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A NOTE ON MEASURABLE FUNCTIONS

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Let S be a σ-algebra of subsets of X. A real-valued function f on X is called measurable (with respect to S) if $f^{-1}(B) \in S$ for every Borel subset of the line. There are at least two generalizations of the notion defined above. P. R. HALMOS in [1] assumes only that S is a σ-ring; S need not contain X. R. SIKORSKI in [4] assumes that $X \in S$ and S is a σ-lattice (of course, according to his terminology); S need not be a σ-ring. In the present paper we construct a more general theory of measurable functions containing both mentioned theories as special cases. In other words we omit in Sikorski's theory the assumption $X \in S$. The idea of producing such a theory is due to T. NEUBRUNN.

In Section 1 we give definitions and examples. In Section 2 we prove that the sum of two measurable functions is a measurable function. In Section 3 we prove that the limit of a sequence of measurable functions is a measurable function and in Section 4 we prove that any measurable function can be approximated by a simple measurable function.

In some theorems we consider functions $f : X \to Y$ where the range space Y is a more general space than the real line.

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P. R. Halmos assumes that S is a σ-ring (i.e. $\emptyset \in S$ and $E_n \in S$ for $n = 1, 2, \ldots$ implies $E_1 - E_2 \in S$, $\bigcup_{n=1}^{\infty} E_n \in S$); a function $f : X \to (-\infty, \infty)$ is called to be measurable if $f^{-1}(B) \cap N(f) \in S$ for every Borel set B, where $N(f) = \{x : f(x) \neq 0\}$.

R. Sikorski assumes that S is a σ-lattice (i.e. $S \neq \emptyset$; $E_n \in S$ for $n = 1, 2, \ldots$ implies $\bigcup_{n=1}^{\infty} E_n \in S$, $\bigcap_{n=1}^{t} E_n \in S$) and $X \in S$. A function $f : X \to (-\infty, \infty)$ is called to be measurable if $f^{-1}(B) \in S$ whenever B has either the form $(-\infty, c)$ or the form (c, ∞), c being any real number.

If S is a σ-lattice but we do not know whether $X \in S$ or not, we can formally use the Halmos' definition.

Definition 1. Let S be a σ-lattice. We shall denote by $\mathcal{M}_1$ the family of all functions $f : X \to (-\infty, \infty)$ such that $N(f) \cap f^{-1}(E) \in S$ whenever $E \in B$ where B is the family of all sets of the form $(-\infty, c)$ or (c, ∞) where c is any real number and $N(f) = \{x : f(x) \neq 0\}$.

It is well known that $\mathcal{M}_1$ coincides with the family of all functions measurable in the Halmos' sense if S is a σ-ring. If S is only a σ-lattice (X need not belong to S) we do not obtain a convenient theory. As M. OKLEŠTEKOVÁ-PLEŠKOVÁ showed in [3] the sum of two functions of $\mathcal{M}_1$ need not belong to $\mathcal{M}_1$.

If S is a σ-ring then the Halmos' definition of measurability is equivalent to the following definition: f is measurable if and only if the following two conditions are satisfied: 1. $N(f) \in S$. 2. $E \in S$, $F \in B \Rightarrow E \cap f^{-1}(F) \in S$. ($B$ has the same meaning as in Definition 1.) It seems that this property is more suitable to be used as a definition in the general case.

Definition 2. Denote by $\mathcal{M}_2$ the family of all functions $f : X \to (-\infty, \infty)$ satisfying the following two conditions:

1. $N(f) \in S$.
2. $E \in S$, $F \in B \Rightarrow E \cap f^{-1}(F) \in S$.

Evidently $\mathcal{M}_2 \subset \mathcal{M}_1$ and $\mathcal{M}_1 = \mathcal{M}_2$ if S is a σ-ring or $X \in S$. In the latter case $f \in \mathcal{M}_1$ if and only if $f^{-1}(E) \in S$ for every $E \in B$. The following proposition may be more interesting.

Proposition 1. *If S is closed under the countable unions and intersections then $\mathcal{M}_1 = \mathcal{M}_2$.*

Proof. Let $f \in \mathcal{M}_1$, $E \in S$, $F \in B$. Evidently $N(f) = N(f) \cap f^{-1}((-\infty, \infty)) = \bigcup_{n=1}^{\infty} (N(f) \cap f^{-1}((-\infty, n))) \in S$. Further

$$(E - N(f)) \cap f^{-1}(F) = E \cap \bigcap_{n=1}^{\infty} f^{-1}\left(F \cap \left(-\frac{1}{n}, \frac{1}{n}\right)\right) \in S,$$

therefore

$$E \cap f^{-1}(F) = (E \cap N(f) \cap f^{-1}(F)) \cup ((E - N(f)) \cap f^{-1}(F)) \in S.$$

If we obtain a suitable theory for $\mathcal{M}_2$ then we shall have a common generalization of the both theories mentioned above as well as a theory with respect to S closed under countable unions and intersections.

