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A REMARK ON SCORZA-DRAGONI THEOREM FOR DIFFERENTIAL INCLUSIONS

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Summary. A new simpler proof of the Scorza-Dragoni theorem for differential inclusions originally proved by Kurzweil and Jarník, is given.

Keywords: differential inclusion, Scorza-Dragoni theorem.

AMS Classification: 34E60.

1. INTRODUCTION

Let $G \subset \boldsymbol{R} \times \boldsymbol{R}^d$. Let $\mathscr{K}$ be the set of all non-empty closed convex subsets of $\boldsymbol{R}^d$. Let $S(x, r)$ denote the open ball in $\boldsymbol{R}^d$ with center at x and radius $r > 0$. For $\Delta \subset \boldsymbol{R}$ let $\mu(\Delta)$ denote the Lebesgue measure of Δ.

Let (Y, d) be a metric space. Recall that $F: Y \to \mathscr{K}$ is called closed (or closed graph) if the set graph $F = \{(y, z): z \in F(y), y \in Y\}$ is closed in $Y \times \boldsymbol{R}^d$. F is called upper semicontinuous (u.s.c.) at a point $y_0 \in Y$ if for every $\varepsilon > 0$ there is $\delta > 0$ such that $d(y, y_0) < \varepsilon$ implies $F(y) \subset F(y_0) + \varepsilon S$ ($S = S(0, 1)$). F is called u.s.c. if it is u.s.c. at each point of Y. Note that each u.s.c. multifunction with closed values is necessarily closed. The reverse is true if, in addition, the set $F(Y)$ is relatively compact.

A multifunction $F: [a, b] \to \mathscr{K}$ is called (Lebesgue) measurable if the set $\{t \mid F(t) \cap D \neq \emptyset\}$ is (Lebesgue) measurable for every closed subset D of $\boldsymbol{R}^d$.

For a given multifunction $F: G \to \mathscr{K}$ consider the differential inclusion

$$x' \in F(t, x) . \tag{1}$$

By a solution of (1) we mean an absolutely continuous function $u: [a, b] \to \boldsymbol{R}^d$ with graph contained in G and such that $u'(t) \in F(t, u(t))$ for a.a. $t \in [a, b]$.

For any function $u: J \to \boldsymbol{R}^d (J \subset \boldsymbol{R})$ and $t_0 \in J$ denote by $\operatorname{Cont} u(t_0)$ the set of all $z \in \boldsymbol{R}^d$ such that $z = \lim (u(t_n) - u(t_0))/(t_n - t_0)$ for some $\{t_n\} \subset J$, $t_n \neq t_0$, $t_n \to t_0$.

In [1] J. Jarník and J. Kurzweil established the following version of Scorza-Dragoni Theorem [5].

Theorem 1. *Let G and $\mathscr{K}$ be as above. Suppose that $F: G \to \mathscr{K}$ is such that*

(i) $F(t, \cdot)$ *is closed for a.a.* t *in* $\text{proj}_{\boldsymbol{R}} G$;

(ii) *for every* $(t_0, x_0) \in G$ *there exist numbers* $\delta_1, \delta_2 > 0$ *and an integrable function* $m: [t_0 - \delta_1, t_0 + \delta_2] \to [0, +\infty)$ *such that* $|F(t, x)| \leqq m(t)$ *for every* $(t, x) \in [t_0 - \delta_1, t_0 + \delta_1] \times S(x_0, \delta_2)$.

Then there exists a set $Q \subset \boldsymbol{R}$ *with* $\mu(Q) = 0$ *such that for every solution* u: $J \to \boldsymbol{R}^d$ *of* (1) *and every* $t \in J \setminus Q$ *we have* $\emptyset \neq \text{Cont}\, u(t) \subset F(t, u(t))$.

The original proof is based on a rather difficult approximation technique. In this note, we give a simpler and shorter proof, by combining some ideas of Opial [3] and Jarník and Kurzweil [1, 2].

Remark 1. Theorem 1 is a slight generalization of Jarník and Kurzweil result. In fact, in [1] F is supposed to be (non-empty convex) compact valued. Moreover, in stead of condition (i) it is supposed that: (i') for every $\varepsilon > 0$ there is a measurable set $A_\varepsilon \subset \boldsymbol{R}$ with $\mu(\boldsymbol{R} \setminus A_\varepsilon) < \varepsilon$ such that F restricted to $G \cap (A_\varepsilon \times \boldsymbol{R}^d)$ is u.s.c. It is easy to see (using the projection theorem) that each F satisfying (i') is Carathéodory, i.e. $F(\cdot, x)$ is (Lebesgue) measurable for each x, and $F(t, \cdot)$ is u.s.c. for a.a. t. Thus (i') implies (i), while (i) does not imply (i').

