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TERNARY RINGS OF PAPPIAN PLANES

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The present article deals with planar ternary rings with a right zero and a left zero associated to the Pappian planes. After preliminaries containing background notions and results, there is given a generalization of results from [4], [5] onto planar ternary rings with a right zero and a left zero together with examples of planar ternary rings considered.

§ 1 Planar ternary rings with a right zero and a left zero and their properties

Definition 1. An ordered pair $(\mathbf{S}, \mathbf{T})$ is called a planar ternary ring (cf. [1]), if $\mathbf{S}$ is a set with at least two elements and $\mathbf{T}$ is a ternary operation on $\mathbf{S}$ such that

A 1. $\forall a, b, c \in \mathbf{S}\ \exists!\ x \in \mathbf{S}\ \mathbf{T}(a, b, x) = c$,

A 2. $\forall a, b, c, d \in \mathbf{S};\ a \neq c\ \exists!\ x \in \mathbf{S}\ \mathbf{T}(x, a, b) = \mathbf{T}(x, c, d)$,

A 3. $\forall a, b, c, d \in \mathbf{S};\ a \neq c\ \exists\ (x, y) \in \mathbf{S}^2\ \mathbf{T}(a, x, y) = b,\ \mathbf{T}(c, x, y) = d$.

If in addition

A 4. $\exists\ 0^L \in \mathbf{S}\ \forall y, z \in \mathbf{S}\ \mathbf{T}(0^L, y, z) = z$,

A 5. $\exists\ 0^R \in \mathbf{S}\ \forall x, z \in \mathbf{S}\ \mathbf{T}(x, 0^R, z) = z$,

then $(\mathbf{S}, \mathbf{T})$ is called the planar ternary ring with a right zero and a left zero.

Consequences

(1) The solution (x, y) from A 3. is determined uniquely.

(2) The element 0^L from A 4. is determined uniquely.

(3) The element 0^R from A 5. is determined uniquely.

(4) $\forall a \in \mathbf{S};\ a \neq 0^L \exists!\ x \in \mathbf{S}\ \mathbf{T}(a, x, b) = c$,

(5) $\forall a \in \mathbf{S};\ a \neq 0^R \exists!\ x \in \mathbf{S}\ \mathbf{T}(x, a, b) = c$.

In what follows let $(\mathbf{S}, \mathbf{T})$ designate always a planar ternary ring with a right zero and left zero. Further define an induced binary multiplication on $\mathbf{S}$ by

$$a \,.\, m :\, = \mathbf{T}(a, m, 0^L), \qquad \forall a, m \in \mathbf{S}.$$

Then

(1) $0^L \,.\, a = 0^L \ \forall a \in \mathbf{S}$,

(2) $a \,.\, 0^R = 0^L \ \forall a \in \mathbf{S}$,

(3) $\forall m \in \mathbf{S}\setminus\{0^R\} \ \exists!\, x \in \mathbf{S} \ x \,.\, m = c$,

(4) $\forall m \in \mathbf{S}\setminus\{0^L\} \ \exists!\, x \in \mathbf{S} \ m \,.\, x = c$.

For each $a \in \mathbf{S}\setminus\{0^L\}$ denote by e_a the solution of $a \,.\, x = a$; additionally define $e_{0^L} : = 0^L$. Now we are able to introduce an induced binary addition $+$ on $\mathbf{S}$ by

$$a + b : = \mathrm{T}(a, e_a, b) \ \forall a, b \in \mathbf{S}.$$

Then

(1) $\forall a, b \in \mathbf{S} \exists!\, x \in \mathbf{S} \ a + x = b$,

(2) $m \,.\, a = n \,.\, a, \ a \neq 0^R \Leftrightarrow m = n$,

(3) $a \,.\, m = a \,.\, n, \ q \neq 0^L \Leftrightarrow m = n$,

(4) $a + b = a + c \Leftrightarrow b = c$,

(5) $0^L + a = a + 0^L = a \ \forall a \in \mathbf{S}$.

Definition 2. $(\mathbf{S}, \mathrm{T})$ is called a generalized Cartesian group if it has following properties:

1. $(\mathbf{S}, +)$ is a group,
2. $\mathrm{T}(a, b, c) = a \,.\, b + c \ \forall a, b, c \in \mathbf{S}$.

§ 2 Coordinatization of projective planes by planar ternary rings with a right zero and a left zero

Let (P, L, I) be a projective plane with a prominent line $\mathbf{n}$ and a prominent point $\mathbf{N} I \mathbf{n}$. Let $\mathscr{A} : = P\setminus\tilde{\mathbf{n}}, \mathscr{B} : L\setminus\tilde{\mathbf{N}}$. It is known that $\#\mathscr{B} = \#\mathbf{S}\times\mathbf{S} = \#\mathscr{A}$, where $\#\mathbf{S}$ is the order of (P, L, I). An ordered quadruple $(\mathbf{n}, \mathbf{N}, \alpha, \beta)$, where α, β are bijections $\alpha : \mathbf{S}\times\mathbf{S} \to \mathscr{A}$, $\beta : \mathbf{S}\times\mathbf{S} \to \mathscr{B}$ will be called a frame.

