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Hui Fang

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POSITIVE PERIODIC SOLUTIONS OF N-SPECIES NEUTRAL DELAY SYSTEMS

HUI FANG, Kunming

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Abstract. In this paper, we employ some new techniques to study the existence of positive periodic solution of n-species neutral delay system

$$N_i'(t) = N_i(t)\Big[a_i(t) - \sum_{j=1}^{n} \beta_{ij}(t)N_j(t) - \sum_{j=1}^{n} b_{ij}(t)N_j(t-\tau_{ij}(t)) - \sum_{j=1}^{n} c_{ij}(t)N_j'(t-\tau_{ij}(t))\Big].$$

As a corollary, we answer an open problem proposed by Y. Kuang.

Keywords: positive periodic solutions, existence, neutral delay system

MSC 2000: 34C25, 34K15, 34A12

1. INTRODUCTION

Consider the following neutral delay model

$$N'(t) = N(t)[a(t) - \beta(t)N(t) - b(t)N(t-\tau(t)) - c(t)N'(t-\tau(t))] \tag{1}$$

where $a(t)$, $\beta(t)$, $b(t)$, $\tau(t)$, $c(t)$ are nonnegative continuous T-periodic functions.

In 1993, Kuang Yang proposed the following open problem (Open Problem 9.2 in [1]): Obtain sufficient conditions for the existence of positive periodic solutions of (1).

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When $a(t)$, $\beta(t)$, $b(t)$, $\tau(t)$ are positive and $c(t) = 0$, such a problem was considered by Freedman and Wu [2]. In this paper, we consider the following more general n-species neutral delay system

$$\begin{aligned}(2)\qquad N_i'(t) = N_i(t)\Bigg[a_i(t) - \sum_{j=1}^{n}\beta_{ij}(t)N_j(t) - \sum_{j=1}^{n}b_{ij}(t)N_j(t-\tau_{ij}(t))\\ - \sum_{j=1}^{n}c_{ij}(t)N_j'(t-\tau_{ij}(t))\Bigg]\end{aligned}$$

where $a_i(t)$, $\beta_{ij}(t)$, $b_{ij}(t)$, $\tau_{ij}(t)$, $c_{ij}(t)$ $(i, j = 1, 2, \ldots, n)$ are nonnegative continuous T-periodic functions.

The purpose of this paper is to establish the existence of positive periodic solutions for neutral delay system (2). As a corollary, we give an answer to the open problem 9.2 in [1]. To show the existence of solutions to the considered problems, we will use an existence theorem developed in [3], [4]. We will state this existence theorem in Section 2.

2. An existence theorem

For a fixed $r \geqslant 0$, let $C =: C([-r, 0]; \mathbb{R}^n)$. If $x \in C([\sigma - r, \sigma + \delta]; \mathbb{R}^n)$ for some $\delta > 0$ and $\sigma \in \mathbb{R}$, then $x_t \in C$ for $t \in [\sigma, \sigma + \delta]$ is defined by $x_t(\theta) = x(t + \theta)$ for $\theta \in [-r, 0]$. The supremum norm in C is denoted by $\|\cdot\|$, i.e. $\|\varphi\| = \max_{\theta\in[-r,0]} |\varphi(\theta)|$ for $\varphi \in C$, where $|\cdot|$ denotes the norm in $\mathbb{R}^n$, and $|u| = \sum_{i=1}^{n} |u_i|$ for $u = (u_1, \ldots, u_n) \in \mathbb{R}^n$.

We consider the following neutral functional differential equation:

$$\frac{\mathrm{d}}{\mathrm{dt}}[x(t) - b(t, x_t)] = f(t, x_t)$$

where $f\colon \mathbb{R} \times C \to \mathbb{R}^n$ is completely continuous and $b\colon \mathbb{R} \times C \to \mathbb{R}^n$ is continuous. Moreover, we assume:

(H1) There exists $T > 0$ such that for every $(t, \varphi) \in \mathbb{R} \times C$, we have $b(t + T, \varphi) = b(t, \varphi)$ and $f(t + T, \varphi) = f(t, \varphi)$.

(H2) There exists a constant $k < 1$ such that $|b(t, \varphi) - b(t, \psi)| \leqslant k\|\varphi - \psi\|$ for $t \in \mathbb{R}$ and $\varphi, \psi \in C$.

