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IDEAL EXTENSIONS OF GRAPH ALGEBRAS

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Abstract. Let $\mathscr{A}$ and $\mathscr{B}$ be graph algebras. In this paper we present the notion of an ideal in a graph algebra and prove that an ideal extension of $\mathscr{A}$ by $\mathscr{B}$ always exists. We describe (up to isomorphism) all such extensions.

Keywords: oriented graph, graph (Shallon) algebra, congruence relation, ideal, quotient graph algebra, ideal extension

MSC 2000: 08A30

0. Introduction

In this paper we study congruence relations on graph (Shallon) algebras and introduce the notion of an ideal in the graph algebra determined by a congruence. Then the aim is, for two given graph algebras $\mathscr{X}$ and $\mathscr{Y}$, to construct a graph algebra $\mathscr{A}$ for which we can find a congrunce Θ on $\mathscr{A}$ such that the ideal of $\mathscr{A}$ determined by the congruence Θ is the algebra $\mathscr{X}$ and the quotient graph algebra $\mathscr{A}/\Theta$ is isomorphic to the graph algebra $\mathscr{Y}$. (The algebra $\mathscr{A}$ will be referred to as an ideal extension of $\mathscr{X}$ by $\mathscr{Y}$.) Our objective is to answer the following questions:

(Q1) is the ideal extension always possible?

(Q2) is it possible to determine all ideal extensions?

We construct a class of ideal extensions of $\mathscr{X}$ by $\mathscr{Y}$ and denote it by $\Gamma_{\mathscr{X},\mathscr{Y}}(Z,\varphi)$.

The construction itself enables us to answer the question (Q1). Afterwards we show that the class $\Gamma_{\mathscr{X},\mathscr{Y}}(Z,\varphi)$ contains all ideal extensions.

Similar ideal related extensions were carried out for other algebraic structures, for instance for lattice ordered groups (cf. [7]), semigroups (cf. [1]), ordered semigroups (cf. [2] and [5]), lattices (cf. [4]) and for partial monounary algebras (cf. [3]).

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1. Preliminaries

Let A be a nonempty set, E a binary relation on A. Then the corresponding relational structure (A, E) is called a (directed) graph. We admit the existence of loops in a graph.

To a directed graph (A, E) there corresponds an algebra $(A \cup \{\infty\}, \circ, \infty)$ where $\infty \notin A$ is a nullary operation and for any $x, y \in A \cup \{\infty\}$, the binary operation $\circ$ is defined by the formula

$$x \circ y = \begin{cases} x & \text{if } (x, y) \in E, \\ \infty & \text{otherwise.} \end{cases}$$

Then $(A \cup \{\infty\}, \circ, \infty)$ is called the graph algebra corresponding to the graph (A, E).

1.1 Definition. Let $\mathscr{A} = (A \cup \{\infty\}, \circ, \infty)$ be an algebra such that $\infty \notin A$ is a nullary operation and $\circ$ is a binary operation satisfying $x \circ y \in \{x, \infty\}$ for any $x, y \in A$ and $x \circ y = \infty$ if at least one of x, y is ∞. Then the algebra $\mathscr{A}$ will be called a *graph* (or *Shallon*) *algebra*.

Remark. By writing $\mathscr{A} = (A \cup \{\infty\}, \circ, \infty)$ we shall always mean that $\infty \notin A$. Thus this assumption will not be repeated.

Graph algebras were introduced by R. C. Shallon [12] and were studied, e.g. in [6], [8]–[11].

1.2 Definition. Let $\mathscr{A} = (A \cup \{\infty\}, \circ, \infty)$ be a graph algebra and let Θ be an equivalence relation on $\mathscr{A}$. The relation Θ is called a *congruence* if

$$(a, b) \in \Theta \text{ and } (c, d) \in \Theta \text{ imply } (a \circ c, b \circ d) \in \Theta \text{ whenever } a, b, c, d \in A \cup \{\infty\}.$$

The class of all equivalence relations on $\mathscr{A}$ will be denoted by $\mathrm{Eq}(\mathscr{A})$ and the class of all congruences on $\mathscr{A}$ will be denoted by $\mathrm{Cong}(\mathscr{A})$.

1.3 Definition. Let $\mathscr{A} = (A \cup \{\infty\}, \circ, \infty)$ be a graph algebra and let $\Theta \in \mathrm{Eq}(\mathscr{A})$. Denote by $\mathrm{Id}(\mathscr{A}, \Theta)$ the class of the equivalence Θ containing the element ∞, i.e.

$$\mathrm{Id}(\mathscr{A}, \Theta) = \{a \in A \cup \{\infty\} \colon (a, \infty) \in \Theta\}.$$

Then the subalgebra $\mathscr{I}(\mathscr{A}) = (\mathrm{Id}(\mathscr{A}, \Theta), \circ, \infty)$ of $\mathscr{A}$ will be referred to as an *ideal in the graph algebra $\mathscr{A}$ determined by the equivalence Θ* or shortly an *ideal in the graph algebra $\mathscr{A}$*.

