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FUNCTIONS OF THE FORM $\sum_{i=1}^{N} f_i(x) g_i(t)$ IN L_2

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We write $\Delta = (\alpha, \beta) \times (\gamma, \delta) \subset \mathbf{R}^2$, the cases $\alpha = -\infty$, $\beta = \infty$, $\gamma = -\infty$, and $\delta = \infty$ are not excluded. Let

$$\mathbf{L}_2 = \{h : \Delta\mathbf{R}; \int_\Delta h^2 < \infty\}.$$

For a fixed positive integer N, let $P_N := \{k \in \mathbf{L}_2; k(x, t) = \sum_{i=1}^{N} f_i(x) g_i(t)\}$. In this paper we shall show that for any N, the functions of P_N do not provide a good approximation in $\mathbf{L}_2$.

Theorem

For every positive integer N and every positive ε, there exists $h \in \mathbf{L}_2$ such that

$$\| h - k \|_{L_2} > \varepsilon$$

for all $k \in P_N$.

The proof of the theorem will be based on the following observations.

Proposition 1. Let I and J be arbitrary subsets of $\mathbf{R}$. A function $h : I \times J \to \mathbf{R}$ can be written in the form

$$k(x, t) = \sum_{i=1}^{N} f_i(x) g_i(t),$$

with linearly independent f_i and g_i ($i = 1, \dots, N$) if and only if the maximum of the rank of the matrices

$$(k(x_i, t_j)), \qquad (i = 1, \dots, r; j = 1, \dots, s)$$

is N when $x_i \in I$, $t_j \in J$, r and s being arbitrary integers.

If the assumption is satisfied, then all such f_i and g_i can be constructed from k in the following way. Let

$$K := \begin{pmatrix} k(x_1, t_1), \dots, k(x_1, t_N) \\ \dots \\ k(x_N, t_1), \dots, k(x_N, t_N) \end{pmatrix}$$

be any (fixed) regular n by n matrix, C any regular n by n matrix. Then

$$(f_1(x), \dots, f_N(x)) = (k(x, t_1), \dots, k(x, t_N)) \,.\, C,$$

and

$$\begin{pmatrix} g_1(t) \\ \dots \\ g_N(t) \end{pmatrix} = C^{-1} \,.\, K^{-1} \,.\begin{pmatrix} k(x_1, t) \\ \dots \\ k(x_N, t) \end{pmatrix}$$

for all $x \in I$, $t \in J$.

Proof was given in [1].

Proposition 2. To each $c \in \mathbf{R}^+$ and $N \in \mathrm{N}$, there exists a constant N by N matrix $M = (m_{ij})$ and a $c_0 > 0$ such that, for every matrix $\bar{M} = (\bar{m}_{ij})$ satisfying

$$|\bar{m}_{ij} - m_{ij}| < c,$$

we have

$$\det \bar{M} > c_0 .$$

Proof. Take an N by N matrix $D = (d_{ij})$, $\det D = c_1 > 0$. Then, due to the continuous dependence of $\det D$ on d_{ij}, there exists an ε_0 such that $\det \bar{D} > c_1/2$, whenever $|\bar{d}_{ij} - d_{ij}| < \varepsilon_0$.

$$M = \left(\frac{c}{\varepsilon_0} d_{ij}\right)$$

satisfies our requirement, since, if $\bar{M} = (\bar{m}_{ij})$ and

$$\left|\bar{m}_{ij} - \frac{c}{\varepsilon_0} d_{ij}\right| < c,$$

then

$$\left|\frac{\varepsilon_0}{c} \bar{m}_{ij} - d_{ij}\right| < \varepsilon_0 .$$

Hence

$$\det\left(\frac{\varepsilon_0}{c} m_{ij}\right) > c_1/2, \qquad \text{or} \qquad \det(\bar{m}_{ij}) > \frac{c_1}{2} \cdot \left(\frac{c}{\varepsilon_0}\right)^N =: c_0 > 0, \qquad \text{q.e.d.}$$

Let μ denote the Lebesgue measure in $\mathbf{R}^2$, μ_1 be the Lebesgue measure in $\mathbf{R}$.

Proposition 3. Let S^* be a measurable subset of the square $(0, a)^2 \subset \mathbf{R}^2$, $\mu(S^*) > a^2 - \delta^2$, $0 < \delta < a$. Let $X = \{x \in \mathbf{R};\ \mu_1\{t; (x, t) \in S^*\} \geqq a - \delta\}$. Then

$$\mu_1(X) \geqq a - \delta.$$

Proof. Suppose $\mu_1(X) < a - \delta$. Then $\mu(S^*) \leqq \mu_1(X) \,.\, a + (a - \mu_1(X)) \,.\, (a - \delta) = a^2 - \delta(a - \mu_1(X)) \leqq a^2 - \delta^2$, which is a contradiction. Hence $\mu_1(X) \geqq a - \delta$.

Proposition 4. For any $\varepsilon_0 \in \mathbf{R}^+$ and $\delta^2 > 0$, there exists a c_0 such that, if h, $k \in \mathbf{L}_2$, $\operatorname{Dom} h = \operatorname{Dom} k \supset (0, a)^2$, and $\| h - k \| < \varepsilon_0$ then

$$\mu\{(x, t) \in (0, a)^2; \quad | h(x, t) - k(x, t) | < c_0\} > a^2 - \delta^2.$$

Proof. Let $c_0 = \varepsilon_0/\delta$. If the last relation is not satisfied, then

$$\mu\{(x, t) \in (0, a)^2; \quad | h(x, t) - k(x, t) | \geqq c_0\} \geqq \delta^2,$$

and

$$\| h - k \|_2 \geqq \sqrt{\{ \int_{(0,a)^2} (h - k)^2\}} \geqq \sqrt{\{\delta^2 \cdot c_0^2\}} = \varepsilon_0,$$

contrary to our assumption.