By using the continuation theorem for composite coincidence degrees, Erbe et al. [3] proved the following existence theorem. See also Theorem 4.7.1 in [4].

Theorem A. *Suppose that there exists a constant $M > 0$ such that:*

(i) *For any $\lambda \in (0,1)$ and any T-periodic solution x of the system*

$$\frac{\mathrm{d}}{\mathrm{dt}}[x(t) - \lambda b(t, x_t)] = \lambda f(t, x_t)$$

we have $|x(t)| < M$ for $t \in \mathbb{R}$;

(ii) *$g(u) =: \int_0^T f(s, \hat{u})\,\mathrm{d}s \neq 0$ for $u \in \partial B_M(\mathbb{R}^n)$, where $B_M(\mathbb{R}^n) = \{u \in \mathbb{R}^n\colon\ |u| < M\}$, and $\hat{u}$ denotes the constant mapping from $[-r, 0]$ to $\mathbb{R}^n$ with the value $u \in \mathbb{R}^n$;*

(iii) $\deg(g, B_M(\mathbb{R}^n)) \neq 0$.

Then there exists at least one T-periodic solution of the system

$$\frac{\mathrm{d}}{\mathrm{dt}}[x(t) - b(t, x_t)] = f(t, x_t)$$

that satisfies $\sup\limits_{t \in \mathbb{R}} |x(t)| < M$.

Remark 1. From the proof of Theorem A (Theorem 4.7.1 in [4]), it is easy to see that if assumption (H2) is replaced by

(H2)$'$ There exists a constant $k < 1$ such that $|b(t, \varphi) - b(t, \psi)| \leqslant k\|\varphi - \psi\|$ for $t \in \mathbb{R}$ and $\varphi, \psi \in \{\varphi \in C\colon\ \|\varphi\| < M\}$ with M as given in condition (i) of Theorem A, then Theorem A still holds.

3. Main results

In order to establish the existence of positive periodic solutions for neutral delay system (2), we first consider the following system

$$\begin{aligned}(3) \qquad x_i'(t) = a_i(t) - \sum_{j=1}^n \beta_{ij}(t)\mathrm{e}^{x_j(t)} - \sum_{j=1}^n b_{ij}(t)\mathrm{e}^{x_j(t-\tau_{ij}(t))} \\ - \sum_{j=1}^n c_{ij}(t) x_j'(t - \tau_{ij}(t))\mathrm{e}^{x_j(t-\tau_{ij}(t))}.\end{aligned}$$

Let C_T^0 denote the linear space of real valued continuous T-periodic functions on $\mathbb{R}$. The linear space C_T^0 is a Banach space with the usual norm for $x(t) = (x_1(t), \ldots, x_n(t)) \in C_T^0$ given by $\|x\|_0 = \max\limits_{t \in R} |x(t)| = \max\limits_{t \in R} \sum\limits_{i=1}^n |x_i(t)|$.

We define the following maps:

$$b\colon\ \mathbb{R}\times C\to\mathbb{R}^n,\quad b(t,\varphi)=(b_1(t,\varphi),\dots,b_n(t,\varphi)),$$
$$b_i(t,\varphi)=-\sum_{j=1}^{n}\frac{c_{ij}(t)}{1-\tau'_{ij}(t)}\mathrm{e}^{\varphi_j(-\tau_{ij}(t))};$$
$$f\colon\ \mathbb{R}\times C\to\mathbb{R}^n,\quad f(t,\varphi)=(f_1(t,\varphi),\dots,f_n(t,\varphi)),$$
$$f_i(t,\varphi)=a_i(t)-\sum_{j=1}^{n}\beta_{ij}(t)\mathrm{e}^{\varphi_j(0)}-\sum_{j=1}^{n}\Big(b_{ij}(t)-\Big(\frac{c_{ij}(t)}{1-\tau'_{ij}(t)}\Big)'\Big)\mathrm{e}^{\varphi_j(-\tau_{ij}(t))},$$
$$i=1,2,\dots,n;\quad t\in\mathbb{R},\quad \varphi=(\varphi_1,\dots,\varphi_n)\in C.$$

Now, the system (3) becomes

$$\frac{\mathrm{d}}{\mathrm{dt}}[x(t)-b(t,x_t)]=f(t,x_t).$$

In the following, we denote

$$\bar g=\frac{1}{T}\int_0^T g(t)\,\mathrm{d}t,\quad g_m=\min_{t\in[0,T]}g(t),\quad |g|_0=\max_{t\in[0,T]}|g(t)|$$

for $g\in\{g\in C(\mathbb{R},\mathbb{R})\colon\ g(t+T)=g(t)\text{ for }t\in\mathbb{R}\}$.

