Eugene M. Norris

Relative Act Ideals

Matematický časopis, Vol. 24 (1974), No. 3, 195--202

Persistent URL: http://dml.cz/dmlcz/126976

Mat. čas. 24, 1974, No. 3, 195—202

RELATIVE ACT IDEALS

EUGENE M. NORRIS

To Professor Alexander Doniphan Wallace

Introduction

An *act* is such a continuous function $m: S \times X \to X$ that S is a topological semigroup, X is a topological space and $m(s, m(t, x)) = m(st, x)$ identically. We write $AB = \{m(s, x): s \in A \text{ and } x \in B\}$ for subsets $A \subseteq S$ and $B \subseteq X$, noting that $AB = \emptyset$ ($\emptyset$ denotes the empty set) if either $A = \emptyset$ or $B = \emptyset$. We denote singleton subsets $\{x\}$ by x, so that $m(s, x)$ is written sx; context will distinguish between multiplication in S and the values of m. All spaces are assumed to be Hausdorff. We use A^*, A° and $A \backslash B$ to denote the closure of A, the interior of A, and the set whose elements are in A but not in B. An act $m: S \times X \to X$ will be denoted by (S, X), as no confusion can arise here; m may be referred to as the *action* of S upon X. We write $(A, B) \leq \leq (C, D)$ just in case $A \subseteq C \subseteq S$ and $B \subseteq D \subseteq X$. The partial order so defined is that of the product of the lattices $\mathscr{P}(S)$ and $\mathscr{P}(X)$ of subsets of S and of X respectively. We recall that the lattice operations $\vee$ and $\wedge$ are given by coordinatewise unions (for $\vee$) and coordinatewise intersections (for $\wedge$) respectively. A *subact* of an act (S, X) is a pair $(T, Y) \leqslant (S, X)$ such that T is a subsemigroup of S and $TY \subseteq Y$. *Throughout this paper,* (T, Y) *is supposed to be a subact of an act* (S, X). For any pair $(A, B) \leqslant (S, X)$ we call A its *input* and B its *state space.* We shall call (A, B) *compact* if both A and B are compact sets, and we extend this convention to include other topological properties, notably *open* and *closed.* A reference for the algebraic theory of semigroups is [2], while [7] serves for topological semigroups. This paper extends some results from the author's dissertation [6]. The encouragement of Professor A. R. Bednarek, the support of the National Science Foundation (GP 6505) and the inspiration of Professor A. D. Wallace are gratefully acknowledged. The author wishes to thank the referee for his invaluable suggestions. The alternate proof of Proposition 4 (i) is due to the referee.

Relative Ideals

A pair of nonvoid sets $(A, B) \leq (S, X)$ is a *left* (respectively, *right*) (T, Y)-*ideal* of (S, X) if A is a left (right) T-ideal of S and $TB \subseteq B$ (respectively, $AY \subseteq B$). The pair (A, B) is a (T, Y)-*ideal* if it is both a left and right (T, Y)-ideal.

The notion of T-ideal (left, right) is due to A. D. Wallace [8, 9]. If a semigroup S is considered as an act (S, S), the action being just semigroup multiplication, then each left or right T-ideal A of S corresponds to the left or right (T, T)-ideal (A, A) of (S, S). The reader is invited to make a comparison with the (H_1, H_2)-ideals of Hrmová [3]. (T, Y)-ideals will be referred to generically as *relative ideals* of (S, X). Of course, each ideal of the semigroup T is a T-ideal, and a T-ideal contained in T is just an ideal of T.

We record first an elementary result which is needed several times in the sequel. Its proof, which we omit, rests only upon the continuity of the action.

Lemma 1. *If* $(A, B) \leqslant (S, X)$ *then* $A^*B^* \subseteq (AB)^*$.

Applying lemma 1 and the definitions of left and right (T, Y)-ideals gives an immediate proof of the next result.

Corollary 2. *If* (A, B) *is a left* (*right*) (T, Y)-*ideal, so is* (A^*, B^*).

