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REMARKS ON STEINHAUS' PROPERTY AND RATIO SETS OF SETS OF POSITIVE INTEGERS

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Abstract. This paper is closely related to an earlier paper of the author and W. Narkiewicz (cf. [7]) and to some papers concerning ratio sets of positive integers (cf. [4], [5], [12], [13], [14]). The paper contains some new results completing results of the mentioned papers. Among other things a characterization of the Steinhaus property of sets of positive integers is given here by using the concept of ratio sets of positive integers.

Introduction

Remember some fundamental notions and results that will be used in what follows.

Definition A. A set $A \subseteq \mathbb{N}$ is said to have Steinhaus property (S) provided that for each $x \in (0, +\infty)$ there are $q_n \in A$ $(n = 1, 2, \ldots)$ such that $\lim\limits_{n\to\infty} \frac{q_n}{n} = x$ (cf. [7]).

The reason of introducing this definition comes from the well-known Steinhaus result (cf. [15], p. 155) according to which for each $x > 0$ there exists a sequence $(q_n)_1^\infty$ of primes such that $\frac{q_n}{n} \to x$ $(n \to \infty)$ (i.e. the set P of all prime numbers has the property (S) by our terminology).

The concept of a ratio set has been introduced in the papers [12], [13]. If $A \subseteq \mathbb{N}$, $B \subseteq \mathbb{N}$, then we put $R(A, B) = \{\frac{a}{b}\colon\ a \in A, b \in B\}$. The set $R(A, B)$ is said to be the ratio set of the sets A, B. In particular for $A = B$ we put $R(A, A) = R(A) = \{\frac{x}{y}\colon\ x \in A, y \in A\}$.

A set $A \subseteq \mathbb{N}$ is said to be (R)-dense provided that the set $R(A)$ is a dense set in $(0, +\infty)$.

The reason for introducing the concept or ratio sets comes from a result of A. Schinzel (cf. [15], p. 155) by which the set of all numbers $\frac{p}{q}, p, q$ are primes, is dense in $(0, +\infty)$ (i.e. the set P of all primes is an (R)-dense set by our terminology).

We remember the notion of asymptotic and uniform densities. If $A \subseteq \mathbb{N}$, then we put $A(n) = \sum\limits_{a \in A, a \leqslant n} 1$. Then $\underline{d}(A) = \liminf\limits_{n\to\infty} \frac{A(n)}{n}$ and $\overline{d}(A) = \limsup\limits_{n\to\infty} \frac{A(n)}{n}$ is said to be lower and the upper asymptotic density of the set A, respectively. If $\underline{d}(A) = \overline{d}(A)$ $(= d(A))$ then the number $d(A)$ is called the asymptotic density of the set A (cf. [8], p. 71).

If s, t are integers, $s \geqslant 0$, $t \geqslant 1$, then $A(s+1, s+t)$ denotes the number of elements $a \in A$ such that $s+1 \leqslant a \leqslant s+t$. Put $\alpha_t = \min\limits_{s\geqslant 0} A(s+1, s+t)$, $\alpha^t = \max\limits_{s\geqslant 0} A(s+1, s+t)$. Then there exist $\underline{u}(A) = \lim\limits_{t\to\infty} \frac{\alpha_t}{t}$ and $\overline{u}(A) = \lim\limits_{t\to\infty} \frac{\alpha^t}{t}$ and these numbers are called the lower and upper uniform density of the set A, respectively. If $\underline{u}(A) = \overline{u}(A)$ $(= u(A))$, then $u(A)$ is called the uniform density of A. Put $\beta_t = \liminf\limits_{s\to\infty} A(s+1, s+t)$, $\beta^t = \limsup\limits_{s\to\infty} A(s+1, s+t)$, then it is well–known that $\underline{u}(A) = \lim\limits_{t\to\infty} \frac{\beta_t}{t}$, $\overline{u}(A) = \lim\limits_{t\to\infty} \frac{\beta^t}{t}$ (cf. [2], [3]).