Theorem B. *Suppose that the following conditions are satisfied:*

(a) $a_i(t)$, $\beta_{ij}(t)$, $b_{ij}(t)$, $\tau_{ij}(t)$, $c_{ij}(t)$ *are T-periodic functions and*

$$a_i(t)\in C(\mathbb{R},(0,+\infty)),\quad \beta_{ij}(t),b_{ij}(t)\in C(\mathbb{R},\mathbb{R}^+),\quad c_{ij}(t)\in C^1(\mathbb{R},\mathbb{R}^+),$$
$$\tau_{ij}(t)\in C^2(\mathbb{R},\mathbb{R}^+),\quad \tau'_{ij}(t)<1,\quad \beta_{ii}(t)\geqslant\beta>0,\quad i,j=1,2,\dots,n;$$

where $\mathbb{R}^+=[0,+\infty)$, β *is a constant;*

(b) *the system*

$$\sum_{j=1}^{n}(\bar\beta_{ij}+\bar b_{ij})u_j=\bar a_i,\quad i=1,2,\dots,n$$

has a unique positive solution $u^*=(u_1^*,\dots,u_n^*)$*;*

(c)

$$\bar a_i>\sum_{j=1,\,j\neq i}\frac{M_{ij}\bar a_j}{m_{jj}},\quad m_{ii}>0,\quad i=1,2,\dots,n;$$

where

$$M_{ij}=|\beta_{ij}|_0+\Big|\frac{b_{ij}-d'_{ij}}{1-\tau'_{ij}}\Big|_0,\quad m_{ij}=(\beta_{ij})_m+\Big(\frac{b_{ij}-d'_{ij}}{1-\tau'_{ij}}\Big)_m,$$
$$d_{ij}(t)=\frac{c_{ij}(t)}{1-\tau'_{ij}(t)},\quad d'_{ij}(t)<b_{ij}(t),\quad i,j=1,2,\dots,n;$$

(d) $k_0 =: c\mathrm{e}^{M_0} < 1$, *where*

$$c = \sum_{i=1}^{n}\sum_{j=1}^{n} |d_{ij}|_0, \quad M_0 = \max\Big\{\sum_{i=1}^{n} |\ln u_i^*|, R, TM_* + \sum_{i=1}^{n} K_i\Big\},$$

$$R = \max_{1\leqslant i\leqslant n}\{R_i\}, \quad R_i = \ln\frac{\bar a_i}{(\beta_{ii})_m} + \sum_{j=1}^{n}\frac{|d_{ij}|_0\bar a_i}{(b_{ij}-d'_{ij})_m} + 2\bar a_i T,$$

$$M_* = \frac{\sum\limits_{i=1}^{n}|a_i|_0 + \sum\limits_{i=1}^{n}\sum\limits_{j=1}^{n}|\beta_{ij}|_0\mathrm{e}^{R_j} + \sum\limits_{i=1}^{n}\sum\limits_{j=1}^{n}|b_{ij}|_0\mathrm{e}^{R_j}}{1 - \sum\limits_{i=1}^{n}\sum\limits_{j=1}^{n}|c_{ij}|_0\mathrm{e}^{R_j}},$$

$$K_i = \max\left\{\left|\ln\frac{\bar a_i}{m_{ii}}\right|, \left|\ln\frac{\bar a_i - \sum\limits_{j=1,\, j\neq i}\frac{M_{ij}\bar a_j}{m_{jj}}}{M_{ii}}\right|\right\}, \quad i = 1, 2, \ldots, n.$$

Then (2) *has at least one positive T-periodic solution.*

Remark 2. For the case $n = 1$, Theorem B gives an answer to the Open Problem 9.2 due to Kuang Y. [1].

Before proving Theorem B, we need the following lemmas.

Lemma 1. *Under the assumptions of Theorem* B, *let* $\Omega = \{\varphi \in C\colon \|\varphi\| < M\}$, *where* $M > M_0$ *is such that* $k =: c\mathrm{e}^{M} < 1$, *then* $|b(t,\varphi) - b(t,\psi)| \leqslant k\|\varphi - \psi\|$ *for* $t \in \mathbb{R}$ *and* $\varphi, \psi \in \Omega$.

