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ON SOME THREE-POINT PROBLEMS FOR THIRD-ORDER DIFFERENTIAL EQUATIONS

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Summary. This paper is concerned with existence and uniqueness of solutions of the three-point mixed problem $u''' = f(t,u,u',u'')$, $u(c) = 0$, $u'(a) = u'(b)$, $u''(a) = u''(b)$, $a \leqslant c \leqslant b$. The problem is at resonance, in the sense that the associated linear problem has non-trivial solutions. We use the method of lower and upper solutions.

Keywords: three-point mixed problem, lower and upper solutions, existence, uniqueness, resonance, Carathéodory conditions.

AMS classification: 34B15, 34B10

1. INTRODUCTION

In this paper we are concerned with the existence and uniqueness of solutions of the three-point mixed problem

$$u''' = f(t,u,u',u'') \tag{1.1}$$

$$u(c) = 0,\ u'(a) = u'(b),\ u''(a) = u''(b),\ a \leqslant c \leqslant b \tag{1.2}$$

where $-\infty < a \leqslant c \leqslant b < +\infty$ and f satisfies the local Carathéodory conditions on $(a,b) \times \mathbf{R}^3$.

The two-point mixed problem for the second order differential equation was solved by V. Lakshmikantham and S. Hu in [15]. They obtained the existence results under the assumption that f is continuous, nonincreasing in its second and third argument and satisfies a certain one-sided condition of the Lipschitz type, and that there exist lower and upper solutions for this problem. In [2], A. R. Aftabizadeh, J. M. Xu and C. P. Gupta considered the three-point problem for the third order differential equation where the boundary condition had the form

$$u'(0) = u'(1) = u(\mu) = 0, \quad 0 \leqslant \mu \leqslant 1. \tag{1.3}$$

Further, C. P. Gupta in [12] studied the questions of existence and uniqueness of solutions of the equation

$$-u''' - \pi^2 u' + g(x, u, u', u'') = e(x) \tag{1.4}$$

or

$$u''' + \pi^2 u' + g(x, u, u', u'') = e(x) \tag{1.5}$$

satisfying (1.3). The existence of a solution for the resonance problem (1.4), (1.3) was obtained provided e is a Lebesgue-integrable function with

$$\int_0^1 e(x) \sin \pi x \, dx = 0$$

and g is a Carathéodory function, bounded on $[0,1] \times B^2 \times \mathbf{R}$ (for every bounded B of $\mathbf{R}$) and

$$g(x, u, v, w) \cdot v \geqslant 0, \; x \in [0,1], \; u, v, w \in \mathbf{R}. \tag{1.6}$$

For the existence of a solution for (1.5), (1.3) g, in addition, has to satisfy

$$\lim_{|x| \to +\infty} \sup \frac{g(x, u, v, w)}{v} = \beta < 3\pi^2.$$

These results were proved by means of the method using second-order integro-differential boundary value problems and the Leray-Schauder continuation theorem.

In contrast to this, here we define lower and upper solutions for (1.1), (1.2) directly, not transforming (1.1), (1.2) into an integro-differential problem. We obtain similar conditions for the existence as in [12], but our sign condition (corresponding to (1.6)) has the form (3.4), i.e. it has to be fulfilled for $v = r_1, r_2, w = 0$ only. Instead of the boundedness of g, we assume the one-sided growth condition (3.1). Our method can be applied to problems (1.1), (1.3) and (1.4), (1.3) as well.

For other authors considering various third-order three-point boundary value problems see for example [1, 3–11, 17–25].

2. Notation and definitions

In what follows let $p, q \in [1, +\infty]$, where $\frac{1}{p} + \frac{1}{q} = 1$, $J = [a, b] \subset \mathbf{R}$, $C^m(J)$ and $L^p(J)$ have the usual meaning and $AC^m(J) = \{f\colon J \to \mathbf{R}\colon f^{(m)}$ is absolutely continuous$\}$.