Denote by U the class of all infinite sets $A \subseteq \mathbb{N}$. If $A \in U$, $A = \{a_1 < a_2 < \ldots < a_n < \ldots\}$ then we put $\varrho(A) = \sum\limits_{k=1}^{\infty} 2^{-a_k} = \sum\limits_{k=1}^{\infty} \varepsilon_k 2^{-k} \in (0,1]$, where $(\varepsilon_k)_1^{\infty}$ is the characteristic function of the set A. The function $\varrho\colon U \to (0,1]$ is a one-to-one mapping of U onto $(0,1]$.

If $S \subseteq U$, then we set $\varrho(S) = \{\varrho(A)\colon\ A \in S\}$. The set $\varrho(S)$ is a tool for "measuring" the greatness of the class S (cf. [8], p. 17-18).

In what follows $\lambda(M)$ denotes the Lebesgue measure of the set $M \subseteq R$ and dim M the Hausdorff dimension of M (cf. [9], [10], [11]).

The symbol $\overline{M}$ denotes the closure of the set $M \subseteq R$. The concepts of the set of the first (Baire) category and the set of the second (Baire) category in $(0,1]$ will be used in the usual sense (cf. [6], p. 43), the interval $(0,1]$ being considering as a metric space with the Euclidean metric. A set $H \subseteq (0,1]$ is said to be a residual set provided that $(0,1] \setminus H$ is a set of the first category.

1. Steinhaus' property, ratio sets and uniform density of sets $A \subseteq \mathbb{N}$

First of all we shall give a characterization of the property (S) based on the concept of ratio sets of sets $A \subseteq \mathbb{N}$.

Theorem 1.1. *A set $A \subseteq \mathbb{N}$ has the property (S) if and only if for each infinite set $B \subseteq \mathbb{N}$ the set $R(A,B)$ is dense* $(0,\infty)$.

P r o o f. 1. Suppose that A has the property (S). Let $B = \{b_1 < b_2 < \ldots < b_n < \ldots\} \subseteq \mathbb{N}$ be an arbitrary infinite set and let $x \in (0,\infty)$. By the assumption there exist $q_n \in A$ $(n = 1, 2, \ldots)$ such that $\frac{q_n}{n} \to x$ $(n \to \infty)$. But then the subsequence

$(\frac{q_{b_n}}{b_n})_{n=1}^{\infty}$ of the sequence $(\frac{q_n}{n})_{n=1}^{\infty}$ converges to x, as well. Now it suffices to observe that $q_{b_n} \in A$, $b_n \in B$ $(n = 1, 2, \ldots)$ and the density of $R(A, B)$ in $(0, \infty)$ follows.

2. Suppose that $A = \{a_1 < a_2 < \ldots < a_n < \ldots\} \subseteq \mathbb{N}$ has not the property (S). In [7] (see Proposition 2 in [7]) is is proved that A has the property (S) if and only if $\lim\limits_{n\to\infty} \frac{a_n+1}{a_n} = 1$. Hence we have $\limsup\limits_{n\to\infty} \frac{a_{n+1}}{a_n} > 1$. Therefore it exist an $\eta > 0$ and $n_1 < n_2 < \ldots$ such that

$$\frac{a_{n_k}+1}{a_{n_k}} \geqslant 1+\eta \quad (k = 1, 2, 3, \ldots). \tag{1}$$

Construct the intervals $I_k = (a_{n_k}, (1+\eta), a_{n_k})$ $(k = 1, 2, \ldots)$. From (1) we see that $I_k \cap R(A, B) = \emptyset$ $(k = 1, 2, \ldots)$, where $B = \{a_{n_1} < a_{n_2} < \ldots < a_{n_k} < \ldots\}$. Hence the set $R(A, B)$ is not dense in $(0, \infty)$. □

The relationship between (R)-density of a set $A \subseteq \mathbb{N}$ and its asymptotic density is established in [12]. It is proved in [12] that if $d(A) > 0$ then A is an (R)-dense set. Simultaneously it is shown in [12] that the condition $\underline{d}(A) > 0$ is not sufficient for the (R)-density of the set A. In connection with these facts the natural question arises whether the positivity of $\underline{u}(A)$ is sufficient for (R)-density of A. The positive answer is contained in the following result.