Proof. For $t \in \mathbb{R}$ and $\varphi, \psi \in \Omega$, we have

$$\begin{aligned}
&|b_i(t,\varphi) - b_i(t,\psi)| \\
&\qquad\leqslant \sum_{j=1}^{n} d_{ij}(t)|\mathrm{e}^{\varphi_j(-\tau_{ij}(t))} - \mathrm{e}^{\psi_j(-\tau_{ij}(t))}| \\
&\qquad\leqslant \sum_{j=1}^{n} d_{ij}(t)\mathrm{e}^{\theta\varphi_j(-\tau_{ij}(t)) + (1-\theta)\psi_j(-\tau_{ij}(t))}|\varphi_j(-\tau_{ij}(t)) - \psi_j(-\tau_{ij}(t))|,
\end{aligned}$$

for some $\theta \in (0,1)$. So, we have

$$|b_i(t,\varphi) - b_i(t,\psi)| \leqslant \sum_{j=1}^{n}|d_{ij}|_0\mathrm{e}^{M}\|\varphi - \psi\|.$$

Therefore,

$$|b(t,\varphi) - b(t,\psi)| \leqslant \sum_{i=1}^{n}\sum_{j=1}^{n}|d_{ij}|_0\mathrm{e}^{M}\|\varphi - \psi\| = k\|\varphi - \psi\|.$$

□

Lemma 2. *If the assumptions of Theorem* B *hold, then every solution* $x(t)\in C_T^0$ *of the system*

$$\frac{\mathrm{d}}{\mathrm{dt}}[x(t)-\lambda b(t,x_t)]=\lambda f(t,x_t),\quad \lambda\in(0,1)$$

satisfies $\|x\|_0\leqslant M_0$.

P r o o f. Let $\frac{\mathrm{d}}{\mathrm{dt}}[x(t)-\lambda b(t,x_t)]=\lambda f(t,x_t)$ for $x(t)\in C_T^0$, that is,

$$\tag{4}\Big[x_i(t)+\lambda\sum_{j=1}^n\frac{c_{ij}(t)}{1-\tau'_{ij}(t)}\mathrm{e}^{x_j(t-\tau_{ij}(t))}\Big]' \\ =\lambda\Big[a_i(t)-\sum_{j=1}^n\beta_{ij}(t)\mathrm{e}^{x_j(t)}-\sum_{j=1}^n\Big(b_{ij}(t)-\Big(\frac{c_{ij}(t)}{1-\tau'_{ij}(t)}\Big)'\Big)\mathrm{e}^{x_j(t-\tau_{ij}(t))}\Big],$$

$i=1,2,\ldots,n$, $\lambda\in(0,1)$.

Integrating these identities, we have

$$\tag{5}\int_0^T\sum_{j=1}^n[\beta_{ij}(t)\mathrm{e}^{x_j(t)}+(b_{ij}(t)-d'_{ij}(t))\mathrm{e}^{x_j(t-\tau_{ij}(t))}]\,\mathrm{d}t \\ =\int_0^T a_i(t)\,\mathrm{d}t,\ \ i=1,2,\ldots,n,$$

where $d_{ij}(t)=:\frac{c_{ij}(t)}{1-\tau'_{ij}(t)}$.

From (4), (5), we have

$$\int_0^T\Big|\Big[x_i(t)+\lambda\sum_{j=1}^n d_{ij}(t)\mathrm{e}^{x_j(t-\tau_{ij}(t))}\Big]'\Big|\,\mathrm{d}t \\ \leqslant\lambda\Big[\int_0^T a_i(t)\,\mathrm{d}t+\int_0^T\sum_{j=1}^n[\beta_{ij}(t)\mathrm{e}^{x_j(t)}+(b_{ij}(t)-d'_{ij}(t))\mathrm{e}^{x_j(t-\tau_{ij}(t))}\Big]\,\mathrm{d}t \\ <2\int_0^T a_i(t)\,\mathrm{d}t=2T\bar a_i.$$

That is,

$$\tag{6}\int_0^T\Big|\Big[x_i(t)+\lambda\sum_{j=1}^n d_{ij}(t)\mathrm{e}^{x_j(t-\tau_{ij}(t))}\Big]'\Big|\,\mathrm{d}t<2T\bar a_i.$$