We say that some property is satisfied on D if it is satisfied for a.e. $t \in J$ and for all $x, y, z \in \mathbf{R}$. Let $s_1, s_2 \in C(J)$, $s_1(t) \leqslant s_2(t)$, $S_i(t) = \int_c^t s_i(\tau)d\tau$, for $t \in J$, $i = 1, 2$. We put

$$D(s_1, s_2) = \{(t, x, y, z) \in D\colon |z| \geqslant 1,\ s_1(t) \leqslant y \leqslant s_2(t),\\ \min(S_1(t), S_2(t)) \leqslant x \leqslant \max(S_1, S_2)\}.$$

We say that $f\colon D \to \mathbf{R}$ satisfies the local Carathéodory conditions on D ($f \in \mathrm{Car}(D)$), if

$$f(\cdot, x, y, z)\colon J \to \mathbf{R} \text{ is measurable on } J \text{ for each } x, y, z \in \mathbf{R},$$
$$f(t, \cdot, \cdot, \cdot)\colon \mathbf{R}^3 \to \mathbf{R} \text{ is continuous for a.e. } t \in J$$

and

$$\sup\{|f(t, x, y, z)|\colon |x| + |y| + |z| \leqslant \rho\} \in L^1(J) \text{ for any } \rho \in (0, +\infty).$$

A function $u \in AC^2(J)$ satisfying (1.1) for a.e. $t \in J$ and fulfilling (1.2) will be called a solution of BVP (1.1), (1.2).

Let $\sigma_1, \sigma_2 \in AC^2(J)$ $m = \min\{\sigma_1, \sigma_2\}$, $M = \max\{\sigma_1, \sigma_2\}$ on J,

$$\alpha(t, x) = \begin{cases} m(t) & \text{for } m(t) > x \\ x & \text{for } m(t) \leqslant x \leqslant M(t) \\ M(t) & \text{for } M(t) < x. \end{cases} \tag{2.1}$$

Functions σ_1, σ_2 will be called lower and upper solutions to (1.1), (1.2), respectively, if

$$[\sigma_i''' - f(t, \alpha(t, x), \sigma_i', \sigma_i'')](-1)^i \leqslant 0 \tag{2.2}$$

for a.e. $t \in J$ and each $x \in \mathbf{R}$,

$$\sigma_i(c) = 0,\ \sigma_i'(a) = \sigma_i'(b),\ [\sigma_i''(a) - \sigma_i''(b)](-1)^i \leqslant 0,\ i = 1, 2. \tag{2.3}$$

For $j = 0$, 1, 2 we denote

$$c_j = \max\left\{|\sigma_1^{(j)}(t)| + |\sigma_2^{(j)}(t)|\colon a \leqslant t \leqslant b\right\}. \tag{2.4}$$

3. Main results

Theorem 1. *Let σ_1 be a lower solution and σ_2 an upper solution of BVP (1.1), (1.2) and let $\sigma_1'(t) \leqslant \sigma_2'(t)$ for each $t \in J$. Let on the set $D(\sigma_1', \sigma_2')$ the inequality*

$$f(t,x,y,z)\operatorname{sign} z \leqslant \omega(|z|)g^{\frac{1}{p}}(t,x)h(y))(1+|z|)^{\frac{1}{q}} \tag{3.1}$$

be satisfied, where $h \in L^q(-c_1, c_1)$, $g \in \mathrm{Car}(J \times \mathbf{R})$ are nonnegative and $\omega \in C(0, +\infty)$ is a positive function with

$$\int_0^\infty \frac{ds}{\omega(s)} = +\infty. \tag{3.2}$$

Then BVP (1.1), (1.2) has a solution u such that

$$\sigma_1' \leqslant u' \leqslant \sigma_2', \ \min\{\sigma_1, \sigma_2\} \leqslant u \leqslant \max\{\sigma_1, \sigma_2\} \text{ on } J. \tag{3.3}$$

Let us recall that c_1 is defined by (2.4). If $\sigma_1' = \sigma_2'$ on J, then $\sigma_1 = \sigma_2$ on J and BVP (1.1), (1.2) has a solution $u = \sigma_1 = \sigma_2$.

Note. Let there exist $r_1, r_2 \in \mathbf{R}$ such that $r_1 < r_2$ and

(3.4) $f(t, \alpha(t,x), r_1, 0) \leqslant 0$, $f(t, \alpha(t,x), r_2, 0) \geqslant 0$ for a.e. $t \in J$ and each $x \in \mathbf{R}$.

Then $\sigma_1 = r_1(t-c)$, $\sigma_2 = r_2(t-c)$. (For $\alpha(t,x)$ see (2.1), where $m(t) = \min\{r_1(t-c), r_2(t-c)\}$, $M(t) = \max\{r_1(t-c), r_2(t-c)\}$.)

Example. Theorem 1 is applicable for example to the functions:

1) $f(t,x,y,z) = e^x\left(y^3 + k(t)\right)\left(1+z^2\right)g(t) + ze^y$, where g, $k \in C(J)$, $g \geqslant 0$.
2) $f(t,x,y,z) = g(t,x)\left(y^5 + k(t) + z^2\right) + ye^{-yz}$, where $k \in C(J)$ and $g \in \mathrm{Car}(J \times \mathbf{R})$ is a nonnegative function bounded on each compact in $J \subset \mathbf{R}$.