The following theorem, stated here in our own notation, is due to A. D. Wallace and is well known to topologists. A proof may be found in [4].

Theorem 3. (Wallace). *Suppose* W *is an open subset of* X *and* $(A, B) \leqslant (S, X)$ *is such a compact pair that* $AB \subseteq W$. *Then there exists such an open pair* $(U, V) \geqslant (A, B)$ *that* $UV \subseteq W$.

Minimal Relative Ideals

A (T, Y)-ideal (A, B) (left, right) is said to be *minimal* if it is minimal in the partial order $\leqslant$, i.e. if, whenever (C, D) is a (T, Y)-ideal (left, right) and $(C, D) \leqslant (A, B)$, then $(C, D) = (A, B)$.

Proposition 4. (i) *If* S *contains a minimal left* T-*ideal then* (S, X) *contains a minimal left* (T, Y)-*ideal; if, further,* T *contains a minimal left* T-*ideal* L *then the minimal left* (T, Y)-*ideals are precisely the class of all* (Ls, Lx) *where* $s \in S$ *and* $x \in X$.

(ii) *If* S *contains a minimal right* T-*ideal then* (S, X) *contains a minimal right* (T, Y)-*ideal;* (A, B) *is a minimal right* (T, Y)-*ideal if and only if* A *is a minimal right* T-*ideal and* $(A, B) = (A, aY)$ *for every* $a \in A$.

Proof. If L and L_1 are any left T-ideals of S and $x \in X$, then certainly

(L_1, Lx) is a left (T, Y)-ideal. We shall see from the following argument, due in its original form to A. D. Wallace, that (L_1, Lx) is minimal, provided that L and L_1 are minimal. If (A, B) is a left (T, Y)-ideal and $(A, B) \leqslant (L_1, Lx)$ then from the hypotheses that $A \subseteq L_1$ and A is a left T-ideal follows the conclusion that $A = L_1$. Now $TB \subseteq B \subseteq Lx$, so the set $N = \{s \in L : sx \in B\}$ is nonvoid. We compute that $(TN)x = T(Nx) \subseteq TB \subseteq B$, so that $TN \subseteq N$; since $N \subseteq L$ then $N = L$ by the minimality of L, which proves that $Lx \subseteq B$. Hence it follows that (L_1, Lx) is a minimal left (T, Y)-ideal. Now suppose that $L \subseteq T$, i.e. that L is a minimal left ideal of T. If (L_1, B) is any minimal left (T, Y)-ideal then L_1 is clearly a minimal left T-ideal. If b is any element of B then $(L_1, Lb) \leqslant (L_1, Tb) \leqslant (L_1, B)$. (L_1, Lb) is a left (T, Y)-ideal, since $T(Lb) = (TL)b \subseteq Lb$, so that $(L_1, B) = (L_1, Lb)$. Since $L_1 = Ls$ for some $s \in S$, as is readily verified, part (i) is proved.

To see part (ii), suppose first that A is a minimal right T-ideal; it is immediate that (A, AY) is a right (T, Y)-ideal which is minimal. If $a \in A$, then aT is a right T-ideal which is contained in A, and so $aT = A$. Then, $AY = (aT)Y = a(TY) \subseteq aY \subseteq AY$, i.e. $AY = aY$. Hence each pair (A, aY) is a minimal right (T, Y)-ideal. Now suppose that (A, B) is any minimal right (T, Y)-ideal. Necessarily A is a minimal right T-ideal, for if A_1 is a right T-ideal contained in A, then, since $(A_1, B) \leqslant (A, B)$ and (A_1, B) is a right (T, Y)-ideal, it follows that $(A_1, B) = (A, B)$, i.e. that $A_1 = A$, proving that A is a minimal right T-ideal. Furthermore, $(A, AY) \leqslant (A, B)$ and (A, AY) is a right (T, Y)-ideal, so $(A, B) = (A, AY) = (A, aY)$ for any $a \in A$ by the previous argument. The proof of (ii) is now clear.