Theorem 1.2. *If $\underline{u}(A) > 0$, then the set A is an (R)-dense set.*

Theorem 1.2 follows immediately from the following lemma.

Lemma 1.1. *If $A \subseteq \mathbb{N}$ and the set $R(A)$ is not dense in $(0, \infty)$, then*

$$\liminf_{t\to\infty} \alpha_t = 0. \tag{2}$$

P r o o f. Let $A = \{a_1 < a_2 < \ldots < a_k < \ldots\} \subseteq \mathbb{N}$ be not (R) dense. Then there exists an interval $(c, d] \leqslant (0, \infty)$ such that $(c, d] \cap R(A) = \emptyset$. From this we get $(ca_k, da_k] \cap A = \emptyset$ $(k = 1, 2, \ldots)$ and hence

$$A([ca_k]+1, [da_k]) = 0 \quad (k = 1, 2, \ldots). \tag{3}$$

Put $s_k = [ca_k]$, $t_k = [da_k] - [ca_k]$ $(k = 1, 2, \ldots)$. Then (3) yields $A(s_k+1, s_k+t_k) = 0$ $(k = 1, 2, \ldots)$ and so $\alpha_{t_k} = 0$ $(k = 1, 2, \ldots)$. From this (2) follows. □

Remark 1.1. a) Note that the condition $\underline{u}(A) > 0$ is only a sufficient but not necessary condition for (R)-density of A. It is namely well-known that $u(P) = 0$ (P being again the set of all primes — cf. [3]), but P is an (R)-dense set (cf. [15], p. 155).

b) Analogously it can be checked that Lemma 1.1 cannot be conversed. It suffices to put $A = P$ and remember that the sequence $1, 2, \ldots, n, \ldots$ of all positive integers contains arbitrarily long "blocks" $b+1, b+2, \ldots, b+m$ that contain no prime number.

c) Note that from the property (S) of a set $A \subseteq \mathbb{N}$ its (R)-density follows (see [7]).

In connection with the mentioned characterization of the property (S) ([7], Proposition 2) we are proving the following result.

Proposition 1.1. *If the set $A = \{a_1 < a_2 < \ldots\} \subseteq \mathbb{N}$ is (R)-dense then*

$$\liminf_{n\to\infty} \frac{a_{n+1}}{a_n} = 1. \tag{4}$$

P r o o f. Suppose that (4) does not hold. Then there exists an $\eta > 0$ such that for each $n = 1, 2, \ldots$ we have $\frac{a_{n+1}}{a_n} > 1 + \eta$. But then $(1, 1+\eta) \cap R(A) = \emptyset$. □

Remark 1.2. Proposition 1.1 cannot be conversed. This can be seen from the example $A = \{2^2, 2^2+1, 2^4, 2^4+1, \ldots, 2^{2n}, 2^{2n}+1, \ldots\} = \{a_1 < a_2 < \ldots\}$. Obviously we have $\liminf\limits_{n\to\infty} \frac{a_{n+1}}{a_n} = 1$ and simultaneously it can be verified that $R(A) \cap (\frac{5}{4}, \frac{16}{5}) = \emptyset$.

In the end of this part we mention a problem from [5]. In this paper on p. 50 the following "Open Problem Two" is introduced which can be formulated in our terminology as follows:

Let $a, b \in \mathbb{N}$, $(a, b) = 1$. Denote by $D(a, b)$ the set of all prime numbers that are contained in the arithmetic progression $(a + bn)_{n=1}^{\infty}$. Is the set $D(a, b)$ an (R)-dense set?

The positive answer to this problem can be derived from an example which is contained on p. 227 of the paper [12]. In this example it is shown that if $A \subseteq \mathbb{N}$ and $A(x) \sim \frac{c_1 x}{\log^\alpha x} (c_1 > 0, \alpha > 0)$, then A is an (R)-dense set. Now, it is wellknown (cf. [1]. p. 154–155) that if $A = D(a, b)$ then

$$A(x) \sim \frac{1}{\varphi(b)} \frac{x}{\log x} \text{ (for } x \to \infty),$$

φ being the Euler function. From this we get the positive answer to the mentioned problem immediately.