   We can see that the existence results hold for arbitrarily rapid growth in the nonlinearity f. For such f the existence theorems of [2] or [12] do not work.

Now we will consider uniqueness.

Theorem 2. *Let there exist a positive function $h \in L^1(J)$ and constants α, β, $\gamma > 0$ satisfying*

$$\alpha\left(\frac{2(b-a)}{\pi}\right)^3 + \beta\left(2\frac{(b-a)}{\pi}\right)^2 + \gamma\left(\frac{2(b-a)}{\pi}\right) < 1, \tag{3.5}$$

such that on D the inequalities

(3.6) $f(t,x,y,z) - f(t,\bar{x},\bar{y},\bar{z}) + h(t)|z-\bar{z}| > 0$ *for* $y > \bar{y}$, $(x-\bar{x})\operatorname{sign}(t-c) \geqslant 0$

and

$$(3.7)\qquad |f(t,x,y,z)-f(t,\bar{x},\bar{y},\bar{z})| \leqslant \alpha|(x-\bar{x})|+\beta|(y-\bar{y})|+\gamma|(z-\bar{z})|$$

are satisfied.

Then BVP (1.1), (1.2) has at most one solution.

The Lipschitz condition (3.7) can be omitted if the sign condition (3.6) is changed as follows.

Theorem 3. *Let there exist a nonnegative function $h \in L^1(J)$ such that for a.e. $t \in J$ and for each $z, \bar{z} \in \mathbf{R}$, φ, $\bar{\varphi} \in C(J)$ the following condition is fulfilled:*

$$\varphi(t) > \bar{\varphi}(t) \Rightarrow f(t, T\varphi, \varphi, z) - f(t, T\bar{\varphi}, \bar{\varphi}, \bar{z}) + h(t)|z-\bar{z}| > 0,$$

where $[Tu](t) = \int\limits_c^t u(s)ds$.

Then BVP (1.1), (1.2) has at most one solution.

4. Lemmas

Lemma 1. *[13, Theorem 256, p. 219]. If $f \in AC(t_1,t_2)$, $f' \in L^2(t_1,t_2)$ and $f(t_0) = 0$, where $-\infty < t_1 \leqslant t_0 \leqslant t_2 < +\infty$, then*

$$\int\limits_{t_1}^{t_2} f^2(t)dt \leqslant \left[\frac{2(t_2-t_1)}{\pi}\right]^2 \int\limits_{t_1}^{t_2} f'^2(t)dt.$$

We will need a certain generalization of the Fredholm Alternative Theorem:

Lemma 2. *[16, Theorem 2.4, p. 25]. Let $h_i \in L(J)$, $i = 1,2,3$ and let g be a function of $\mathrm{Car}(D)$ and let the equation*

$$(4.1)\qquad u''' = \sum_{i=1}^{3} h_i(t)u^{(i-1)}(t)$$

have only the trivial solution satisfying (1.2). If there exists $h \in L^1(J)$ such that

$$|g(t,x,y,z)| \leqslant h(t) \text{ on } D,$$

then the equation

$$(4.2)\qquad u''' = \sum_{i=1}^{3} h_i(t)u^{(i-1)}(t) + g(t,u,u',u'')$$

has a solution satisfying (1.2).

Lemma 3 (On a priori estimates). *Let $r_1, r_2 \in \mathbf{R}$, $r_1 < r_2$, $g \in$ Car$(J \times \mathbf{R})$, $h \in L^q(r_1, r_2)$, and let $\omega \in C(0, +\infty)$ be a positive function satisfying (3.2).*

Then there exists $r^* \in (1, +\infty)$ such that for any function $u \in AC^2(J)$ the conditions (1.2),

$$r_1 \leqslant u'(t) \leqslant r_2 \text{ for every } t \in J, \tag{4.3}$$

$$u''' \operatorname{sign} u'' \leqslant \omega(|u''|) g^{\frac{1}{p}}(t, u) h(u')(1 + |u''|)^{\frac{1}{q}} \text{ for a.e. } t \in J,\ |u''(t)| \geqslant 1, \tag{4.4}$$

imply the estimate

$$|u''(y)| \leqslant r^* \text{ for every } t \in J. \tag{4.5}$$

Proof. Let G be the set of all functions $v \in AC^2(J)$ satisfying (1.2) and (4.3), Then

$$|v(t)| \leqslant \sigma, \quad \text{where } \sigma = (b - a) \max\{|r_1|, |r_2|\}. \tag{4.6}$$