Of course, we can give a general definition including all the classical cases.

Definition 3. Let X, Y be arbitrary non-empty sets, S, T families of subsets of X, B a family of subsets of Y. Let N be a map $N : Y^X \to 2^X$ associating with any function $f : X \to Y$ a subset $N(f)$ of X. Then we say that a function f is measurable (with respect to S, T, B and N) if

1. $N(f) \in S$.
2. $E \cap f^{-1}(F) \in S$ for all $E \in T$, $F \in B$.

Denote the family of all measurable functions by $\mathcal{M}(S, T, B, N)$.

Examples. 1. $S = T$ is a σ-lattice, $Y = (-\infty, \infty)$, $B = \{(-\infty, c) : c$ real number$\} \cup \{(c, \infty) : c$ real number$\}$, $N(f) = \{x : f(x) \neq 0\}$. We obtain the system $\mathcal{M}_2$ including both Halmos' and Sikorski's theory.

2. We can obtain Sikorski's theory also in another way, if we put $N(f) = X$ for all $f : X \to Y$.

3. Let Y be a metric space, B a base of open sets in Y, S be a σ-ring of subsets of X, $N(f) = \emptyset$ for all $f : X \to Y$, $T = \{X\}$. f is measurable if $f^{-1}(E) \in S$ for all $E \in B$ (see [2]).

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In this section we assume that Y is an Abelian topological group satisfying the second axiom of countability.

Theorem 1. *Let S be a σ-lattice, B a base of open sets. Let $f, g : X \to Y$ be such functions that $f^{-1}(F), g^{-1}(F) \in S$ for all $F \in B$. Then also $(f + g)^{-1}(F) \in S$ for all $F \in B$.*

Proof. Let $\{V_n\}$ be a countable base consisting of elements of B. Let $U \in B$. We must prove $(f + g)^{-1}(U) \in S$. Put $\gamma = \{(m, n) : V_n + V_m \subset U\}$. First we prove

$$(f + g)^{-1}(U) = \bigcup_{(m,n)\in\gamma} f^{-1}(V_n) \cap g^{-1}(V_m)\,.$$

If $x \in f^{-1}(V_n) \cap g^{-1}(V_m)$, $(m, n) \in \gamma$, then $f(x) \in V_n$, $g(x) \in V_m$ and hence $f(x) + g(x) \in U$. Let $f(x) + g(x) \in U$. Then there are open sets V, W such that $f(x) \in V$, $g(x) \in W$, $V + W \subset U$. Take V_n, V_m from the base such that $f(x) \in V_n$, $g(x) \in V_m$ and $V_n \subset V$, $V_m \subset W$. Then $V_n + V_m \subset U$ and hence $(m, n) \in \gamma$.
Now we see that $(f + g)^{-1}(U) \in S$ for all $U \in B$ and Theorem 1 is proved.

Theorem 2. *Let S be a σ-lattice, B a base of open sets in Y. Let for any $f : X \to Y$ be $N(f) = \{x : f(x) \neq 0\}$. Then $\mathcal{M} = \mathcal{M}(S, S, B, N)$ is closed under the operation of addition, i.e. $f, g \in \mathcal{M} \Rightarrow f + g \in \mathcal{M}$.*

Proof. First we prove that $N(f + g) \in S$. Let $\{V_n\}$ be a base of open sets that are elements of B. Put

$$\delta = \{(m, n) : V_n + V_m \subset U = Y - \{0\}\}.$$

Then

$$N(f + g) = \bigcup_{(m,n)\in\delta} f^{-1}(V_n) \cap g^{-1}(V_m) \in S.$$

Now let $E \in S$. Put $S' = \{E \cap F : F \in S\}$. S' is a σ-lattice. By the assumption, $f^{-1}(F) \in S'$ for all $F \in B$ and $f \in \mathcal{M}$. Hence by Theorem 1 we have $(f + g)^{-1}(F) \in S'$ for any $f, g \in \mathcal{M}$ and any $F \in B$. Therefore

$$E \cap (f + g)^{-1}(F) \in S$$

for all $E \in S$, $F \in B$ and Theorem 2 is proved.

Corrolary. *The family $\mathcal{M}_2$ from Definition 2 is closed under the operation of addition.*

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In the last two sections we shall use the following notation.