By (5), we have

$$\int_0^T a_i(t)\,\mathrm{d}t\geqslant\int_0^T\Big[\sum_{j=1}^n(\beta_{ij})_m\mathrm{e}^{x_j(t)}+\sum_{j=1}^n(b_{ij}-d'_{ij})_m\mathrm{e}^{x_j(t-\tau_{ij}(t))}\Big]\,\mathrm{d}t \\ \geqslant T\Big[\sum_{j=1}^n(\beta_{ij})_m\mathrm{e}^{x_j(\xi_i)}+\sum_{j=1}^n(b_{ij}-d'_{ij})_m\mathrm{e}^{x_j(\xi_i-\tau_{ij}(\xi_i))}\Big],$$

for some $\xi_i \in [0,T]$. Therefore, we have

$$x_i(\xi_i) \leqslant \ln \frac{\bar{a}_i}{(\beta_{ii})_m}, \quad \mathrm{e}^{x_j(\xi_i - \tau_{ij}(\xi_i))} \leqslant \frac{\bar{a}_i}{(b_{ij} - d'_{ij})_m}. \tag{7}$$

From (6), (7), we have

$$\begin{aligned} &x_i(t) + \lambda \sum_{j=1}^{n} d_{ij}(t)\mathrm{e}^{x_j(t-\tau_{ij}(t))} \\ &\quad \leqslant x_i(\xi_i) + \lambda \sum_{j=1}^{n} d_{ij}(\xi_i)\mathrm{e}^{x_j(\xi_i-\tau_{ij}(\xi_i))} \\ &\qquad + \int_0^T \left| \left[x_i(t) + \lambda \sum_{j=1}^{n} d_{ij}(t)\mathrm{e}^{x_j(t-\tau_{ij}(t))} \right]' \right| \mathrm{d}t \\ &\quad \leqslant \ln \frac{\bar{a}_i}{(\beta_{ii})_m} + \sum_{j=1}^{n} \frac{\bar{a}_i |d_{ij}|_0}{(b_{ij} - d'_{ij})_m} + 2\bar{a}_i T =: R_i, \quad i = 1,2,\ldots,n; \end{aligned}$$

hence, we have $x_i(t) < R_i$, $i = 1,2,\ldots,n$.

From (4), we have

$$\begin{aligned} x'_i(t) = \lambda \Big[& a_i(t) - \sum_{j=1}^{n} \beta_{ij}(t)\mathrm{e}^{x_j(t)} \\ & - \sum_{j=1}^{n} b_{ij}(t)\mathrm{e}^{x_j(t-\tau_{ij}(t))} - \sum_{j=1}^{n} c_{ij}(t) x'_j(t-\tau_{ij}(t))\mathrm{e}^{x_j(t-\tau_{ij}(t))} \Big], \end{aligned}$$

$$\begin{aligned} |x'_i(t)| \leqslant \lambda \Big[& a_i(t) + \sum_{j=1}^{n} \beta_{ij}(t)\mathrm{e}^{x_j(t)} + \sum_{j=1}^{n} b_{ij}(t)\mathrm{e}^{x_j(t-\tau_{ij}(t))} \\ & + \sum_{j=1}^{n} c_{ij}(t) |x'_j(t-\tau_{ij}(t))| \mathrm{e}^{x_j(t-\tau_{ij}(t))} \Big] \\ < & |a_i|_0 + \sum_{j=1}^{n} |\beta_{ij}|_0 \mathrm{e}^{R_j} + \sum_{j=1}^{n} |b_{ij}|_0 \mathrm{e}^{R_j} + \sum_{j=1}^{n} |c_{ij}|_0 |x'_j|_0 \mathrm{e}^{R_j}. \end{aligned}$$

So, we have

$$\begin{aligned} \|x'\|_0 &\leqslant \sum_{i=1}^{n} |x'_i(t)|_0 \\ &\leqslant \sum_{i=1}^{n} |a_i|_0 + \sum_{i=1}^{n}\sum_{j=1}^{n} |\beta_{ij}|_0 \mathrm{e}^{R_j} + \sum_{i=1}^{n}\sum_{j=1}^{n} |b_{ij}|_0 \mathrm{e}^{R_j} + \sum_{i=1}^{n}\sum_{j=1}^{n} |c_{ij}|_0 |x'_j|_0 \mathrm{e}^{R_j} \\ &\leqslant \sum_{i=1}^{n} |a_i|_0 + \sum_{i=1}^{n}\sum_{j=1}^{n} |\beta_{ij}|_0 \mathrm{e}^{R_j} + \sum_{i=1}^{n}\sum_{j=1}^{n} |b_{ij}|_0 \mathrm{e}^{R_j} + \sum_{i=1}^{n}\sum_{j=1}^{n} |c_{ij}|_0 \|x'\|_0 \mathrm{e}^{R_j}. \end{aligned}$$

