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Czechoslovak Mathematical Journal, Vol. 50 (2000), No. 1, 15–24

Czechoslovak Mathematical Journal, 50 (125) (2000), 15–24

THE IMPACT OF UNBOUNDED SWINGS OF THE FORCING TERM ON THE ASYMPTOTIC BEHAVIOR OF FUNCTIONAL EQUATIONS

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(Received May 17, 1996)

Abstract. Necessary and sufficient conditions have been found to force all solutions of the equation

$$(r(t)y'(t))^{(n-1)} + a(t)h(y(g(t))) = f(t),$$

to behave in peculiar ways. These results are then extended to the elliptic equation

$$|x|^{p-1}\Delta y(|x|) + a(|x|)h(y(g(|x|))) = f(|x|)$$

where Δ is the Laplace operator and $p \geqslant 3$ is an integer.

Keywords: oscillatory, nonoscillatory, exterior domain, elliptic, functional equation

MSC 2000: 34K25, 35B40

1. Introduction

Recently this author [14] studied the functional equation

$$(r(t)y'(t))^{(n-1)} + a(t)h(y(g(t))) = f(t), \tag{1}$$

for some integer $n > 0$. In this work, we had found conditions subject to which *all* oscillatory solutions of equation (1) approach zero as $t \to \infty$. Some of these conditions were somewhat weakened to achieve necessity and sufficiency, especially when $a(t) > 0$. Our work in [14] had improved upon our earlier work in [9, 13]. In fact, in our work in [9, 13], we had showed that subject to

$$\int^{\infty} t^{n-2}|a(t)|\,\mathrm{d}t < \infty, \tag{2}$$

$$\int^{\infty} t^{n-2}|f(t)|\,\mathrm{d}t < \infty, \tag{3}$$

and

$$\frac{t^{n-k}}{r(t)} \leqslant Q, \tag{4}$$

where $Q > 0$, $0 \leqslant k < 1$, for $t \in [T, \infty)$, $T > 0$, all oscillatory solutions of equation (1) tend to zero as $t \to \infty$. We pursued this study further, and in [16], we strengthened conditions (2) and (3) to require

$$\int^{\infty} |a(t)| \, \mathrm{d}t < \infty, \tag{5}$$

$$\int^{\infty} |f(t)| \, \mathrm{d}t < \infty, \tag{6}$$

and (4). It was observed, via an example in [16] that the condition on the constant k, namely

$$0 \leqslant k < 1 \tag{7}$$

could not be weakened. In other words, subject to (5) and (6), it was not possible to have $k \geqslant 1$ in order to achieve the same results.

Our main goal in this work is twofold:

First we build on the condition (7) and obtain results which force all solutions of equation (1) to be *oscillatory* and *unbounded* as $t \to \infty$. Secondly we apply these results to the oscillatory behavior of the elliptic equation

$$\Delta y(|x|) + \frac{1}{|x|^{p-1}} a(|x|) h(y(g(|x|))) = \frac{1}{|x|^{p-1}} f(|x|). \tag{8}$$

In equation (8), Δ is the Laplace operator

$$\Delta y = \sum_{i=1}^{p} \frac{\delta^{2y}}{\delta x_i^2} \tag{9}$$

and $p \geqslant 3$, an integer. Elliptic equations such as (8) are natural extensions of certain forms of ordinary differential equations.

For related results also see Chen [1], Hammett [3], Kusano and Onose [4, 5], and this author [10–13]. For oscillation theorems related to integral equations see this author [15]. Our work here complements the work of these authors.

2. Definitions and assumptions

Unless otherwise stated, the following assumptions and definitions apply throughout this work:

(i) $g(t)$, $r(t)$, $a(t)$, $f(t)$, and $h(t)$ are $\mathbb{R} \to \mathbb{R}$ and continuous, $\mathbb{R}$ being the real line;

(ii) $r(t) > 0$ on $\mathbb{R}^+$ where $\mathbb{R}^+$ is the positive half real line for $t \geqslant N > 0$. The constant N is now fixed for the rest of this work;

(iii) $th(t) > 0$, and there exists an $m > 0$ such that

$$\frac{h(t)}{t} \leqslant m, \quad t \geqslant N; \tag{10}$$

(iv) $0 < g(t) \leqslant t$, $g(t) \to \infty$ as $t \to \infty$.