A little different solution of the mentioned problem form [5] is given in [14].

Using the method of dyadic numbers $\varrho(A)$ of sets $A \in U$ we shall investigate some classes of sets $A \subseteq \mathbb{N}$ that are related to the (R)-density and Steinhaus property (S).

Denote by T_R^* the class of all $A = \{a_1 < a_2 < \ldots\} \in U$ with $\liminf\limits_{n\to\infty} \frac{a_{n+1}}{a_n} = 1$. Denote by T_R the class of all $A \in U$ that are (R)-dense (*cf.*[13]) and by T_S the class of all $A \in U$ having the property (S) (cf. [7]). Further denote by T_B the class of all $A \in U$ that are bases for Q^+ (i.e. for which the following holds: Every $r \in Q^+$ can be expressed in the form $r = \frac{a}{b}$, where $a \in A$, $b \in A$). The symbol T_{B_0} denotes the class of all $A \in U$ that are strong bases for Q^+ (for which the following holds: For each $r \in Q^+$ there exists an infinite number of pairs $(a, b) \in A \times A$ such that $r = \frac{a}{b}$) (cf. [13]).

Obviously we have $T_{B_0} \subseteq T_B$ and $T_R \subseteq T_R^*$ (see Proposition 1.1). Hence

$$\varrho(T_R) \subseteq \varrho(T_R^*) \tag{5}$$

Now the natural question arises how great is the difference $T_R^* \setminus T_R$. A certain information about this is given in the following theorem (see part (ii) and (iii) of Theorem 2.1).

Theorem 2.1. (i) *The set* $\varrho(T_R^*)$ *is a union of a* G_δ*-set and an* F_σ*-set in* $(0, 1]$.
(ii) *The set* $\varrho(T_R^*)$ *is residual in* $(0, 1]$.
(iii) *We have* $\lambda(\varrho(T_R^* \setminus T_R)) = 0$.
(iv) *We have* $\dim \varrho(T_R^* \setminus T_R) \geqslant \sqrt{2} - 1 > 0$.

P r o o f.
(i) For $n, k \in \mathbb{N}$, $s \geqslant 0$ we put $B(n, s, k) = \Big\{x = \sum\limits_{k=1}^{\infty} \varepsilon_k(x) 2^{-k} \in (0, 1]\colon \varepsilon_n(x) = 1$, $\varepsilon_{n+1}(x) = \ldots = \varepsilon_{n+s-1}(x) = 0$, $\varepsilon_{n+s}(x) = 1$, $|\frac{n+s}{n} - 1| < \frac{1}{k}\Big\}$.
Then we get

$$\varrho(T_R^*) = \bigcap_{k=1}^{\infty} \bigcap_{m=1}^{\infty} \bigcup_{n,s \geqslant m} B(n, s, k). \tag{6}$$

Put $E = (0, 1] \setminus D$, where D is the set of dyadic rationals. Then $B(n, s, k) \cap E$ is an open set in E and so by (6) the set $E \cap \varrho(T_R^*)$ is a G_δ-set in E and in $(0, 1]$, as well. Consider that D is a countable set, thus $D \cap \varrho(T_R^*)$ is an F_δ-set. The assertion follows from the following obvious equality

$$\varrho(T_R^*) = [E \cap \varrho(T_R^*)] \cup [D \cap \varrho(T_R^*)].$$

(ii) Since $\varrho(T_R)$ is residual in $(0,1]$ (cf. [13]), the part (ii) of Theorem follows from (5).

(iii) Since ϱ is a one-to-one mapping of U onto $(0,1]$ we get

$$\varrho(T_R^* \setminus T_R) = \varrho(T_R^*) \setminus \varrho(T_R) \tag{7}$$

But $\lambda(\varrho(T_R)) = 1$ (cf. [13]) and so by (5) we have $\lambda(\varrho(T_R^*)) = 1$. The part (iii) of Theorem follows from (7).