Therefore $g_0(t) = \sup\{|g(t, v(t))| : v \in G\} \in L^1(J)$. Let us put

$$k_0 = 2\|g_0^{\frac{1}{p}}\|_{L^p(J)} \|h\|_{L^q(r_1, r_2)} \tag{4.7}$$

and

$$\Omega(x) = \int_0^x \frac{ds}{\omega(|s|)} \text{ for } x \in \mathbf{R}. \tag{4.8}$$

From (3.2) and (4.8) it follows that Ω is an odd function, $\Omega(\mathbf{R}) = \mathbf{R}$ and the inverse mapping Ω^{-1} exists.

Let $u \in AC^2(J)$ satisfy (1.2), (4.3) and (4.4). By (1.2), there exists $a_0 \in J$ such that

$$u''(a_0) = 0.$$

Let us suppose that there exists $t_1 \in (a_0, b]$ such that

$$|u''(t_1)| > k_1, \tag{4.9}$$

where

$$k_1 = \Omega^{-1}(\Omega(1) + k_0). \tag{4.10}$$

Let $[a_1, b_1] \subset [a_0, b]$ be the maximal interval containing t_1 in which $|u''(t)| \geqslant 1$. Let $s_1 \in (a_1, b_1]$ be such a point that $|u''(s_1)| = \sigma_1 = \max\{|u''(t)| : a_1 \leqslant t \leqslant b_1\}$. Then (4.4) yields

$$\int_{a_1}^{s_1} \frac{u'''(t) \operatorname{sign} u''(t)}{\omega(|u''(t)|)}\, dt \leqslant \int_{a_1}^{s_1} g^{\frac{1}{p}}(t, u(t)) h(u'(t)) (1 + |u''(t)|)^{\frac{1}{q}} dt.$$

In the case $u''(t) \geqslant 1$ on $[a_1, s_1]$, using the Hőlder inequality, we can obtain $\Omega(\sigma_1) - \Omega(1) \leqslant k_0$, which implies, by (4.7), (4.10),

$$\sigma_1 \leqslant k_1. \tag{4.11}$$

Inequality (4.11) contradicts (4.9). Similarly, supposing $u''(t) \leqslant -1$ on $[a_1, s_1]$ we can get $\Omega(-\sigma_1) - \Omega(-1) \geqslant -k_0$ and $s_0 - \sigma_1 \geqslant -k_1$, which also contradicts (4.9). Therefore we have

$$|u''(t)| \leqslant k_1 \qquad \text{for each } t \in [a_0, b] \text{ and } |u''(a)| \leqslant k_1. \tag{4.12}$$

Now, let us auppose that there exists $t_2 \in (a, a_0)$ with

$$|u''(t_2)| > r^*, \tag{4.13}$$

where $r^* = \Omega^{-1}(\Omega(1) + 2k_0)$. Let $[a_2, b_2] \subset [a, a_0]$ be the maximal interval containing t_2 in which $|u''(t)| \geqslant k_1$. Let $s_2 \in (a_2, b_2)$ be such that

$$|u''(s_2)| = \sigma_2 = \max\{|u''(t)| : a_2 \leqslant r \leqslant b_2\}.$$

Then (4.4) yields

$$\int_{a_2}^{s_2} \frac{u'''(t) \operatorname{sign} u''(t)}{\omega(|u''(t)|)}\, dt \leqslant k_0.$$

In the same way as in the first part of the proof we get either $\sigma_2 \leqslant r^*$ or $-\sigma_2 \geqslant -r^*$. Both of the inequalities contradict (4.13). Hence

$$|u''(t)| \leqslant r^* \text{ for every } t \in [a, a_0]. \tag{4.14}$$

From (4.12) and (4.14) the estimate (4.5) follows. □

5. Auxiliary existence result

Proposition. *Let σ_1 be a lower solution and σ_2 an upper solution of BVP (1.1), (1.2) and $\sigma_1'(t) \leqslant \sigma_2'(t)$ on J. Let there exist $h_0 \in L^1(J)$ such that on D the function f satisfies*

$$|f(t,x,y,z)| \leqslant h_0(t) \text{ for } \sigma_1'(t) \leqslant y \leqslant \sigma_2'(t). \tag{5.1}$$

Then BVP (1.1), (1.2) has a solution u satisfying (3.3).