For every couple $(\mathbf{n}, \mathbf{N})$, $\mathbf{N} I \mathbf{n}$ there exists a couple of bijections α, β so that $(\mathbf{S}, \mathrm{T})$, where $(x, y)^\alpha I (u, v)^\beta \Leftrightarrow : y = \mathrm{T}(x, u, v)$ is a planar ternary ring with a right zero and a left zero. This frame $(\mathbf{n}, \mathbf{N}, \alpha, \beta)$ will be called cartesian and $(\mathbf{S}, \mathrm{T})$ is said to be corresponding to this frame (compare with terminology in [3]).

Lines $l \# \mathbf{n}$ through $\mathbf{N}$ will be called vertical. Points of $\mathbf{n}$ are called improper, other points are called proper.

In the forthcoming text we shall write (x, y) instead of $(x, y)^\alpha$ and $[x, y]$ instead of $(x, y)^\beta$.

If $(\mathbf{S}, \mathrm{T})$ corresponds to a Cartesian frame, then:

(1) $[a, b]$, $[a', b']$ carry the same improper point if and only if $a = a'$.

(2) (x, y), (x', y') are on the same vertical line if and only if $x = x'$.

Lines different from $\mathbf{n}$ carrying the same improper point are called parallel. The vertical line carrying points $(0^L, a) \ \forall a \in \mathbf{S}$ is said to be a vertical axis. Lines $[0^R, v]$ for all $v \in \mathbf{S}$ are called horizontal. We easily see that:

(3) $[a, b]$, $[a', b']$ carry the same point of the vertical axis if and only if $b = b'$.
(4) (x, y), (x', y') lie on the same horizontal line if and only if $y = y'$.

§ 3 Pappian planes

Definition 3. A projective plane is called $(\mathbf{A}, \mathbf{b})$ – transitive, if for any different points $\mathbf{B}$, $\mathbf{C}$ with $\mathbf{B} \neq \mathbf{A}$, $\mathbf{C} \neq \mathbf{A}$; $\mathbf{B}X\mathbf{b}$, $\mathbf{C}X\mathbf{b}$, $\mathbf{A}/\mathbf{BC}$ there is a perspective collineation with an axis $\mathbf{b}$ and a centre $\mathbf{A}$, which maps $\mathbf{B}$ into $\mathbf{C}$.

In the sequel we shall investigate a fixed projective plane Π with Cartesian frame $(\mathbf{n}, \mathbf{N}, \alpha, \beta)$; the corresponding planar ternary ring will be designated by $(\mathbf{S}, \mathbf{T})$.

Instead of $(\mathbf{N}, \mathbf{n})$ – transitive we shall use also the notation vertically transitive. Π is called a translation plane if it is $(\mathbf{A}, \mathbf{n})$-transitive for all points $\mathbf{A}$ on $\mathbf{n}$.

Theorem 1. Π is vertically transitive if and only if $(\mathbf{S}, \mathbf{T})$ is a generalized Cartesian group.

Proof: Denote by $\mathbf{G_N}$ the group of all perspective collineations with a centre $\mathbf{N}$ and an axis $\mathbf{n}$. It can be shown, that this group is isomorphic with $(\mathbf{S}, +)$ with a neutral element 0^L and so $(\mathbf{S}, +)$ is a group. From the condition that $\mathbf{G_N}$ operates transitively on proper points of one (and consequently each) vertical line it follows the linearity property. For details see [4] p. 621–622.

Theorem 2. Π is a translation plane if and only if
(A) $(\mathbf{S}, \mathbf{T})$ is a generalized Cartesian group,
(B) for arbitrary $a, b, c \in \mathbf{S}$ the equation $a \,.\, x + b \,.\, x = c \,.\, x$ has only the trivial solution 0^R or it is fulfilled identically.

The proof of this theorem is analogous to the proof of Theorem 2 of [4]. It is only necessary to substitute the two-sided zero by a right zero.

Let Π be a translation plane and $\mathbf{A}$ one of its proper points. It is known, that Π is the Desarguesian if and only if it is $(\mathbf{A}, \mathbf{n})$ – transitive. So for verification whether Π is the Desarguesian it suffices to find a proper point (for instance $\mathbf{O} = (0^L, 0^L)$) such that Π is $(\mathbf{O}, \mathbf{n})$ – transitive. This fact will be used in the proof of the following theorem. But first some conventions:

If $z \,.\, x = y$ then $z =: y/x$.
If $x \,.\, z = y$ then $z =: x\backslash y$.