We digress briefly to indicate how 4(i) follows from Theorem 2.2 and Lemma 2.1 of [3]. Suppose (S, X) is an act, and $X' = \{1\} \times X$. We let S act on X' as follows: $m'(s, (1, x)) = (1, m(s, x)) = (1, sx)$ for all $(s, (1, x)) \in S \times X'$. It is clear that (S, X', m') is an act. Let $H = S \cup X' \cup \{0\}$, where $0 \notin S \cup X'$. Define on H the binary operation

$$a \circ b = \begin{cases} ab, \text{ if } a, b \in S \text{ (semigroup product in } S) \\ m'(a, b), \text{ if } a \in S \text{ and } b \in X' \\ 0, \text{ otherwise} \end{cases}$$

Then $(H, \circ)$ is easily seen to be a semigroup and S is a subsemigroup of H. If $(A, B) \leqslant (S, X)$, then (A, B) is a left (T, X)-ideal of (S, Y) if and only if (A, B') is a pair of left T-ideals of H, where $B' = \{1\} \times B$.

It is well known [8] that if T is a compact subsemigroup of S then each left (right) T-ideal contains a minimal such. This leads immediately to a proof of the next result.

Corollary 5. *If T is compact, then each (T, Y)-ideal (left, right) contains a minimal such.*

The next result is concerned only with (S, X)-ideals.

Theorem 6. *Let $\mathscr{L}$ (respectively, $\mathscr{R}$) denote the collection of all minimal left (right) (S, X)-ideals and suppose both $\mathscr{L}$ and $\mathscr{R}$ are nonvoid. Then (S, X) has a unigue minimal (S, X)-ideal, namely, (K, KX) where K is the minimal ideal of S; furthermore, $\vee\mathscr{L} = (K, KX) = \vee\mathscr{R}$.*

Proof. The existence of a minimal left and a minimal right S-ideal imply the existence of the minimal ideal K of S as is well known [7]. We first show that (K, KX) is the minimal (S, X)-ideal. That (K, KX) is an (S, X)-ideal is clear, since $S(KX) = (SK)X \subseteq KX$. Now suppose that (A, B) is any (S, X)-ideal; then A is an S-ideal and therefore contains K. Furthermore, $KX \subseteq AX \subseteq SB \cup AX \subseteq B$, so that $(K, KX) \leqslant (A, B)$. It follows that (K, KX) is the unique minimal (S, X)-ideal.

We prove next the assertion about $\mathscr{L}$. It is well known in the theory of semigroups [2] that K is the union of all minimal left S-ideals (any two of which are disjoint) and all minimal right S-ideals. Hence, letting $(A, B) = \vee\mathscr{L}$, we have that $A = K$. This fact and Proposition 4(i) imply together that $(A, B) \leqslant (K, KX)$. To see that $KX \subseteq B$, suppose that $y = kx$ for some $k \in K$ and that L is the minimal left S-ideal containing K; then $y \in Lx$ and $(L, Lx) \in \mathscr{L}$ by Proposition 4(i), so that $Lx \subseteq B$. Therefore $KX \subseteq B$ and hence we see that $(K, KX) = \vee\mathscr{L}$.

The assertion concerning $\mathscr{R}$ is easily seen to hold, for if $\vee\mathscr{R} = (C, D)$ then, as mentioned above, $C = K$. The union of the collection of all sets aX where a belongs to some minimal right S-ideal is seen to be KX and, by virtue of Proposition 4(ii) is also D. Hence $(C, D) = (K, KX)$.

Corollary 7. *If (S, X) is compact and connected, then (K, KX) is also compact and connected.*

Proof. Since S is compact, the minimal ideal K exists. K is compact and connected since S is [7]. KX is the continuous image under the action of the compact connected set $K \times X$.