Proof. Let us choose $m \in N$ and put (on D)

$$w_1(t,x,y,z) = -m\,(y-\sigma_1')\left[f(t,\sigma_1,\sigma_1',\sigma_1'') - f(t,\alpha(t,x),\sigma_1',z) - \frac{c_1}{m}\right],$$
$$w_2(t,x,y,z) = m\,(y-\sigma_2')\left[f(t,\sigma_2,\sigma_2',\sigma_2'') - f(t,\alpha(t,x),\sigma_2',z) + \frac{c_1}{m}\right],$$

$$f_m(t,x,y,z) = \begin{cases} f(t,\sigma_1,\sigma_1',\sigma_1'') - \dfrac{c_1}{m} & \text{for } y \leqslant \sigma_1' - \dfrac{1}{m} \\ f(t,\alpha(t,x),\sigma_1',z) + w_1 & \text{for } \sigma_1' - \dfrac{1}{m} < y < \sigma_1' \\ f(t,\alpha(t,x),y,z) & \text{for } \sigma_1' \leqslant y \leqslant \sigma_2' \\ f(t,\alpha(t,x),\sigma_2',z) + w_2 & \text{for } \sigma_2' < y < \sigma_2' + \dfrac{1}{m} \\ f(t,\sigma_2,\sigma_2',\sigma_2'') + \dfrac{c_1}{m} & \text{for } y \geqslant \sigma_2' + \dfrac{1}{m}, \end{cases} \tag{5.2}$$

where c_1 is defined by (2.4) and $\alpha(t,x)$ by (2.1).

From (5.1), (5.2) it follows that

$$|f_m(t,x,y,z)| \leqslant h_0(t) + \frac{c_1}{m} \text{ on } D.$$

Let us consider the differential equation

$$u''' = \frac{u'}{m} + f_m(t,u,u',u''). \tag{5.3}$$

According to Lemma 2, BVP (5.3), (1.2) has a solution u_m. We shall show that u_m satisfies

$$\sigma_1'(t) - \frac{1}{m} \leqslant u_m'(t) \leqslant \sigma_2'(t) + \frac{1}{m} \tag{5.4}$$

for every $t \in [a,b]$. Put

$$v(t) = (-1)^i \left(u_m'(t) - \sigma_i'(t)\right) - \frac{1}{m}$$

for $t \in [a,b]$ and $i \in \{1,2\}$. Then, by (1.2), (2.3),

$$v(a) = v(b), \ v'(a) \geqslant v'(b). \tag{5.5}$$

Let $v(t) > 0$ for every $t \in [a,b]$. Then, by (2.2) and (5.2), we have

$$\begin{aligned} v''(t) &= (-1)^i \left(u_m'''(t) - \sigma_i'''(t)\right) \\ &= (-1)^i \left(\frac{u_m'}{m} + f_m(t, u_m, u_m', u_m'') - \sigma_i'''(t)\right) \geqslant \frac{(-1)^i}{m} u_m' + \frac{c_1}{m} > \frac{1}{m^2} \end{aligned}$$

for a.e. $t \in (a,b)$. Thus $v'(b) - v'(a) > \frac{(b-a)}{m^2}$, which contradicts (5.5). Therefore there exists $t_0 \in (a,b)$ such that

$$v(t_0) \leqslant 0. \tag{5.6}$$

First, suppose that (5.4) is not satisfied on (t_0, b), i.e. there exists $t^* \in (t_0, b)$ such that

$$v(t^*) > 0. \tag{5.7}$$

Let $(\alpha, \beta) \subset (t_o, b)$ be the maximal interval containing t^* in which $v(t) > 0$. Then $v(\alpha) = 0$, $v'(\alpha) \geqslant 0$ and

$$v''(t) > \frac{1}{m^2} \text{ for a.e. } t \in [\alpha, \beta]. \tag{5.8}$$

If $\beta < b$, then $v(\beta) = 0$ and $v'(\beta) \leqslant 0$. On the other hand, by (5.8), $v'(\beta) > \frac{(\beta-\alpha)}{m^2} > 0$, and we get a contradiction. Therefore $\beta = b$ and, according to (5.5),

$$v(b) > 0, \ v'(b) > 0, \ v(a) > 0, \ v'(a) > 0.$$

Let $(a, a_0) \subset (a, t_0)$ be the maximal interval in which $v(t) > 0$. Analogously as above we can prove $a_0 = t_0$ and $v(t_0) > 0$, which contradicts (5.6). Consequently,

$$v(t) \leqslant 0 \text{ for every } t \in [t_0, b]. \tag{5.9}$$

In view of (5.5) and (5.9) we have $v(a) \leqslant 0$.