A function continuous on $\mathbb{R}$ is said to be oscillatory on $\mathbb{R}^+$ if it has arbitrarily large zeros on $\mathbb{R}^+$. Otherwise it is called nonoscillatory.

The preceding conditions guarantee that all solutions of equation (1) can be continuously extended to all of $\mathbb{R}^+$. This fact was observed in Theorem 2.1 of [14]. From this point on, the term "solution" only applies to a solution which is continuously extendable for all $t \geqslant N$.

3. Main results

The following Lemma 1 is Theorem 3.1 in [16].

Lemma 1. *In addition to* (i) *through* (iv), *suppose*

$$\int^{\infty} |a(t)| \, \mathrm{d}t < \infty, \tag{11}$$

$$\int^{\infty} |f(t)| \, \mathrm{d}t < \infty \tag{12}$$

and there exists a constant $Q > 0$ such that

$$\frac{1}{r(t)} \leqslant \frac{Q}{t^{n-k}}, \quad 0 \leqslant k < 1 \tag{13}$$

for $t \geqslant N$. Then all oscillatory solutions of equation (1) *approach zero as $t \to \infty$.*

Proof. Follows from Theorem 3.1 of [16]. □

Theorem 1. *Suppose assumptions* (i) *through* (iv) *and conditions* (11) *and* (13) *of Lemma* 1 *hold. Further suppose that*

$$\limsup_{t\to\infty} \int^t \frac{1}{r(s)} \int^s \frac{(s-x)^{n-2}}{(n-2)!} f(x)\,\mathrm{d}x\,\mathrm{d}s = \infty \tag{14}$$

and

$$\liminf_{t\to\infty} \int^t \frac{1}{r(s)} \int^s \frac{(s-x)^{n-2}}{(n-2)!} f(x)\,\mathrm{d}x\,\mathrm{d}s = -\infty. \tag{15}$$

Let $y(t)$ be any solution of equation (1). *Then*

$$\limsup_{t\to\infty} |y(t)| = \infty. \tag{16}$$

P r o o f. Let $t_0 > N$ and fixed. Suppose $S_0 > t_0$ is sufficiently large so that $g(t) \geqslant t_0$ for $t \geqslant S_0$. On repeated integration from equation (1), we obtain

$$\begin{aligned} r(t)y'(t) = {} & \sum_{i=1}^{n-1} C_i (t-t_0)^{i-1} \\ & - \int_{t_0}^t \frac{(t-x)^{n-2}}{(n-2)!} a(x)h(y(g(x)))\,\mathrm{d}x \\ & + \int_{t_0}^t \frac{(t-x)^{n-2}}{(n-2)!} f(x)\,\mathrm{d}x \end{aligned} \tag{17}$$

where

$$C_i = \frac{(r(t_0)y'(t_0))^{(i-1)}}{(i-1)!}, \qquad i = 1, 2, \ldots, n-1.$$

Equation (17) yields

$$\begin{aligned} y(g(t)) = y(g(t_0)) & + \sum_{i=1}^{n-1} C_i \int_{t_0}^{g(t)} \frac{(s-t_0)^{i-1}}{r(s)}\,\mathrm{d}s \\ & - \int_{t_0}^{g(t)} \frac{1}{r(s)} \int_{t_0}^s \frac{(s-x)^{n-2}a(x)h(y(g(x)))}{(n-2)!}\,\mathrm{d}x\,\mathrm{d}s \\ & + \int_{t_0}^{g(t)} \frac{1}{r(s)} \int_{t_0}^s \frac{(s-x)^{n-2}f(x)}{(n-2)!}\,\mathrm{d}x\,\mathrm{d}s. \end{aligned} \tag{18}$$