Now, we prove our theorem with given N and ε.

Without loss of generality, let $\Delta = [0, a(N + 1))^2$. For $1 \leqq i, j \leqq N + 1$, define

$$S_{ij} = \{(x, t); \quad a(i - 1) \leqq x < ai, a(j - 1) \leqq t < aj\}.$$

Using the determinant M from Proposition 2 for $c = \varepsilon/\delta = \varepsilon a/(2N)$, define h on $\Delta \backslash S_{N+1,N+1}$ by $h(x, t) = m_{ij}$ for $(x, t) \in S_{ij}$. Consider each $S_{ij} \subset \Delta \backslash S_{N+1,N+1}$ separately. Due to Proposition 4, there exists

$$S_{ij}^* \subset S_{ij}, \qquad \mu(S_{ij}^*) > a^2 - \frac{a^2}{(2N)^2}, \qquad \delta := \frac{a}{2N}$$

for $k \in \mathbf{L}_2$, $\| h - k \| < \varepsilon$, so that we have

$$| h - k | < c \qquad \text{on } S_{ij}^*.$$

Let

$$X_{ij} = \{x \in (a(i - 1), ai); \mu_1\{t; (x, t) \in S_{ij}^*\} > a - \delta\}$$

and

$$T_{ij} = \{t \in (a(j - 1), aj); \mu_1\{x; (x, t) \in S_{ij}^*\} > a - \delta\}$$

for all $1 \leqq i, j \leqq N + 1$, $S_{N+1,N+1}$ being $S_{N+1,N+1}^*$ for this definition. Since $\delta = a/(2N)$, for

$$X_i^* := \bigcap_{j=1}^{N+1} X_{ij}, \qquad T_j^* := \bigcap_{i=1}^{N+1} T_{ij},$$

we have, from Proposition 3 and de Morgan's rule,

$$\mu_1(X_i^*) > a/2 \qquad \text{and} \qquad \mu_1(T_j^*) > a/2, \qquad i, j \leqq N + 1.$$

Fix $x_i \in X_i^*$ for $1 \leqq i \leqq N$ and $t_j \in T_j^*$ for $1 \leqq j \leqq N$. Let $x \in X_{N+1}^*$, $t \in T_{N+1}^*$. Let $k \in \mathbf{L}_2$ be also of the form $k(x, t) = \sum_{i=1}^{N} f_i(x)\, g_i(t)$.

In accordance with Proposition 1,

$$k(x, t) = (k(x, t_1), \dots, k(x, t_N)) \, . \, K^{-1} \, . \begin{pmatrix} k(x_1, t) \\ \dots \\ k(x_N, t) \end{pmatrix}, \tag{1}$$

for $x \in X^*_{N+1}$, $t \in T^*_{N+1}$,

$$K = \begin{pmatrix} k(x_1, t_1), \dots, k(x_1, t_N) \\ \dots \\ k(x_N, t_1), \dots, k(x_N, t_N) \end{pmatrix}.$$

Since $x_i \in X^*_i$ and $t_j \in T^*_j$, we have

$$| \, h(x_i, t_j) - k(x_i, t_j) \, | < \varepsilon a/(2N).$$

Hence $\det K > c_0 > 0$.

We can conclude that $k(x, t)$, satisfying (1) on $X^*_{N+1} \times T^*_{N+1} \subset S_{N+1,N+1}$, is expressible as a polynomial of order $N + 2$ in $k(x_i, t_j)$, $k(x, t_j)$, $k(x_i, t)$ divided by $\det K > c_0 > 0$.

On $X^*_{N+1} \times T^*_{N+1}$ we also have

$$| \, k(x_i, t_j) - m_{ij} \, | < \varepsilon a/(2N),$$
$$| \, k(x, t_j) - m_{N+1,j} \, | < \varepsilon a/(2N)$$

and

$$| \, k(x_i, t) - m_{i,N+1} \, | < \varepsilon a/(2N).$$

Hence $k(x, t) \in \mathbf{L}_2$ is bounded on the set $\Delta^* := X^*_{N+1} \times T^*_{N+1}$ with $\mu(\Delta^*) \geqq$ $\geqq a^2/4 > 0$: $k(x, t) < L$, where L depends upon h on $\Delta \setminus S_{N+1,N+1}$ (i.e. upon m_{ij}, but not upon $m_{N+1,N+1}$), and upon ε and N. However, h is not defined on $S_{N+1,N+1}$ yet. If $h(x) := b$ here, b being a constant, $b > L + 4\varepsilon/a^2$, h remains in the class $\mathbf{L}_2$, nothing from our construction is changed and

$$\| \, h - k \, \|_{\mathbf{L}_2} \geqq \sqrt{\{\int_{\Delta^*} (h - k)^2\}} = \| \, h - k \, \|_{\mathbf{L}_2/\Delta^*} \geqq$$

$\geqq | \, \| \, h \, \|_{\mathbf{L}_2/\Delta^*} - \| \, k \, \|_{\mathbf{L}_2/\Delta^*} \, | \geqq | \, b\mu(\Delta^*) - L\mu(\Delta^*) \, | > \varepsilon$, a contradiction Q.E.D.

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