Since

$$\sum_{i=1}^{n}\sum_{j=1}^{n}|c_{ij}|_0 \mathrm{e}^{R_j} \leqslant \sum_{i=1}^{n}\sum_{j=1}^{n}|c_{ij}|_0 \mathrm{e}^{M_0} \leqslant \sum_{i=1}^{n}\sum_{j=1}^{n}|d_{ij}|_0 \mathrm{e}^{M_0} < 1,$$

we have

$$\|x'\|_0 < \frac{\sum_{i=1}^{n}|a_i|_0 + \sum_{i=1}^{n}\sum_{j=1}^{n}|\beta_{ij}|_0 \mathrm{e}^{R_j} + \sum_{i=1}^{n}\sum_{j=1}^{n}|b_{ij}|_0 \mathrm{e}^{R_j}}{1 - \sum_{i=1}^{n}\sum_{j=1}^{n}|c_{ij}|_0 \mathrm{e}^{R_j}} =: M_*. \tag{8}$$

Let $s = t - \tau_{ij}(t)$, $t = \sigma_{ij}(s)$ be the inverse function of $s = t - \tau_{ij}(t)$ $(t \in [0, T])$. Then, we have

$$\begin{aligned}\int_0^T (b_{ij}(t) - d'_{ij}(t))\mathrm{e}^{x_j(t-\tau_{ij}(t))}\,\mathrm{d}t &= \int_{-\tau_{ij}(0)}^{T-\tau_{ij}(T)} \frac{b_{ij}(\sigma_{ij}(s)) - d'_{ij}(\sigma_{ij}(s))}{1 - \tau'_{ij}(\sigma_{ij}(s))}\mathrm{e}^{x_j(s)}\,\mathrm{d}s \\ &= \frac{b_{ij}(\eta_{ij}) - d'_{ij}(\eta_{ij})}{1 - \tau'_{ij}(\eta_{ij})} \int_{-\tau_{ij}(0)}^{T-\tau_{ij}(T)} \mathrm{e}^{x_j(s)}\,\mathrm{d}s \\ &= \frac{b_{ij}(\eta_{ij}) - d'_{ij}(\eta_{ij})}{1 - \tau'_{ij}(\eta_{ij})} \int_0^T \mathrm{e}^{x_j(s)}\,\mathrm{d}s,\end{aligned}$$

for some $\eta_{ij} \in [0, T]$; and $\int_0^T \beta_{ij}(t)\mathrm{e}^{x_j(t)}\,\mathrm{d}t = \beta_{ij}(\mu_{ij})\int_0^T \mathrm{e}^{x_j(t)}\,\mathrm{d}t$, for some $\mu_{ij} \in [0, T]$; hence, from (5), we have

$$\sum_{j=1}^{n}\left[\beta_{ij}(\mu_{ij}) + \frac{b_{ij}(\eta_{ij}) - d'_{ij}(\eta_{ij})}{1 - \tau'_{ij}(\eta_{ij})}\right]\int_0^T \mathrm{e}^{x_j(t)}\,\mathrm{d}t = \int_0^T a_i(t)\,\mathrm{d}t, \quad i = 1, 2, \ldots, n.$$

Since $\int_0^T \mathrm{e}^{x_j(t)}\,\mathrm{d}t = T\mathrm{e}^{x_j(\delta_j)}$, for some $\delta_j \in [0, T]$ $(j = 1, 2, \ldots, n)$, we have

$$\sum_{j=1}^{n}\left[\beta_{ij}(\mu_{ij}) + \frac{b_{ij}(\eta_{ij}) - d'_{ij}(\eta_{ij})}{1 - \tau'_{ij}(\eta_{ij})}\right]\mathrm{e}^{x_j(\delta_j)} = \bar{a}_i, \quad i = 1, 2, \ldots, n. \tag{9}$$

From (9), we have

$$m_{ii}\mathrm{e}^{x_i(\delta_i)} \leqslant \bar{a}_i, \quad i = 1, 2, \ldots, n.$$