Maximal Relative Ideals

A (T, Y)-ideal is *proper* if both its input and state space are proper subsets of S and of X respectively. It is *maximal proper* if it is proper and is contained in no other proper (T, Y)-ideal. We extend the methods of Koch and Wallace [5] to prove the following result.

Theorem 8. *If (T, Y) is closed and (S, X) is compact, then each proper (T, Y)-ideal is contained in a maximal proper (T, Y)-ideal and each maximal proper (T, Y)-ideal is open.*

We shall defer the proof of this theorem until we have established some notation and proved a lemma.

For any pair (A, B) satisfying $(\emptyset, \emptyset) < (A, B) \leqslant (S, X)$ let $(\hat{A}, \hat{B}) = \vee \{(P, Q) : (P, Q)$ is a (T, Y)-ideal and $(P, Q) \leqslant (A, B)\}$. Clearly, $(\hat{A}, \hat{B}) \leqslant (A, B)$; if (A, B) contains no (T, Y)-ideal then $(\hat{A}, \hat{B}) = (\emptyset, \emptyset)$, otherwise $(\hat{A}, \hat{B})$ is the largest (T, Y)-ideal contained in (A, B). We omit the proof of these simple remarks, observing only that the join of any collection of (T, Y)-ideals is again a (T, Y)-ideal.

We denote by $J_1(A)$ the set $A \cup AT \cup TA \cup TAT$, i.e. the T-ideal generated by A, and we set $J_2(A, B) = B \cup TB \cup J_1(A)Y$. For brevity, put $J(A, B) = (J_1(A), J_2(A, B))$. Of course, $J(A, B) = (\emptyset, \emptyset)$ if and only if $(A, B) = (\emptyset, \emptyset)$. In the nonvoid case, $J(A, B)$ is a (T, Y)-ideal which is contained in any (T, Y)-ideal containing (A, B).

Lemma 9. *Let* $(A, B) \leqslant (S, X)$. Then

(i) *If (A, B) is closed, so is $(\hat{A}, \hat{B})$.*

(ii) *If (S, X) is compact, (T, Y) is closed and (A, B) is open, then $(\hat{A}, \hat{B})$ is open.*

Proof. Both assertions certainly hold if $(A, B) = (\emptyset, \emptyset)$, so we may assume without loss of generality that (A, B) contains at least one (T, Y)-ideal. If (A, B) is closed, then $(\hat{A}^*, \hat{B}^*) \leqslant (A^*, B^*) = (A, B)$, so that $(\hat{A}^*, \hat{B}^*)$ is a (T, Y)-ideal contained in (A, B); therefore, $(\hat{A}^*, \hat{B}^*) \leqslant (\hat{A}, \hat{B})$, proving that $(\hat{A}, \hat{B})$ is closed. To see (ii), let $(t, x) \in \hat{A} \times \hat{B}$. Necessarily there is a (T, Y)-ideal $(P, Q) \leqslant (A, B)$ with $t \in P$ and $x \in Q$; indeed, from the definition of $(\hat{A}, \hat{B})$ there are (T, Y)-ideals (P_i, Q_i), $i = 1, 2$, so that $t \in P_1$ and $x \in Q_2$. But $(P_1 \cup P_2, Q_1 \cup Q_2)$ is also a (T, Y)-ideal contained in (A, B). By direct computation we can see that $J(t, x) \leqslant (P, Q) \leqslant (A, B)$, so that $J_1(t) \subseteq A$ and $J_2(t, x) \subseteq B$. Since A and B are open sets by hypothesis, we may apply Theorem 3 to each of the compact sets whose union is $J_1(t)$ and to each of the compact sets whose union is $J_2(t, x)$ to obtain a pair (U, V) of open neighborhoods of t and of x with the property that $J(U, V) \leqslant (A, B)$. Since $J(U, V)$ is a (T, Y)-ideal then $(t, x) \leqslant (U, V) \leqslant J(U, V) \leqslant (\hat{A}, \hat{B})$, so that $t \in U \subseteq \hat{A}$ and $x \in V \subseteq \hat{B}$, proving (ii).

