^+} \overline{G}_\alpha$. The family $\{X \setminus G_\alpha\}_{\alpha \in m^+}$ is an increasing sequence of closed subsets of X and, since $T(X) \leq m$ and $cf(m^+)=m^+$, the set $\bigcup_{\alpha \in m^+}(X \setminus G_\alpha)=X \setminus H$ is closed. Observe that the family $\{X \setminus \overline{G}_\alpha\}_{\alpha \in m^+}$ is an open cover of the subspace $X \setminus H$. As $X \setminus H$ is closed and open in X, we have $wL(X \setminus H) \leq wL(X) \leq m$ and then there exists a set of ordinal numbers $A \subset m^+$ such that $|A| \leq m$ and $X \setminus H = \overline{\bigcup_{\alpha \in A} (X \setminus \overline{G}_\alpha)}$. By virtue of the regularity of m^+ there exists an ordinal $\tilde{\alpha} \in m^+$ such that $A \subset \tilde{\alpha}$. We have $G_{\tilde{\alpha}} \subset \overline{G}_{\tilde{\alpha}} \subset G_\alpha$ for every $\alpha \in A$ and thus $\overline{\bigcup_{\alpha \in A}(X \setminus \overline{G}_\alpha)} \subset X \setminus G_{\tilde{\alpha}}$. This implies $X \setminus H \subset X \setminus G_{\tilde{\alpha}}$, i.e. $G_{\tilde{\alpha}} \subset H$. This is a contradiction because H is a proper subset of $G_{\tilde{\alpha}}$. Therefore the theorem is proved.

Corollary 4.5. If X is a topological space then $k(X) \leq L(X)T(X)$.

Remark 3. The above corollary xan be deduced from Theorem 1 in [5] in which it is proved that $F(X) \leq L(X)T(X)$, recalling that, for any space X, $k(X) \leq F(X)$.

Remark 4. Note that Theorem 4.4 fails if we replace k(X) with F(X) at least when X is not a normal space. In fact, the space Y constructed in [2, example 2.4] is completely regular and $wL(Y)=T(Y)=\aleph_0$. For any $q \in Q$ the set $\{q\} \times \varkappa$ is a closed discrete subspace of cardinality $\varkappa$ of Y, and, since a closed discrete subspace can be regarded as a free sequence, we have $F(Y)=\varkappa > \aleph_0$.

Question 4.6. Under which conditions other than paracompactness, does the inequality $F(X) \leq wL(X)T(X)$ hold?

Added in proof. In [4] a notion of local T-tightness is introduced and a result similar to Theorem 2.1 is proved.

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