Suppose to the contrary that $y(t)$ is bounded as $t \to \infty$. Since condition (13) of Lemma 1 holds, the first two terms on the right side of (18) are finite as $t \to \infty$. As for the third term, we notice that

$$\left|\int_{t_0}^{g(t)} \frac{1}{r(s)} \int_{t_0}^{s} \frac{(s-x)^{n-2} a(x) h(y(g(x)))}{(n-2)!} \, \mathrm{d}x \, \mathrm{d}s\right| \leqslant Qm \int_{t_0}^{s} \int_{t_0}^{t} \frac{(s-x)^{n-2} |a(x)| |y(g(x))|}{(n-2)! s^{n-k}} \, \mathrm{d}x \, \mathrm{d}s. \tag{19}$$

Changing the order of integration in the right side of (19) and using the fact that $y(t)$ is bounded we get

$$\left|\int_{t_0}^{g(t)} \frac{1}{r(s)} \int_{t_0}^{s} \frac{(s-x)^{n-2} a(x) h(y(g(x)))}{(n-2)!} \, \mathrm{d}x \, \mathrm{d}s\right| \leqslant M_0 \int_{t_0}^{t} \left(\int_{x}^{t} \frac{1}{s^{2-k}} \, \mathrm{d}s\right) |a(x)| \, \mathrm{d}x < \infty, \quad \text{since } 0 \leqslant k < 1;$$

where

$$M_0 = Qmd, \ |y(t)| \leqslant d \quad \text{for } t \geqslant t_0$$

for some $d > 0$. Thus the third term on the right side of (18) remains finite as $t \to \infty$. Since the last integral on the right side of (18) swings between $-\infty$ and ∞ and the left side of (18) remains finite as $t \to \infty$, we run into a contradiction. This completes the prof of Theorem 1.

Remark 1. Since $0 \leqslant k < 1$ in condition (13), which in accordance with our work in [16] cannot be weakened, it is not clear from the preceding proof if conditions (14) and (15) could be simply replaced by requiring

$$\int^{\infty} |f(t)| \, \mathrm{d}t = \infty. \tag{20}$$

A partial answer to this query is provided by Theorem 3 later. Our next theorem gives sufficient conditions for equation (1) to be oscillatory, that is, all solutions of equation (1) are oscillatory. □

Theorem 2. *Suppose assumptions* (i) *through* (iv) *and condition* (13) *of Lemma* 1 *hold. Further suppose that* $a(t) \geqslant 0$ *for* $t \geqslant 0$, (14), (15) *hold, and*

$$\int^{\infty} a(t) \, \mathrm{d}t < \infty. \tag{21}$$

Then all solutions of equation (1) *are oscillatory and unbounded.*

P r o o f. Suppose to the contrary that $y(t)$ is a nonoscillatory solution of equation (1). Let $t_0 > N$ be fixed. Suppose $S_0 > t_0$ is sufficiently large so that $g(t) \geqslant t_0$

for $t \geqslant S_0$. Without any loss of generality suppose $y(t) \geqslant 0$ and $y(g(t)) \geqslant 0$ for $t \geqslant S_0$. Following the proof of Theorem 1, we arrive at (18).

Rewriting (18) we have

$$
\begin{aligned}
(22) \qquad & y(g(t)) - y(g(t_0)) - \sum_{i=1}^{n-1} C_i \int_{t_0}^{g(t)} \frac{(s-t_0)^{i-1}}{r(s)} \, \mathrm{d}s \\
& + \int_{t_0}^{g(t)} \frac{1}{r(s)} \int_{t_0}^{s} \frac{(s-x)^{n-2} a(x) h(y(g(x)))}{(n-2)!} \, \mathrm{d}x \, \mathrm{d}s \\
& = \int_{t_0}^{g(t)} \frac{1}{r(s)} \int_{t_0}^{s} \frac{(s-x)^{n-2} f(x)}{(n-2)!} \, \mathrm{d}x \, \mathrm{d}s.
\end{aligned}
$$

Now the second and third term on the left side of (22) are finite due to condition (13). The fourth term is not only finite as shown before but also positive. Thus all terms on the left side of (22) are either bounded or nonnegative. Since the right side of (22) swings between $-\infty$ and ∞ as $t \to \infty$, a contradiction is reached. The proof is essentially complete since the unboundedness of $y(t)$ follows from Theorem 1. $\square$

Example 1. The equation

$$
(23) \qquad (t^2 y')' + \frac{1}{t^3} y(t-\pi) = 4t^2 \cos t + (2t - t^3) \sin t - \frac{t-\pi}{t^3} \sin(t), \ t \geqslant \pi
$$

satisfies the conditions and conclusion of Theorem 1 and 2. It has $y = t \sin t$ as a solution.