Definition 4. Let X, Y be any non-void sets, S and B families of subsets of X, Y respectively. By $\mathcal{M}'_3$ we denote the family of all functions satisfying the implication $E \in B \Rightarrow f^{-1}(E) \in S$. Let $y \in Y$ be a fixed point. For any $f : X \to Y$ put $N(f) = \{x : f(x) \neq y\}$. Then by $\mathcal{M}'_1$ ($\mathcal{M}'_2$) we denote the family of all functions satisfying the implication $E \in B \Rightarrow N(f) \cap f^{-1}(E) \in S$ ($E \in B$, $F \in S \Rightarrow N(f) \in S$, $f^{-1}(E) \cap F \in S$).

Theorem 3. *Let Y be a regular topological space satisfying the second axiom of countability. Let B be a countable base in Y, let S be closed under countable unions and intersections. Let $\{f_n\}$ be a sequence of functions of $\mathcal{M}'_3$ converging to a function f. Then $f \in \mathcal{M}'_3$. If $f_n \in \mathcal{M}'_2$ $(n = 1, 2, \ldots)$ and moreover Y is Hausdorff space then $f \in \mathcal{M}'_2$.*

Proof. The first conclusion follows immediately from the equality

$$\{x : \lim f_n(x) \in E\} = \bigcup_{\bar{A}\subset E, A\in B} \bigcup_{N=1}^{\infty} \bigcap_{n=N}^{\infty} \{x : f_n(x) \in A\}.$$

To prove the second assertion we show first that

$$N(\lim f_n) = \bigcup_{A\in B, y\notin \bar{A}} \bigcup_{N=1}^{\infty} \bigcap_{n=N}^{\infty} \{x : f_n(x) \in A\}.$$

Since $N(\lim f_n) \subset \bigcup_{n=1}^{\infty} N(f_n)$, we have

$$N(\lim f_n) = \bigcup_{A\in B, y\notin \bar{A}} \bigcup_{N=1}^{\infty} \bigcap_{n=N}^{\infty} [\{x : f_n(x) \in A\} \cap \bigcup_{n=1}^{\infty} N(f_n)] .$$

Since $f_n \in \mathcal{M}'_2$ $(n = 1, 2, \ldots)$ we have $\bigcup_{n=1}^{\infty} N(f_n) \in S$, hence also $\{x : f_n(x) \in A\} \cap \cap \bigcup_{n=1}^{\infty} N(f_n) \in S$. Therefore $N(\lim f_n) \in S$.

To prove the second property of $\mathcal{M}'_2$, for fixed $F \in S$ put $S' = \{G \cap F : G : S\}$. If $f_n \in \mathcal{M}'_2$ with respect to S then $f_n \in \mathcal{M}'_3$ with respect to S' and hence also $f \in \mathcal{M}'_3$ with respect to S' and $F \cap f^{-1}(E) \in S$ for all $F \in S$, $E \in B$.

Now let Y be the real line, $y = 0$.

Theorem 4. *Let S be closed under the countable unions and intersections. Let B consist of all open intervals in Y. If $f_n \in \mathcal{M}'_3$ or $f_n \in \mathcal{M}'_2 = \mathcal{M}_2 = \mathcal{M}_1$ $(n = 1, 2, \ldots)$ then $\sup f_n \in \mathcal{M}'_3$, $\inf f_n \in \mathcal{M}'_3$ or $\sup f_n \in \mathcal{M}'_2$, $\inf f_n \in \mathcal{M}'_2$, respectively.*

Proof. Let $a < b$, $f_n \in \mathcal{M}'_3$ $(n = 1, 2, \ldots)$. Then

$$\{x : \sup f_n(x) > a\} = \bigcup_{n=1}^{\infty} \{x : f_n(x) > a\} \in S ,$$

$$\{x : \sup f_n(x) < b\} = \bigcup_{m=1}^{\infty} \bigcap_{n=1}^{\infty} \bigcap_{k=1}^{\infty} \left\{x : f_n(x) < b - \frac{1}{m} + \frac{1}{k}\right\} \in S .$$

Similar assertions hold for $\inf f_n$.

If $f_n \in \mathcal{M}'_2$ $(n = 1, 2, \ldots)$, $E \in S$ and we want to prove that $E \cap \{x : \sup f_n(x) \in \in (a, b)\} \in S$, we can proceed similarly as in the previous theorem. Further $N(f_n) \in S$, $N(\sup f_n) \subset \bigcup_{n=1}^{\infty} N(f_n) \in S$ and

$$N(\sup f_n) = [\bigcup_{n=1}^{\infty} N(f_n) \cap \{x : \sup f_n(x) > 0\}] \cup [\bigcup_{n=1}^{\infty} N(f_n) \cap \{x : \sup f_n(x) < 0\}] \in S .$$

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Definition 5. A function $f : X \to Y$ is called simple if there is a decomposition $\{E_1, \ldots, E_n\}$ of X such that f is constant on every E_i.

