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# Note on special arithmetic and geometric means 

Horst Alzer

> Abstract. We prove: If $A(n)$ and $G(n)$ denote the arithmetic and geometric means of the first $n$ positive integers, then the sequence $n \mapsto n A(n) / G(n)-(n-1) A(n-1) / G(n-1)$ $(n \geq 2)$ is strictly increasing and converges to $e / 2$, as $n$ tends to $\infty$.

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In this paper we denote by $A(n)$ and $G(n)$ the arithmetic and geometric means of the first $n$ positive integers, that is,

$$
A(n)=\frac{1}{n} \sum_{i=1}^{n} i=\frac{n+1}{2} \quad \text { and } \quad G(n)=\prod_{i=1}^{n} i^{1 / n}=(n!)^{1 / n}
$$

In 1964 H. Minc and L. Sathre [2] published several remarkable inequalities involving $G(n)$. "Probably the most interesting of them, and certainly the hardest to prove" [2, p. 41], is

$$
\begin{equation*}
1<n \frac{G(n+1)}{G(n)}-(n-1) \frac{G(n)}{G(n-1)} \quad(n \geq 2) \tag{1}
\end{equation*}
$$

It is the aim of this paper to present a closely related result. We prove the following counterpart of inequality (1):

$$
\begin{equation*}
\frac{3}{\sqrt{2}}-1 \leq n \frac{A(n)}{G(n)}-(n-1) \frac{A(n-1)}{G(n-1)}<\frac{e}{2} \quad(n \geq 2) \tag{2}
\end{equation*}
$$

Both bounds are best possible. The double-inequality (2) is an immediate consequence of the following
Theorem. The sequence

$$
n \mapsto n \frac{A(n)}{G(n)}-(n-1) \frac{A(n-1)}{G(n-1)} \quad(n \geq 2)
$$

is strictly increasing and converges to $e / 2$, as $n$ tends to $\infty$.
Proof: In the first part of the proof we show that the function

$$
f(x)=x(x+1)(\Gamma(x+1))^{-1 / x} \quad(0<x \in \mathbb{R})
$$

is strictly convex on $[4, \infty)$. In what follows we assume $x \geq 4$. Differentiation yields

$$
x^{2}(x+1) \frac{f^{\prime \prime}(x)}{f(x)}=2 x-2 x \Psi(x+1)+2 \log \Gamma(x+1)+(x+1)(\Psi(x+1))^{2}
$$

$$
\begin{align*}
& -\frac{2(x+1)}{x} \Psi(x+1) \log \Gamma(x+1)+\frac{x+1}{x^{2}}(\log \Gamma(x+1))^{2}  \tag{3}\\
& -x(x+1) \Psi^{\prime}(x+1)
\end{align*}
$$

where $\Psi=\Gamma^{\prime} / \Gamma$ designates the logarithmic derivative of the gamma function. Using the inequalities

$$
\begin{aligned}
0 & <(x-1 / 2) \log (x)-x+\log \sqrt{2 \pi} \\
& <\log \Gamma(x)<1 /(12 x)+(x-1 / 2) \log (x)-x+\log \sqrt{2 \pi} \\
0 & <\log (x)-1 /(2 x)-1 /\left(12 x^{2}\right)<\Psi(x)<\log (x)-1 /(2 x)
\end{aligned}
$$

and

$$
\Psi^{\prime}(x)<1 / x+1 /\left(2 x^{2}\right)+1 /\left(6 x^{3}\right)
$$

(see [1, p. 820 ff.$]$ ), we get from (3):

$$
\begin{align*}
x^{2}(x+1) \frac{f^{\prime \prime}(x)}{f(x)}> & 2 x-2 x\left[\log (x+1)-\frac{1}{2(x+1)}\right]  \tag{4}\\
& +2[(x+1 / 2) \log (x+1)-(x+1)+\log \sqrt{2 \pi}] \\
& +(x+1)\left[\log (x+1)-\frac{1}{2(x+1)}-\frac{1}{12(x+1)^{2}}\right]^{2} \\
& -\frac{2(x+1)}{x}\left[\log (x+1)-\frac{1}{2(x+1)}\right] \times \\
& \times\left[\frac{1}{12(x+1)}+(x+1 / 2) \log (x+1)-(x+1)+\log \sqrt{2 \pi}\right] \\
& +\frac{x+1}{x^{2}}[(x+1 / 2) \log (x+1)-(x+1)+\log \sqrt{2 \pi}]^{2} \\
& -x(x+1)\left[\frac{1}{x+1}+\frac{1}{2(x+1)^{2}}+\frac{1}{6(x+1)^{3}}\right] \\
& =\frac{1}{2}+\frac{1}{x}\left[\frac{25}{12}-\frac{3}{2} \log (2 \pi)+(\log \sqrt{2 \pi})^{2}\right]-\frac{1}{2(x+1)} \\
& +\frac{1}{x^{2}}\left[1-\log (2 \pi)+(\log \sqrt{2 \pi})^{2}\right]+\frac{1}{4(x+1)^{2}}+\frac{1}{144(x+1)^{3}} \\
& +\frac{x+1}{4 x^{2}}(\log (x+1))^{2}+\log (x+1)\left\{\frac{1}{x}\left[\frac{1}{2} \log (2 \pi)-\frac{5}{3}\right]\right. \\
& \left.-\frac{1}{6(x+1)}+\frac{1}{x^{2}}\left[\frac{1}{2} \log (2 \pi)-1\right]\right\} .
\end{align*}
$$