Remark 2. The concern raised in Remark 1 about condition (20) is partially answered by our next theorem. In this regard Lemma 1 is useful below.

Theorem 3. *Suppose assumption* (i) *through* (iv) *and conditions* (11) *and* (13) *of Lemma* 1 *hold. Further suppose that whenever the condition* (20) *holds, then it also implies that conditions* (14) *and* (15) *of Theorem* 1 *hold. Then a necessary and sufficient condition for all oscillatory solutions of equation* (1) *to not approach zero is that condition* (20), *namely*

$$
(20) \qquad \int^{\infty} |f(t)| \, \mathrm{d}t = \infty.
$$

is satisfied.

Proof. Necessity follows by Lemma 1.

Sufficiency. Since (20) holds, conditions (14) and (15) of Theorem 1 are implied. Let $y(t)$ be an oscillatory solution of equation (1) such that $y(t) \to 0$ as $t \to \infty$.

We follow the proof of Theorem 1 and arrive at conclusion (18). Now, under the conditions of this theorem, the first three terms on the right side of (18) have been shown to be finite in the proof of Theorem 1. Since the last term on the right side of (18) swings between $-\infty$ and ∞; and the left side of (18) approaches zero, we reach a contradiction which completes the proof of this theorem. $\square$

Remark 3. Theorem 3 essentially states that when $f(x)$ is sufficiently large so that conditions (20), (14), and (15) hold, only then would it imply that an oscillatory solution of equation (1) would not converge to zero asymptotically. The equation (23) of Example 1 also satisfies the conclusion of Theorem 3. It can easily be verified that all conditions of this theorem are satisfied.

4. Impact on an elliptic equation

We now turn our attention to the application of the preceding three theorems to the elliptic equation (8), namely

$$\Delta y(|x|) + \frac{1}{|x|^{p-1}} a(|x|) h(y(g(|x|))) = \frac{1}{|x|^{p-1}} f(|x|) \tag{8}$$

for an integer $p \geqslant 3$. We study the asymptotic behavior of any solution $y(|x|)$ of (8) which exists in the domain Ω where

$$x \in \Omega = \{x\colon\ |x| > N_0\} \subset \mathbb{R}^p, \quad N_0 > 0, \quad \text{an integer,} \tag{24}$$

and $|x|$ is the Euclidean distance defined by

$$|x| = \sqrt{\sum_{i=1}^{p} x_i^2}. \tag{25}$$

Δ as noted before is the Laplace operator.

Lemma 2. *A radially symmetric function $y(|x|)$ is a solution of the elliptic equation (8) if and only if $y(t)$ is a solution of the differential equation*

$$(t^{p-1} y'(t))' + a(t) h(y(g(t))) = f(t) \tag{26}$$

where $t = |x|$.

P r o o f. Suppose first that $y(|x|)$ is a solution of equation (8). Then

$$\begin{aligned}\frac{\delta^2 y}{\delta x_1^2} &= \frac{\delta}{\delta x_1}\Big(\frac{\delta y}{\delta x_1}\Big)\\ &= \frac{\delta}{\delta x_1}\Big(y'(t)\cdot\frac{x_1}{t}\Big)\\ &= \Big(y''(t)\cdot\frac{x_1}{t} - y'(t)\frac{x_1}{t^2}\Big)\frac{x_1}{t} + \frac{y'(t)}{t}.\end{aligned}$$

Therefore

$$\Delta y = \sum_{i=i}^{p}\frac{\delta^2 y}{\delta x_i^2} = y''(t) - \frac{y'(t)}{t} + \frac{py'(t)}{t}. \tag{27}$$

(27) now yields

$$t^{p-1}\Delta y = t^{p-1}y''(t) - t^{p-2}y'(t) + pt^{p-2}y'(t) = (t^{p-1}y'(t))' \tag{28}$$

from which we get

$$t^{p-1}\Delta y + a(t)h(y(g(t))) = (t^{p-1}y'(t))' + a(t)h(y(g(t))) = f(t). \tag{29}$$

Similarly if $y(t)$ satisfies equation (26), then it readily follows that $y(|x|)$ satisfies equation (8). This completes the proof of Lemma 2.