Finally let us remark that the assumption of Theorem 1 does not assure the existence of solutions of (1).

2. Proof of Theorem 1. Following [1] (owing Lindelöf property) it suffices to prove the following local version of Theorem 1.

Theorem 2. *Let* U *be an open subset of* $\boldsymbol{R}^d$. *Let* $I = [a, b]$. *Let* $F: I \times U \to \mathscr{K}$ *be such that* $F(t, \cdot)$ *is closed for a.a.* t *in* I *and* $|F(t, x)| \leqq m(t)$ *for a.a.* $t \in I$ *and all* $x \in U$, *where* m *is integrable on* I.

Then there is a set $I_0 \subset I$ *with* $\mu(I_0) = 0$ *such that for every solution* $u: J \to \boldsymbol{R}^d$ $(J \subset I)$ *of* (1) *and every* $t \in J \setminus I_0$ *we have* $\emptyset \neq \text{Cont}\, u(t) \subset F(t, u(t))$.

Proof. By [4, Theorem 1] there exists a multifunction $\tilde{F}: I \times U \to \mathscr{K} \cup \{\emptyset\}$ such that:

(α) $\tilde{F}(t, x) \subset F(t, x)$ for every $(t, x) \in I \times U$;

(β) if $\Delta \subset I$ is a measurable set, $u, v: D \to \boldsymbol{R}^d$ are measurable functions, then $v(t) \in F(t, u(t))$ a.e. in Δ implies $v(t) \in \tilde{F}(t, u(t))$ a.e. in Δ;

(γ) for every $\varepsilon > 0$ there is a closed set $I_\varepsilon \subset I$ with $\mu(I \setminus I_\varepsilon) < \varepsilon$ such that $\tilde{F}$ restricted to $I_\varepsilon \times U$ is closed.

By virtue of (α) and (β) it suffices to verify the statement of Theorem 2 for $\tilde{F}$.

Let $\varepsilon > 0$. Let I_ε be as in (γ). By virtue of Lusin's Theorem we can assume that m restricted to I_ε is continuous. Thus $M = \sup \{m(t): t \in I_\varepsilon\} < +\infty$.

Denote by χ the characteristic function of the set $I \setminus I_\varepsilon$. Clearly, for a.a. $t \in I_\varepsilon$ we have

$$(2) \qquad \lim_{h \to 0} \frac{1}{h} \int_t^{t+h} \chi(s)\, m(s)\, \mathrm{d}s = \frac{\mathrm{d}}{\mathrm{d}t} \int_a^t \chi(s)\, m(s)\, \mathrm{d}s = 0\,.$$

Let I_ε^* denote the set of all points of (Lebesgue) density of I_ε for which (2) is fulfilled.

Let $t^* \in I_\varepsilon^*$. Let $u\colon J \to \mathbf{R}^d$ be a solution of (1) (if (1) has no solution there is nothing to prove).

Claim 1. Cont $u(t^*) \neq \emptyset$.

Indeed, let $v\colon J \to U$ be a measurable function such that $v(s) \in F(t, u(s))$ for every $s \in J$ and

$$u(t) = u(t_0) + \int_{t_0}^{t} v(s)\,\mathrm{d}s\,, \quad t \in J\,.$$

We have

$$\frac{u(t^* + h) - u(t^*)}{h} = \frac{1}{h}\int_{t^*}^{t^*+h} v(s)\,\mathrm{d}s = \frac{1}{h}\int_{t^*}^{t^*+h} (1 - \chi(s))\,v(s)\,\mathrm{d}s + \frac{1}{h}\int_{t^*}^{t^*+h} \chi(s)\,v(s)\,\mathrm{d}s\,.$$

From (2) and the boundedness of m on I_ε it follows that

$$\left|\frac{u(t^* + h) - u(t^*)}{h}\right| \leqq M + 1 \quad \text{for} \quad 0 < h \leqq h_0\,, \quad h_0 > 0\,.$$

Consequently, there is a sequence $\{h_i\} \subset (0, h_0]$ with $h_i \to 0$ such that the sequence $\{(u(t^* + h_i) - u(t^*))/h_i\}$ is convergent. Thus Cont $u(t^*) \neq \emptyset$.

Claim 2. Cont $u(t^*) \subset \tilde{F}(t, u(t^*))$.