Since

$$
1-\log (2 \pi)+(\log \sqrt{2 \pi})^{2}>0
$$

and

$$
\frac{x+1}{4 x^{2}}(\log (x+1))^{2}>\frac{1}{2 x} \log (x+1)
$$

we conclude from (4):

$$
\begin{equation*}
x^{2}(x+1) \frac{f^{\prime \prime}(x)}{f(x)}>\frac{1}{2}+\frac{a}{x}-\frac{1}{2(x+1)}-\log (x+1)\left[\frac{b}{x}+\frac{1}{6(x+1)}+\frac{c}{x^{2}}\right] \tag{5}
\end{equation*}
$$

where

$$
\begin{aligned}
& a=\frac{25}{12}-\frac{3}{2} \log (2 \pi)+(\log \sqrt{2 \pi})^{2}=0.170 \ldots \\
& b=\frac{7}{6}-\frac{1}{2} \log (2 \pi)=0.247 \ldots, \quad c=1-\frac{1}{2} \log (2 \pi)=0.081 \ldots
\end{aligned}
$$

Using $\log (x+1)<x$, we obtain from (5):

$$
\begin{aligned}
x^{2}(x+1) \frac{f^{\prime \prime}(x)}{f(x)} & >\frac{1}{3}-b+\left(a-c-\frac{1}{3}\right) \frac{1}{x} \\
& \geq \frac{1}{3}-b+\left(a-c-\frac{1}{3}\right) \frac{1}{4}=0.024 \ldots
\end{aligned}
$$

valid for all $x \geq 4$.
Thus, $f$ is strictly convex on $[4, \infty)$. From Jensen's inequality we get

$$
2 f(n)<f(n-1)+f(n+1)
$$

for all integers $n \geq 5$. This implies that the sequence

$$
n \mapsto[f(n)-f(n-1)] / 2=n A(n) / G(n)-(n-1) A(n-1) / G(n-1)
$$

is strictly increasing for $n \geq 5$. The approximate values of $n A(n) / G(n)-$ $(n-1) A(n-1) / G(n-1)$ for $n=2,3,4,5$, are $1.121,1.180,1.216,1.239$, respectively. Hence, $n \mapsto n A(n) / G(n)-(n-1) A(n-1) / G(n-1)$ is strictly increasing for all $n \geq 2$.

Finally we prove that

$$
\begin{equation*}
\lim _{n \rightarrow \infty}[n A(n) / G(n)-(n-1) A(n-1) / G(n-1)]=e / 2 \tag{6}
\end{equation*}
$$

If we set

$$
\alpha(n)=n / G(n) \quad \text { and } \quad \beta(n)=G(n) / G(n-1)
$$

then we have for $n \geq 2$ :

$$
\begin{aligned}
n \frac{A(n)}{G(n)}- & (n-1) \frac{A(n-1)}{G(n-1)} \\
& =\frac{1}{2}\left[\alpha(n)+\frac{n}{n-1} \alpha(n-1)-\frac{n}{n-1} \alpha(n) \frac{\beta(n)-1}{\log \beta(n)} \log \alpha(n)\right]
\end{aligned}
$$

Since

$$
\lim _{n \rightarrow \infty} \alpha(n)=e \quad \text { and } \quad \lim _{n \rightarrow \infty} \beta(n)=1
$$

we obtain (6). This completes the proof of the Theorem.

## References

[1] Fichtenholz G.M., Differential - und Integralrechnung, II, Dt. Verlag Wissensch., Berlin, 1979.
[2] Minc H., Sathre L., Some inequalities involving $(r!)^{1 / r}$, Edinburgh Math. Soc. 14 (1964/65), 41-46.

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