Proof of Theorem 8. The theorem requires Zorn's Lemma for its proof. If (P, Q) is a proper (T, Y)-ideal and $(s, x) \in S \backslash P \times X \backslash Q$ then $(P, Q) \leqslant (\widehat{S/s}, \widehat{X/x})$ which is an open proper (T, Y)-ideal, in light of lemma 9; hence the collection $(\mathscr{B}, \leqslant)$ of all open proper (T, Y)-ideals containing (P, Q) is nonvoid. Suppose $\mathscr{C} = \{(C_\lambda, D_\lambda) : \lambda \in \Lambda\}$ is a maximal linearly ordered subcollection of $\mathscr{B}$ and let $(A, B) = \vee\mathscr{C}$. (A, B) is open, being the join of open pairs, and is a (T, Y)-ideal containing (P, Q). If (A, B) were not proper then either $A = S$ or $B = X$, and the compactness of S or of X, as the case may be, would imply that S (or X) is the union of a finite linearly ordered collection of proper open subsets, which is absurd. Hence (A, B) is a proper (T, Y)-ideal containing (P, Q); it is clearly a maximal such.

The relation between maximal proper T-ideals and maximal proper (T, Y)-ideals is not as clear as in the case of minimal ideals. Examples show that a maximal proper T-ideal M need not be the input to any proper (T, Y)-ideal, even if S, T, X and Y are all compact, for it can happen that $MY = X$. The following results illuminate the matter somewhat. For convenience, we adopt the terminology of Paalman-de Miranda [7] and say that (S, X) has the *maximal property* with respect to a subact (T, Y) if each proper (T, Y)-ideal is contained in a maximal proper (T, Y)-ideal.

Proposition 10. *If (S, X) has the maximal property relative to (T, Y) and M is a maximal proper T-ideal, then M is the input to a maximal proper (T, Y)-ideal if and only if $MY \neq X$.*

Proof. If (M, N) is a maximal proper (T, Y)-ideal then $MY \subseteq N \neq X$, so the necessity of the condition is clear. For the converse, one need only observe that if $MY \neq X$, then (M, MY) is a proper (T, Y)-ideal and is therefore contained in a maximal proper (T, Y)-ideal (P, Q). But then $M \subseteq P$ and P is a proper T-ideal, so the maximality of M implies that $M = P$. Hence (M, Q) is a maximal proper (T, Y)-ideal.

For the sake of brevity, we shall use the following notation of A. D. Wallace [8, 9]: if $A \subseteq S$ and if B and C are subsets of X, let $A^{[-1]}B = \{x \in X : Ax \subseteq B\}$ and $BC^{[-1]} = \{s \in S : sC \subseteq B\}$.

Proposition 11. *Suppose (A, B) is a left (respectively, right) (T, Y)-ideal. Then*

(i) *$BY^{[-1]}$ is a left (right) T-ideal*
(ii) *$B \subseteq T^{[-1]}B$ (respectively, $A \subseteq BY^{[-1]}$).*

Proof. If (A, B) is a left (T, Y)-ideal and if $s \in BY^{[-1]}$ and $t \in T$, then $(ts)Y = t(sY) \subseteq tB \subseteq TB \subseteq B$. Hence $ts \in BY^{[-1]}$, so $BY^{[-1]}$ is a left T-ideal and $B \subseteq T^{[-1]}B$. If (A, B) is a right (T, Y)-ideal, $s \in BY^{[-1]}$ and $t \in T$, then $(st)Y = s(tY) \subseteq s(TY) \subseteq sY \subseteq B$, so that $BY^{[-1]}$ is a right T-ideal. Of course, $AY \subseteq B$, so that $A \subseteq BY^{[-1]}$.