Indeed, let $z \in$ Cont $u(t^*)$. Let $\{t^* + h_i\} \subset J$, $h_i \to 0$ be such that

$$\frac{u(t^* + h_i) - u(t^*)}{h_i} \to z\,.$$

Suppose that $h_i > 0$, $i = 1, 2, \ldots$ (in the case $h_i < 0$ the arguments is similar). Set $\Delta_i^\varepsilon = [t^*, t^* + h_i] \cap I_\varepsilon$. As above, we have

$$(3) \qquad \frac{u(t^* + h_i) - u(t^*)}{h_i} = \frac{1}{h_i}\int_{t^*}^{t^*+h_i} (1 - \chi(s))\,v(s)\,\mathrm{d}s + \frac{1}{h_i}\int_{t^*}^{t^*+h_i} \chi(s)\,v(s)\,\mathrm{d}s =$$

$$= \frac{\mu(\Delta_i^\varepsilon)}{h_i}\,\frac{1}{\mu(\Delta_i^\varepsilon)}\int_{\Delta_i^\varepsilon} v(s)\,\mathrm{d}s + \frac{1}{h_i}\int_{t^*}^{t^*+h_i} \chi(s)\,v(s)\,\mathrm{d}s\,.$$

By (2) the last term in (3) tends to zero as $i \to +\infty$. Moreover, $m(\Delta_i^\varepsilon)/h_i \to 1$ as $i \to +\infty$, because t^* is a point of density of I_ε.

Since $\tilde{F}(\cdot, u(\cdot))$ is closed and uniformly bounded on $J \cap I_\varepsilon$, it is u.s.c. on $J \cap I_\varepsilon$. Thus, for given $\eta > 0$ there is i_0 such that $\tilde{F}(t, u(t)) \subset \tilde{F}(t^*, u(t^*)) + \eta S$ for every $t \in \Delta_i^\varepsilon$, $i \geqq i_0$. This and the mean value theorem imply

$$\frac{1}{\mu(\Delta_i^\varepsilon)}\int_{\Delta_i^\varepsilon} v(s)\,\mathrm{d}s \in \tilde{F}(t^*, u(t^*)) + \eta S\,, \quad i \geqq i_0\,.$$

Since η is arbitrary, we have

$$\lim_{i \to +\infty} \frac{1}{\mu(\Delta_i^\varepsilon)} \int_{\Delta_i^\varepsilon} v(s)\, ds \in \tilde{F}(t^*, u(t^*)) .$$

Consequently, $z \in \tilde{F}(t^*, u(t^*))$. Since z is arbitrary in Cont $u(t^*)$, Claim 2 is proved.

Let $\varepsilon_n \downarrow 0$. Set $I^* = \bigcup I_{\varepsilon_n}$. Since $\mu(I_\varepsilon^*) \geqq \mu(I) - \varepsilon_n$, we have $\mu(I^*) = \mu(I)$. Clearly, for every solution $u: J \to \mathbf{R}^d$ of (1) and every $t^* \in J \cap I^*$, $\emptyset \neq \operatorname{Cont} u(t^*) \subset \subset F(t^*, u(t^*))$. This completes the proof.

Remark 2. Theorem 2 fails if we drop the assumption that F is convex valued. Indeed, let $v: [0, 1] \to \mathbf{R}$ be such that $v(t) = 1$ if $t \in (1/3^k, 2/3^k)$, $v(t) = -1$ if $t \in \in (2/3^k, 3/3^k)$, $k = 1, 2, \ldots$ and, $v(t) = 0$ otherwise. Obviously the function

$$u(t) = \int_0^t v(s)\, ds$$

is a solution of the differential inclusion

$$x' \in \{-1, 1\}\,, \quad t \in [0, 1] \tag{4}$$

and Count $u(0) = [0, 1/2]$. A slight modification of the above construction furnishes a solution of (4) such that for given $t_0 \in [0, 1]$, Cont $u(t_0) = [0, 1/2]$.

Remark 3. Adopting the argument of [1] one can extend immediately the above result to the case of functional differential inclusions

$$x' \in F(t, x_t)$$

where $x_t(\theta) = x(t + \theta)$, $\theta \in [-a, 0]$ and $F: I \times C([-a, 0], \mathbf{R}^d) \to \mathscr{K}$.

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Souhrn

POZNÁMKA KE SCORZA-DRAGONIOVĚ VĚTĚ PRO DIFERENCIÁLNÍ INKLUZE

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Je podán nový a jednodušší důkaz Scorza-Dragoniovy věty pro diferenciální inkluze, původně dokázané J. Kurzweilem a J. Jarníkem.

Резюме

ЗАМЕЧАНИЕ ПО ТЕОРЕМЕ СКОРЦА-ДРАГОНИ ДЛЯ ДИФФЕРЕНЦИАЛЬНОГО ВКЛЮЧЕНИЯ

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Дано новое и более простое доказательство теоремы Скорца-Драгони для дифференциального включения, первоначально доказаной Я. Курцвейлем и И. Ярником.

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