Now, suppose that (5.4) is not satisfied on (a, t_0), i.e. there exists $t^* \in (a, t_0)$ fulfilling (5.7), and let $(\alpha, \beta) \subset (a, t_0)$ be the maximal interval containing t^* in which $v(t) > 0$. Analogously as above we get $v(t_0) > 0$ which contradicts (5.6). Hence

$$v(t) \leqslant 0 \text{ for every } t \in [a, t_0]. \tag{5.10}$$

From (5.9) and (5.10) it follows that u'_m satisfies (5.4). Therefore

$$(5.11) \qquad |u'_m(t)| \leqslant c_1 + \frac{1}{m} \text{ for every } t \in [a,b].$$

Hence, by (1.2),

$$(5.12) \qquad |u_m(t)| \leqslant (b-a)(c_1 + \frac{1}{m}) \text{ for every } t \in [a,b].$$

Integrating (5.3) from t to a_0, where $a_0 \in (a,b)$ is such that $u''(a_0) = 0$, we get

$$(5.13) \qquad |u''_m(t)| \leqslant r_0 \text{ for every } t \in [a,b],$$

where $r_0 = (\frac{1}{m})(b-a)(c_1 + \frac{1}{m}) + (\frac{c_1}{m})(b-a)) + \int\limits_a^b h_0(t)\,dt$.

From (5.11), (5.12) and (5.13) it follows that the sequences $(u_m)_{m=1}^{\infty}$, $(u'_m)_{m=1}^{\infty}$, and $(u''_m)_{m=1}^{\infty}$ are uniformly bounded and equi-continuous on $[a,b]$, and by the Arzelà-Ascoli Theorem, without loss of generality, we may suppose that they are uniformly converging on $[a,b]$. By (5.2), (5.3) and (5.4), the function $u(t) = \lim\limits_{m\to\infty} u_m(t)$ on $[a,b]$ is a solution of BVP (1.1), (1.2) and satisfies (3.3). □

6. Proofs of theorems

Proof of Theorem 1. Without loss of generality we may suppose $c_1 > 0$. Let r^* be the constaant found by Lemma 3 for $r_1 = -c_1$, $r_2 = c_1$. Let us put

$$\rho_0 = r^* + c_0 + c_1 + c_2,$$

$$\chi(\rho_0, s) = \begin{cases} 1 \text{ for } 0 \leqslant s \leqslant \rho_0 \\ 2 - \dfrac{s}{\rho_0} \text{ for } \rho_0 < s < 2\rho_0 \\ 0 \text{ for } s \geqslant 2\rho_0, \end{cases}$$

$$(6.1) \qquad g(t,x,y,z) = \chi(\rho_0, |x| + |y| + |z|) f(t,x,y,z,) \text{ on } D,$$

and consider the equation

$$(6.2) \qquad u''' = g(t,u,u',u'').$$

Since $\max\{|\sigma_i(t)| + |\sigma'_i(t)| + |\sigma''_i(t)| : a \leqslant t \leqslant b\} < \rho_0$ for $i = 1,2$, σ_1 is a lower solution and σ_2 an upper solution of BVP (6.2), (1.2). Further, $|g(t,x,y,z)| \leqslant g^*(t)$ on D, where

$$g^*(t) = \sup\{|f(t,x,y,z)| : |x| + |y| + |z| \leqslant 2\rho_0\} \in L^1(J).$$

Thus, by Proposition, BVP (6.2), (1.2) has a solution u satisfying (3.3). Consequently, u fulfils (4.3) for $r_1 = -c_1$, $r_2 = c_1$.

According to (3.1), (6.2) we have

$$u''' \operatorname{sign} u'' = \chi\left(\rho_0, \sum_{i=0}^{2} |u^{(i)}(t)|\right) f(t, u, u', u'') \operatorname{sign} u''$$
$$\leqslant \omega(|u''|) g^{\frac{1}{p}}(t, u) h(u')(1 + |u''|)^{\frac{1}{q}}$$

for a.e. $t \in (a, b)$, $|u''(t)| \geqslant 1$. Therefore, by Lemma 3, u satisfies (4.5). Consequently, according to (1.2), (4.3), (4.5) we get

$$|u(t)| + |u'(t)| + |u''(t)| \leqslant \rho_0 \quad \text{for every } t \in J. \tag{6.3}$$