Proposition 12. *Suppose (M, N) is a maximal proper (T, Y)-ideal. Then*

(i) *Either $NY^{[-1]} = M$ or $NY^{[-1]} = S$*

(ii) *Either* $T^{[-1]}N = N$ *or* $T^{[-1]}N = X$
(iii) *If* $NY^{[-1]} = S$ *then* M *is a maximal proper* T*-ideal*
(iv) *If* $T^{[-1]}N = X$ *then* N *is a maximal proper subset of* X *for which* $TN \subseteq N$.

Proof. Since (M, N) is both a left and a right (T, Y)-ideal, Proposition 11 implies that $NY^{[-1]}$ is a T-ideal containing M. Now, $(NY^{[-1]}, N)$ is a (T, Y)-ideal, for $(NY^{[-1]})Y \subseteq TN \subseteq N$, and since $(M, N) \leqslant (NY^{[-1]}, N)$, (i) follows from the maximality of (M, N). Part (ii) follows similarly since $(M, T^{[-1]}N)$ is easily seen to be a (T, Y)-ideal containing (M, N). Next, suppose M is contained in a T-ideal I, and suppose that $NY^{[-1]} = S$. Then $IY \subseteq N$ and hence (I, N) is a (T, Y)-ideal containing (M, N); the maximality of (M, N) then implies that either $M = I$ or $I = S$. Hence M is a maximal proper T-ideal, proving (iii). To see (iv), suppose $N \subseteq Q \subseteq X$ and $TQ \subseteq Q$. Since $T^{[-1]}N = X$ then $TQ \subseteq N$, which implies that (M, Q) is a proper (T, Y)-ideal containing (M, N). Hence either $Q = N$ or $Q = X$.

We conclude with a generalization of a result of Koch and Wallace [5], as an application of these ideas.

Proposition 13. *If* (T, Y) *is a closed subact of a compact act* (S, X) *and if* (M, N) *is such a maximal proper* (T, Y)*-ideal that the idempotents of* T *are contained in* M, *then* $TY \subseteq N$.

Proof. If, to the contrary, there is some $(t, y) \in T \times Y$ such that $ty \notin N$, then necessarily $y \notin N$. Since $(M, N \cup Ty)$ is a (T, Y)-ideal property containing (M, N) then $N \cup Ty = X$ necessarily. Hence $y \in Ty$, so that $y = sy$ for some $s \in T$, and hence the closed set $yy^{[-1]}$ contains the set $\{s^n : n \geqslant 1\}$ and therefore contains its closure $\Gamma(s)$, a compact semigroup containing an idempotent e, which satisfies $y = ey$; but $ey \in My \subseteq N$, implying $y \in N$.

REFERENCES

[1] BEDNAREK, A. R. — NORRIS, E. M.: Congruences and Ideals in Machines. Rev. roumaine math. pures et app., *15*, 1970, 193—199.
[2] CLIFFORD, A. H. — PRESTON, G.B.: Algebraic Theory of Semigroups. volume I, Mathematical Surveys number 7, Amer. Math. Soc.
[3] HRMOVÁ, R.: Relative Ideals in Semigroups. Mat. Čas., *17*, 1967, 206—223.
[4] KELLEY, J. L.: General Topology. D. Van Nostrand Company, New York, 1955.
[5] KOCH, R. J. — WALLACE, A. D.: Maximal Ideals in Compact Semigroups. Duke Math. J. *21*, 1954, 681—685.
[6] NORRIS, E. M.: Some Structure Theorems for Topological Machines. Dissertation, University of Florida, 1969.
[7] PAALMAN- DE MIRANDA, A. B.: Topological Semigroups. Mathematisch Centrum, Amsterdam, 1964.

[8] WALLACE, A. D.: Relative Ideals in Semigroups I. Colloq. math. *9*, 1962, 55–61,
[9] WALLACE, A. D.: Relative Ideals in Semigroups II. Acta math. scient. hung. *14*. 1963, 137–148.

Received September 3, 1969
in revised form November 14, 1973

University of South Carolina
Columbia, South Carolina
United States