The following theorem is an analogue of Theorem 1 as applied to the elliptic equation (8). □

Theorem 4. *Suppose assumptions* (i) *through* (iv) *and condition* (11) *of Lemma* 1 *hold. Further suppose that*

$$\limsup_{t\to\infty}\int^{t}\frac{1}{s^{p-1}}\int^{s} f(x)\,\mathrm{d}x\,\mathrm{d}s = \infty \tag{30}$$

and

$$\liminf_{t\to-\infty}\int^{t}\frac{1}{s^{p-1}}\int^{s} f(x)\,\mathrm{d}x\,\mathrm{d}s = -\infty. \tag{31}$$

Let $y(|x|)$ be any solution of equation (8) *which exists in the domain $\Omega = \{x\colon |x| > N_0\}$, $N_0 > 0$ for some integer N_0. Then $y(|x|)$ is unbounded.*

P r o o f. We notice that $p \geqslant 3$. Therefore,

$$r(t) = \frac{1}{t^{p-1}} \leqslant \frac{1}{t^{2-k}},$$

$0 \leqslant k < 1$. Thus condition (13) of Lemma 1 holds. Since all conditions of Theorem 1 for $n = 2$ are now satisfied, all solutions of the equation

$$(t^{p-1}y'(t))' + a(t)h(y(g(t))) = f(t)$$

are unbounded. The conclusion now applies to its companion elliptic equation (8). This completes the proof. $\square$

Example 2. Consider the equation

$$|x|^3\Delta y(|x|) + \frac{1}{|x|^2}y(|x|) = (3|x|^2 - |x|^4)\sin(|x|) + 5|x|^3\cos(|x|) + \frac{\sin(|x|)}{|x|} \tag{32}$$

for $p = 4$. Equation (32) satisfies the conditions of Theorem 4. Thus all solutions of (32) defined in the exterior domain $\Omega = \{x\colon\ |x| > N_0\}$ where $N_0 > 0$ some integer, are unbounded. In fact,

$$y(|x|) = |x|\sin(|x|)$$

is one such solution. Its companion equation

$$(t^3y'(t))' + \frac{1}{t^2}y(t) = (3t^2 - t^4)\sin t + 5t^3\cos t + \frac{\sin t}{t}, \quad t > 0 \tag{33}$$

has the solution $y(t) = t\sin t$.

The following is the analogue of Theorem 2 in relation to equation (8).

Theorem 5. *Suppose assumptions* (i) *through* (iv), (30) *and* (31) *hold. Further suppose that* $a(t) \geqslant 0$ *for* $t \geqslant 0$ *and* $\int^\infty a(t)\,\mathrm{d}t < \infty$. *Then all solutions of equation* (8) *existing in the exterior domain* $\Omega = \{x\colon\ |x| > N_0\}$ *for some integer* $N_0 > 0$ *are oscillatory and unbounded.*

Proof. Follows in a manner similar to the proof of Theorem 4. $\square$

Remark 4. The equation (32) of Example 2 also satisfies the conditions and the conclusion of Theorem 5.

Finally, we have Theorem 6 as an analogue of Theorem 3 as applied to equation (8).

Theorem 6. *Suppose assumptions* (i) *through* (iv) *and condition* (11) *of Lemma* 1 *hold. Further suppose that whenever*

$$\int^\infty |f(t)|\,\mathrm{d}t = \infty$$

it implies that conditions (30) *and* (31) *of Theorem* 4 *also hold. Then a necessary and sufficient condition for all oscillatory solutions of the elliptic equation* (8)*, existing in the exterior domain* $\Omega = \{x\colon |x| > N_0\}$ *for some integer* $N_0 > 0$*, to not approach zero is that*

$$\int^{\infty} |f(t)|\,\mathrm{d}t = \infty.$$

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