(iv) Construct the sets

$$A_k = \{2^k+1, 2^k+2, \ldots, 2^k+[t2^k]\} \quad (k = 1,2,\ldots),$$
$$0 < t < \sqrt{2}-1 \quad \text{and put } A = \bigcup_{k=1}^{\infty} A_k.$$

Obviously the set A belongs to T_R^*. We show that A does not belong to T_R.

Let $c, d \in A$, $c \geqslant d$. There are two possibilities here:

a) There exist $k, j, k \neq j$, such that $c \in A_k$, $d \in A_j$

b) There exists a k such that $c, d \in A_k$.

a) We have $k > j$ and so

$$\frac{c}{d} \geqslant \frac{2^k+1}{2^j+[t2^j]} \geqslant \frac{2^{k-j}}{1+t} \geqslant \frac{2}{1+t}.$$

Since $t < \sqrt{2}-1$, we have $1+t < \frac{2}{1+t}$.

b) By a simple estimation we get

$$\frac{c}{d} < \frac{2^k+[t2^k]}{2^k} \leqslant 1+t$$

According to previous inequalities we get $R(A) \cap (1+t, \frac{2}{1+t}) = \emptyset$. Thus $A \notin T_R$.

So we have $A \in T_R^* \setminus T_R$. Obviously no subset A' of A belongs to T_R. Further a subset B of A belongs certainly to T_R^* if

$$\bigcup_{k=1}^{\infty} \{2^k+1, 2^k+2\} \subseteq B \subseteq \bigcup_{k=1}^{\infty} \{2^k+1, \ldots, 2^k+[t2^k]\}. \tag{8}$$

Denote by W the class of all $B \subseteq A$ satisfying (8). Then

$$W \subseteq T_R^* \setminus T_R \tag{9}$$

In what follows we shall use the following consequence of Theorem 2.7 from [11]:

Let $I \subseteq \mathbb{N}$ and $(\varepsilon_k^0)_{k\in I}$ be given sequence of 0's and 1's. Denote by $Z = Z(I, (\varepsilon_k^0), k \in I)$ the set of all numbers $x \in \sum\limits_{k=1}^{\infty} \varepsilon_k(x)\cdot 2^{-k} \in (0,1]$ for which $\varepsilon_k(x) = \varepsilon_k^0$ if $k \in I$ and $\varepsilon_k(x) = 0$ or 1 for $k \in \mathbb{N} \setminus I$. Then we have

$$\dim Z(I, (\varepsilon_k^0), k \in I) = \liminf_{n\to\infty} \frac{\log \prod\limits_{j\leqslant n, j\in\mathbb{N}\setminus I} 2}{n \log 2} = \underline{d}(\mathbb{N} \setminus I)$$

Using the previous result from [11] we get an estimation for the Hausdorff dimension of set $\varrho(W)$.

By notation used in [11] (Theorem 2.7) we put

$$I = \bigcup_{j=1}^{\infty} \{2^j + 1, 2^j + 2\} \cup (\mathbb{N} \setminus A), \varepsilon_k^0 = 1 \text{ if}$$

$$k \in \bigcup_{j=1}^{\infty} \{2^j + 1, 2^j + 2\} \text{ and } \varepsilon_k^0 = 0 \text{ for } k \in \mathbb{N} \setminus A.$$

Then $\varrho(W) = Z(I, (\varepsilon_k^0), k \in I)$. Hence by definition of I we have

$$\mathbb{N} \setminus I = \bigcup_{j=1}^{\infty} \{2^j + 3, 2^j + 4, \ldots, 2^j + [t2^j]\}.$$

Minimal values of the quotient $\frac{(\mathbb{N}\setminus I)(n)}{n}$ are attained at the numbers $n = 2^{j+1} + 2$ $(j = 1, 2, \ldots)$.

Therefore we have

$$\liminf_{n\to\infty} \frac{(\mathbb{N} \setminus I)(n)}{n} = \lim_{j\to\infty} \frac{\sum\limits_{k=3}^{j} [t2^k]}{2^{j+1} + 2} = t.$$

So we get dim $\varrho(W) = \underline{d}(\mathbb{N} \setminus I) = t$. This together with (9) yields dim $\varrho(T_R^* \setminus T_R) \geqslant t$. Since this holds for every $t, 0 < t < \sqrt{2} - 1$, the assertion follows. $\square$

In what follows we shall investigate the relationship between T_S, T_B and T_{B_0} from metric and topological point of view.