In view of (6.1)–(6.3), u is a solution of BVP (1.1), (1.2). □

Proof. of Theorem 2. Let u_1, u_2 be two solutions of (1.1), (1.2). Put $v = u_1 - u_2$. Then

$$v(c) = 0, \; v'(a) = v'(b), \; v''(a) = v''(b). \tag{6.4}$$

By (6.4) there exists $t_0 \in (a, b)$ such that

$$v''(t_0) = 0. \tag{6.5}$$

Now, suppose that $v'(t) > 0$ for every $t \in [a, b]$. Then $v(t) \cdot$ sign $(t - c) \geqslant 0$ for every $t \in [a, b]$, and (3.6) implies

$$v'''(t) + h(t)|v''(t)| > 0 \quad \text{for a.e. } t \in (a, b). \tag{6.6}$$

Inequality (6.6) can be written in the form

$$\left(\left(\exp \int_a^b \tilde{h}(s)\, ds\right) v''(t)\right)' > 0 \quad \text{for a.e. } t \in (a, b), \tag{6.7}$$

where $\tilde{h}(t) = h(t) \cdot \operatorname{sign} v''(t)$. Integrating (6.7) from a to t_0 and from t_0 to b we get $v''(a) < 0$ and $v''(b) > 0$, which contradicts (6.4). So there exists $t_1 \in (a, b)$ such that

$$v'(t_1) = 0. \tag{6.8}$$

Put $\sigma = \left(\int_a^b (v''')^2(t)\, dt\right)^{\frac{1}{2}}$. Then, by Lemma 1, $\|v''\|_{L^2(J)} \leqslant \sigma \cdot \frac{2(b-a)}{\pi}$, $\|v'\|_{L^2(J)} \leqslant \sigma(\frac{2(b-a)}{\pi})^2$, $\|v\|_{L^2(J)} \leqslant \sigma(\frac{2(b-a)}{\pi})^3$. Therefore we can find from (3.7)

$$\sigma \leqslant \sigma \left[\alpha(\frac{2(b-a)}{\pi})^3 + \beta(\frac{2(b-a)}{\pi})^2 + \gamma(\frac{2(b-a)}{\pi})\right].$$

Consequently, by (3.5), $\sigma = 0$. Thus $v(t) = 0$ for each $t \in [a, b]$. □

Proof of Theorem 3. Let v be the function from the proof of Theorem 2 and let us suppose that there exists $\bar{t} \in (a,b)$ such that $v'(\bar{t}) > 0$ and $(\alpha,\beta) \subset (a,b)$ is the maximal interval containing $\bar{t}$ with $v'(t) > 0$ for each $t \in (\alpha,\beta)$. From the first part of the previous proof we can obtain $(\alpha,\beta) \neq (a,b)$. 1. Let $\alpha > a$, $\beta < b$. In this case the inequality (6.7) is fulfilled on (α,β) and

$$v'(\alpha) = v'(\beta) = 0,\ v''(\alpha) \geqslant 0,\ v''(\beta) \leqslant 0. \tag{6.9}$$

Therefore there exists $a_1 \in (\alpha,\beta)$ such that $v''(a_1) = 0$. Integrating (6.7) from α to a_1 and from a_1 to β, we get $v''(\alpha) < 0$ and $v''(\beta) > 0$, respectively. This contradicts (6.9). 2. Let $\alpha > a$, $\beta = b$. Then (6.7) is satisfied on (α, b) and

$$v'(\alpha) = 0,\ v'(\beta) \geqslant 0,\ v''(\alpha) \geqslant 0. \tag{6.10}$$

If $v''(b) \leqslant 0$, then there exists $b_1 \in [\alpha, b]$ such that $v''(b_1) = 0$. Supposing $b_1 > \alpha$ and integrating (6.7) from α to b_1, we get $v''(\alpha) < 0$—a contradiction to (6.10). Analogously, supposing $b_1 < b$ and integrating (6.7) from b_1 to b, we get $v''(b) > 0$—a contradiction to (6.10). If $v''(b) > 0$, then from (6.4), (6.10) the inequalities

$$v''(a) > 0,\ v'(a) \geqslant 0 \tag{6.11}$$

follow. According to $v'(\alpha) = 0$, we have $a_2 \in (a,\alpha)$ such that $v'(a_2) > 0$, $v''(a_2) = 0$, $v'(t) > 0$ on (a, a_2). Thus (6.7) is also satisfied on (a, a_2) and integrating of (6.7) from a to a_2 we have $v''(a) < 0$ which contradicts (6.11). 3. The case $\alpha = a$ and $\beta < b$ can be proved similarly. Thus we have proved $v'(t) = 0$ on $[a,b]$ and, in view of (6.4), $v(t) = 0$ on $[a,b]$. The uniqueness is proved. □

References

[1] *A. R. Aftabizadeh, J. Wiener*: Existence and uniqueness theorems for third order boundary value problems, Rend. Sem. Mat. Univ. Padova *75* (1986), 130–141.