Therefore, we have

$$x_i(\delta_i) \leqslant \ln\frac{\bar{a}_i}{m_{ii}}, \quad i = 1, 2, \ldots, n. \tag{10}$$

From (9), (10), we get

$$\begin{aligned}\bar{a}_i &\leqslant M_{ii}\mathrm{e}^{x_i(\delta_i)} + \sum_{j=1,\, j\neq i} M_{ij}\mathrm{e}^{x_j(\delta_j)}\\ &\leqslant M_{ii}\mathrm{e}^{x_i(\delta_i)} + \sum_{j=1,\, j\neq i} \frac{M_{ij}\bar{a}_j}{m_{jj}}, \quad i = 1, 2, \ldots, n.\end{aligned}$$

Therefore, we have

$$x_i(\delta_i) \geqslant \ln \frac{\bar{a}_i - \sum\limits_{j=1,\, j\neq i} \frac{M_{ij}\bar{a}_j}{m_{jj}}}{M_{ii}}, \quad i = 1, 2, \ldots, n. \tag{11}$$

From (10), (11), we have

$$|x_i(\delta_i)| \leqslant \max\left\{\left|\ln\frac{\bar{a}_i}{m_{ii}}\right|, \left|\ln \frac{\bar{a}_i - \sum\limits_{j=1,\, j\neq i} \frac{M_{ij}\bar{a}_j}{m_{jj}}}{M_{ii}}\right|\right\} =: K_i, \quad i = 1, 2, \ldots, n. \tag{12}$$

Combining (8), (12), we have

$$|x_i| \leqslant |x_i(\delta_i)| + \int_0^T |x_i'| \,\mathrm{d}t \leqslant K_i + \int_0^T |x_i'| \,\mathrm{d}t, \quad i = 1, 2, \ldots, n;$$

hence,

$$\|x\|_0 \leqslant \sum_{i=1}^n K_i + \int_0^T \|x'\|_0 \,\mathrm{d}t = \sum_{i=1}^n K_i + TM_* \leqslant M_0.$$

Obviously, M_0 is independent of λ. This completes the proof. $\square$

Proof of Theorem B. We apply Theorem A to (3). Clearly, for M as given in Lemma 1, the condition (i) in Theorem A is satisfied. Let $g(u) = (g_1(u), \ldots, g_n(u))$. Since

$$\begin{aligned}g_i(u) &= \int_0^T f_i(s, \hat{u}) \,\mathrm{d}s = \int_0^T a_i(t)\,\mathrm{d}t - \sum_{j=1}^n \int_0^T \beta_{ij}(t)\,\mathrm{d}t\,\mathrm{e}^{u_j} - \sum_{j=1}^n \int_0^T b_{ij}(t)\,\mathrm{d}t\,\mathrm{e}^{u_j}\\ &= T\left[\bar{a}_i - \sum_{j=1}^n (\bar{\beta}_{ij} + \bar{b}_{ij})\mathrm{e}^{u_j}\right],\end{aligned}$$

and $M > \sum\limits_{i=1}^n |\ln u_i^*|$, we have $g(u) \neq 0$ for any $u \in \partial B_M(\mathbb{R}^n)$. Thus, the condition (ii) in Theorem A holds. Next we show that condition (iii) also holds. By (b) and the formula for Brouwer degree (see Theorem 2.2.3, [4]), we have

$$\deg(g, B_M(\mathbb{R}^n)) = \sum_{u \in g^{-1}(0) \cap B_M(\mathbb{R}^n)} \operatorname{sign} \det Dg(u) = (-1)^n \text{ or } (-1)^{n+1}.$$

Therefore, all the conditions required in Theorem A hold. It follows by Theorem A and Remark 1 that (3) has a T-periodic solution $(x_1^*(t), \ldots, x_n^*(t))$. Therefore, (2) has a positive T-periodic solution $(\mathrm{e}^{x_1^*(t)}, \ldots, \mathrm{e}^{x_n^*(t)})$. This finishes the proof of Theorem B. $\square$

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Author's address: Department of Systems Science and Applied Mathematics, Kunming University of Science and Technology, Kunming, Yunnan 650051, P. R. China, and Department of Applied Mathematics, Hunan University, Changsha 410082, P. R. China, e-mail: `huifang@public.km.yn.cn`.