Observe that the set P of all primes belongs to T_S but obviously it does not belong to T_B (and so it does not belong to T_{B_0}, as well). Hence $T_S \setminus T_B \neq \emptyset \neq T_S \setminus T_{B_0}$.

Note that the inclusion $T_B \subseteq T_S$ does not hold. This is a simple consequence of two topological results on sets $\varrho(T_B)$, $\varrho(T_S)$ (cf. [7] and [13]). By these results the set $\varrho(T_B)$ is residual in $(0,1]$ and $\varrho(T_S)$ is a set of the first category in $(0,1]$.

Proposition 2.1. *Each of the sets* $\varrho(T_B \setminus T_S)$, $\varrho(T_{B_0} \setminus T_S)$ *is residual in* $(0,1]$.

P r o o f. It suffices to prove the part concerning the second set. Since ϱ is one-to-one mapping, we have

$$\varrho(T_{B_0}) = \varrho(T_{B_0} \setminus T_S) \cup \varrho(T_{B_0} \cap T_S). \tag{10}$$

The set $\varrho(T_{B_0})$ is residual (cf. [13]) and $\varrho(T_S)$ is a set of the first category in $(0,1]$ (cf. [7]). From these facts that assertion follows from (10). $\square$

The sets $\varrho(T_B)$, $\varrho(T_{B_0})$ and $\varrho(T_S)$ have the Lebesgue measure 1 (cf. [7], [13]). From this we get immediately

Proposition 2.2. *The sets* $\varrho(T_S \setminus T_B)$, $\varrho(T_B \setminus T_S)$, $\varrho(T_{B_0} \setminus T_S)$, $\varrho(T_S \setminus T_{B_0})$ *have the Lebesgue measure* 0.

Proposition 2.2 evokes the question what is the Hausdorff dimension of sets mentioned in this proposition. In this connection we give a lower estimation for dim $\varrho(T_S \setminus T_B)$.

Theorem 2.2. *We have* dim $\varrho(T_S \setminus T_B) \geqslant \frac{1}{2}$.

Corollary. *We have* dim $\varrho(T_S \setminus T_{B_0}) \geqslant \frac{1}{2}$.

P r o o f. Observe that the set $\mathbb{N}_1 = \{1, 3, \ldots, 2k-1, \ldots\}$ of all odd positive integers belongs to T_S (cf. Proposition 2 in [7]), but it does not belong to T_B and consequently no subset of $\mathbb{N}_1$ belongs to T_B.

Let $d \in \mathbb{N}$. Put $M_0 = \{1, 1+2d, 1+2d\cdot 2, \ldots, 1+2d\cdot n, \ldots\} \subseteq \mathbb{N}_1$.

Denote by S_d the class of all sets M satisfying the inclusions $M_0 \subseteq M \subseteq M_0 \cup M_1$, where

$$M_1 = \bigcup_{n=0}^{\infty} \{1+2d\cdot n+2, 1+2d\cdot n+4, \ldots, 1+2d\cdot n+2(d-1)\}.$$

Using proposition 2 from [7] one can easily check that

$$S_d \subseteq T_S \setminus T_B. \tag{11}$$

The Hausdorff dimension of the set $\varrho(S_d)$ can be determined on the basis of Theorem 2.7 from [11]. We get

$$\dim \varrho(S_d) = \liminf_{n\to\infty} \frac{\log \prod\limits_{k\leqslant n, k\in M_1} 2}{n \log 2} = \underline{d}(M_1).$$

We have obviously

$$d(M_1) = d(\mathbb{N}_1) - d(M_0) = \frac{1}{2} - \frac{1}{2d}.$$

Owing to (11) we have $\varrho(T_S \setminus T_B) \geqslant \frac{1}{2} - \frac{1}{2d}$. This holds for every $d \in \mathbb{N}$. Thus by $d \to \infty$ the theorem follows. $\square$

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