Theorem 3. Π is the Desarguesian if and only if $(\mathbf{S}, \mathbf{T})$ satisfies conditions (A), (B) and moreover
(C) for arbitrary $a, b, c \in \mathbf{S}$, the equation $x \,.\, a + x \,.\, b = x \,.\, c$ has only the trivial solution 0^L or it is fulfilled identically;
(D) for arbitrary $a, b, c, d \in \mathbf{S}\backslash\{0^L\}$, the equation $a\backslash(b \,.\, x) = c\backslash(d \,.\, x)$ has only the trivial solution 0^R or it is fulfilled identically.

The proof is similar to the proof of Theorem 2 in [5], it is only necessary to put a left zero instead of the two-sided zero and consider the elements x different from 0^R.

Theorem 4. Π is the Pappian if and only if $(\mathbf{S}, \mathbf{T})$ satisfies the conditions $(A)-(D)$ and additionally the condition

$$a \,.\, (c \backslash (b \,.\, c)) = b \,.\, (c \backslash (a \,.\, c)) \quad \forall a, b, c \in \mathbf{S} \backslash \{0^L\}.$$

The proof coincides with the proof of Theorem 3 in [5]. Only one adaption is necessary: instead of the two-sided zero 0 we need to put a left zero 0^L.

§ 4 Examples

I. Let $\mathbf{S} := \{0, 1, 2, 3, 4\}$ and let $+$ be addition modulo 5. The multiplication will be defined by

	0	1	2	3	4
0	0	0	0	0	0
1	2	0	4	3	1
2	1	0	2	4	3
3	4	0	3	1	2
4	3	0	1	2	4

Now define a ternary operation $\mathbf{T}$ on $\mathbf{S}$ by $\mathbf{T}(x, u, v) := x \,.\, u + v \ \forall x, u, v \in \mathbf{S}$. It is trivial that $0^L = 0$, $0^P = 1$ and that $(\mathbf{S}, \mathbf{T})$ is a generalized Cartesian group with a commutative addition. The multiplication is not associative and the distributive laws do not hold as the following examples show:

$$2 \,.\, (3.3) = 2.1 = 0, \qquad (2.3) \,.\, 3 = 4.3 = 2,$$
$$(3 + 1) \,.\, 3 = 4.3 = 2, \qquad 3.3 + 1.3 = 1 + 3 = 4,$$
$$3 \,.\, (1 + 3) = 3.4 = 2, \qquad 3.1 + 3.3 = 0 + 1 = 1.$$

We can verify that all conditions (A) – (E) are satisfied so that $(\mathbf{S}, \mathbf{T})$ corresponds to a Pappian plane.

II. Let $\mathbf{R}$ designate the set of all reals. Define the ternary operation on $\mathbf{R}$ as follows: for a fixed $\alpha \in \mathbf{R}$ we put $y = \mathbf{T}(x, u, v) :\Leftrightarrow y + \alpha \,.\, x = x \,.\, u + v$, where $+, .$ means the addition and the multiplication on $\mathbf{R}$. Then

$$\mathbf{T}(x, u, v) = x \,.\, (u - \alpha) + v.$$

It is trivial to verify the validity of A 1. – A 5. from § 1 with $0^L = 0$, $0^R = \alpha$. Now let $\oplus$, $\odot$ denote the induced addition and multiplication of $(\mathbf{R}, \mathbf{T})$. We know that

$$a \odot b := \mathbf{T}(a, b, 0^L), \qquad a \oplus b := \mathbf{T}(a, e_a, b) \ \forall a, b \in \mathbf{R},$$

which means

$$a \odot b = a \,.\, (b - \alpha), \qquad a \odot b = a + b \ \forall a, b \in \mathbf{R}.$$

So we immediately see that the conditions $(A)-(E)$ of § 3 are valid.

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Souhrn

TERNÁRNÍ OKRUHY PAPPOVSKÝCH ROVIN

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V článku se studují planární ternární okruhy s levou a pravou nulou bez jednotky. Vyslovují se nutné a postačující podmínky pro to, aby takový planární ternární okruh koordinatizoval vertikálně transitivní, translační, desarguesovskou a pappovskou rovinu.

Резюме

ТЕРНАРЫ ПАППОВСКИХ ПЛОСКОСТЕЙ

ЛИБУШЕ МАРКОВА

В работе изучаются тернары с левым и правым нулевым элементом без единицы. Показываются необходимые и достаточные условия для того, чтобы такой тернар служил к введению координат в вертикально-транситивную, трансляционную, Дезаргову и Паппову плоскость.