Definition 6. We shall say that the family B of subsets of Y satisfies the condition (A) if for any $U, V \in B$ there are $U_1, \ldots, U_n \in B$ such that $\bigcup_{i=1}^{n} U_i = U \cup V$ and $\{U_i\}$ is a *refinement* of the system of all non-void sets among $U - V$, $U \cap V$, $V - U$ (i.e. any U_i is a subset of some of them).

Theorem 5. *Let Y be Hausdorff space satisfying the second axiom of countability. Let S be closed under finite unions. Let B be a countable base of neighbourhoods in Y (the elements of B need not be open), and let either B satisfy (A) or S be a ring.*

If $\emptyset \in S$ then to any $f \in \mathcal{M}'_1$ there is a sequence $\{f_n\}$ of simple functions of $\mathcal{M}'_1$ such that $f_n \to f$ i.e. $f_n(x) \to f(x)$ for any $x \in X$.

If $Y \in B$ or $X \in S$ then to any $f \in \mathcal{M}'_2$ ($f \in \mathcal{M}'_3$) there is a sequence $\{f_n\}$ of simple functions of $\mathcal{M}'_2$ ($\mathcal{M}'_3$) such that $f_n \to f$.

Proof. Put $B' = \{E \in B : y \notin E\}$ in the case $f \in \mathcal{M}'_1$ or $f \in \mathcal{M}'_2$ and $B' = B$ in the case $f \in \mathcal{M}'_3$. Let $B' = \{V_i\}_{i=1}^{\infty}$. Construct the sequence $\{f_n\}$ as follows.

There are W_i ($i = 1, 2, \ldots, k$) such that $\bigcup_{i=1}^{n} V_i = \bigcup_{i=1}^{k} W_i$ and each W_i is a subset of a set $\bigcap_{i=1}^{q} V_{k_i} \cap \bigcap_{j=1}^{p} V'_{n_j}$; moreover, either $W_i \in B$ (according to (A)) or $f^{-1}(W_i) \in S$ (if S is a ring). Choose arbitrary $y_i \in W_i$, $y_i \neq y$. Then put $f_n(x) = y_i$ for $x \in f^{-1}(W_i)$, $i = 1, \ldots, k$, and $f_n(x) = y$ for $x \notin \bigcup_{i=1}^{n} f^{-1}(V_i)$.

The functions f_n are simple. We have to prove that $f_n(x) \to f(x)$ for any $x \in X$. Let U be a neighbourhood of $f(x)$. Choose N such that $f(x) \in V_N \subset U$. Let $n > N$. Then $f_n(x) = y_i$ where $f(x) \in W_i$, $f(x) \in V_N$, hence $W_i \subset V_N \subset U$. Then $f_n(x) \in U$ for any $n > N$. This means that $f_n(x) \to f(x)$.

Now let $f \in \mathcal{M}'_1$. If $y \notin E$, then $N(f_n) \cap f_n^{-1}(E) = \bigcup_{i\in\alpha} (N(f) \cap f^{-1}(W_i)) \in S$, since α is the finite set of indices i for which $f_n^{-1}(E) = \bigcup_{i\in\alpha} f^{-1}(W_i)$. If $y \in E$ then $f_n^{-1}(E) \cap \cap N(f_n) = \emptyset \in S$ or $\bigcup f^{-1}(W_i) \in S$. Hence $f_n \in \mathcal{M}'_1$.

If $f \in \mathcal{M}'_3$, $X \in S$ or $Y \in B$ then $f^{-1}(W_i)$, $X - f^{-1}(W_i) \in S$, hence $f_n \in \mathcal{M}'_3$.

If $f \in \mathcal{M}'_2$, then similarly $E \cap f^{-1}(W_i) \in S$, $E - f^{-1}(W_i) \in S$. $N(f_n) = \{x : f_n(x) \neq y\} = \bigcup f^{-1}(W_i) = \bigcup f^{-1}(W_i) \cap N(f) \in S$.

Corrolary ([2], lemma 3). *Let Y be a separable metric space, X a topological space, S the σ-algebra of all Borel subsets of X, B a base of neighbourhoods in Y. Then to any $f \in \mathcal{M}'_3$ there is a sequence $\{f_n\}$ of functions of $\mathcal{M}'_3$ converging to f.*

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