[2] *A. R. Aftabizadeh, J. M. Xu, C. P. Gupta*: Existence and uniqueness theorems for three-point boundary value problems, SIAM J. Math. Anal. *20* (1989), 716–726.

[3] *R. P. Agarwal*: On boundary value problems for $y''' = f(x,y,y',y'')$, Bull. of the Inst. Math. Acad. Sinica *12* (1984), 153–157.

[4] *R. P. Agarwal*: Existence-uniqueness and iterative methods for third order boundary value problems, J. Comp. Anal. Math., to appear.

[5] *J. Andres*: On a boundary value problem for $x''' = f(t,x,x',x'')$, Acta UPO, ser. mat. *27* (1988), 289–298.

[6] *D. Barr, T. Sherman*: Existence and uniqueness of solutions of three-point boundary value problems, J. Diff. Eqs. *13* (1973), 197–212.

[7] *S. A. Bespalova, J. A. Klokov*: A three-point boundary value problem for a third-order nonlinear ordinary differential equation, Diff. uravn. *12* (1976), 963–970. (In Russian.)

[8] *G. Carristi*: A three-point boundary value problem for a third order differential equation, Boll. Um. Mat. Ital., C 4 *1* (1985), 259–269.
[9] *K. M. Das, B. S. Lalli*: Boundary value problems for $y''' = f(x, y, y', y'')$, J. Math. Anal. Appl. *81* (1981), 300–307.
[10] *A. Granas, R. Guenther, J. Lee*: Nonlinear Boundary Value Problems for Ordinary Differential Equatins, Polish Acad. of Sciences, 1985.
[11] *M. Greguš*: Third Order Linear Boundary Value Problems, D. Reidel Publishing Co., 1987.
[12] *C. P. Gupta*: On a third-order three-point boundary value problem at resonance, Diff. Int. Equations *2* (1989), 1–12.
[13] *G. H. Hardy, J. E. Littlewood, G. Polya*: Inequalities, IL, Moscow, 1970. (In Russian.)
[14] *J. Henderson*: Best interval lengths for boundary value problems for third order Lipschitz equations, SIAM J. Math Anal. *18* (1987), 293–305.
[15] *S. Hu, V. Lakshmikantham*: Periodic boundary value problems for integro-differential equations of Volterra type, Nonlinear Anal. *10* (1986), 1203–1208.
[16] *I. T. Kiguradze*: Some Singular Boundary Value Problems for Ordinary Differential Equations, Univ. Press, Tbilisi, 1975. (In Russian.)
[17] *E. Lepina, A. Lepin*: Existence of a solution of the three-point BVP for a nonlinear third-order ordinary differential equation, Latv. M. E. *4* (1986), 247–256. (In Russian.)
[18] *E. Lepina, A. Lepin*: Necessary and sufficient conditions for existence of a solution of a three-point BVP for a nonlinear third order differential equation, Latv. M. E. *8* (1970), 149–154. (In Russian.)
[19] *K. N. Murthy, D. R. K. S. Rao*: On existence and uniqueness of solutions of two and three point boundary value problems, Bull. Calcuta Math. Soc. *73,3* (1981), 164–172.
[20] *K. N. Murthy, B. D. C. N. Prasad*: Three-point boundary value problems, existence and uniqueness, Yokohama Math. J. *29* (1981), 101–105.
[21] *K. N. Murthy, B. D. C. N. Prasad*: Application of Lyapunov theory to three-point boundary value problems, J. Math. Phys. Sci. *19* (1985), 225–234.
[22] *L. I. Pospelov*: Necessary and sufficient conditions for existence of a solution for some BVPs for the third order nonlinear ordinary differential equation, Latv. M. E. *8* (1970), 205–213. (In Russian.)
[23] *D. J. O'Regan*: Topological transversality: Applications to third order boundary value problems, SIAM J. Math. Anal. *18* (1987), 630–641.
[24] *J. Rusnák*: A three-point boundary value problem for third order differential equations, Math. Slovaca *33* (1983), 307–320.
[25] *N. I. Vasiljev, J. A. Klokov*: Elements of the Theory of Boundary Value Problems for Ordinary Differential Equations, Zinatne, Riga, 1978. (In Russian.)

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