Remark. The ideal $\mathscr{I}(\mathscr{A})$ in the graph algebra $\mathscr{A}$ is a graph algebra as well.

1.4 Notation. Let $\mathscr{A} = (A \cup \{\infty\}, \circ, \infty)$ be a graph algebra. We introduce "successors" $\mathscr{S}_{\mathscr{A}}(x)$ and "predecessors" $\mathscr{P}_{\mathscr{A}}(x)$ of an element $x \in A$ by

$$\mathscr{S}_{\mathscr{A}}(x) = \{y \in A \colon\ x \circ y = x\},$$
$$\mathscr{P}_{\mathscr{A}}(x) = \{y \in A \colon\ y \circ x = y\}.$$

For completeness we put

$$\mathscr{S}_{\mathscr{A}}(\infty) = \mathscr{P}_{\mathscr{A}}(\infty) = \emptyset.$$

If there is no danger of confusion, we shall ommit indices "$\mathscr{A}$" in $\mathscr{S}_{\mathscr{A}}(x)$ and $\mathscr{P}_{\mathscr{A}}(x)$.

Next, let $x \in A$. Then $\mathscr{A}(x)$ will denote the set of those elements y in A for which there exists exist $n \in \mathbb{N}$ and a sequence $\{x_i\}_{i=0}^n$ with $x_0, \ldots, x_n \in A$ such that

$$\begin{aligned} x_0 &= x, \\ x_n &= y, \\ x_{i+1} \circ x_i &= x_{i+1} \quad \text{for } i = 0, \ldots, n-1. \end{aligned}$$

For an element ∞ we put $\mathscr{A}(\infty) = \emptyset$.

Remark. It is worth realizing that for $x \in A$ we have $\mathscr{P}_{\mathscr{A}}(x) \subseteq \mathscr{A}(x)$.

2. Congruences on graph algebras

2.1 Theorem. *Let $\mathscr{A} = (A \cup \{\infty\}, \circ, \infty)$ be a graph algebra, $\Theta \in \mathrm{Eq}(\mathscr{A})$ and let the subalgebra $(X \cup \{\infty\}, \circ, \infty)$ of $\mathscr{A}$ be the ideal in $\mathscr{A}$ determined by the equivalence Θ. Then $\Theta \in \mathrm{Cong}(\mathscr{A})$ if and only if the following three conditions hold:*

(i) *$\mathscr{A}(x) \subseteq X$ whenever $x \in X \cup \{\infty\}$,*
(ii) *if $(x, y) \in \Theta$ then $\mathscr{S}(x) = \mathscr{S}(y)$ whenever $x, y \in A \setminus X$,*
(iii) *if $(x, y) \in \Theta$ then $\mathscr{P}(x) \setminus X = \mathscr{P}(y) \setminus X$ whenever $x, y \in A \cup \{\infty\}$.*

Proof. First let Θ be a congruence on $\mathscr{A}$.

(i) If $x = \infty$ then $\mathscr{A}(x) = \emptyset \subseteq X$. Let $x \in X$ and $y \in \mathscr{A}(x)$. Then $(x, \infty) \in \Theta$ and there exist $n \in \mathbb{N}$ and a sequence $\{x_i\}_{i=0}^n$ with $x_0, \ldots, x_n \in A$ such that $x_0 = x$, $x_n = y$, and $x_{i+1} \circ x_i = x_{i+1}$ for $i = 0, \ldots, n-1$. We shall proceed by induction with respect to n, the length of the sequence.

If $n = 1$ then $y \circ x = y$. The relation Θ is reflexive, thus $(y, y) \in \Theta$. Since Θ is a congruence, $(y \circ x, y \circ \infty) = (y, \infty) \in \Theta$, which implies that $y \in X$.

Let $n \in \mathbb{N}$, $n \geqslant 1$. Suppose that for all elements y in $\mathscr{A}(x)$ with the corresponding sequence of length at the most n we have $y \in X$. If $\tilde{y} \in \mathscr{A}(x)$ and $x_0 = x$, $x_{n+1} = \tilde{y}$,

$x_{i+1} \circ x_i = x_{i+1}$ for $i = 0, \ldots, n$ then the element $y = x_n \in \mathscr{A}(x)$ belongs to X and by the assumption for sequences of length 1, namely for $\tilde{x}_0 = y$, $\tilde{x}_1 = \tilde{y}$ and

$$\tilde{x}_1 \circ \tilde{x}_0 = \tilde{y} \circ y = x_{n+1} \circ x_n = x_{n+1} = \tilde{y} = \tilde{x}_1,$$

we obtain $\tilde{y} \in X$. Therefore $\mathscr{A}(x) \subseteq X$.

(ii) Now take $x, y \in A \setminus X$ such that $(x, y) \in \Theta$. If $x = y$ then obviously $\mathscr{S}(x) = \mathscr{S}(y)$. Let $x \neq y$. If both the sets $\mathscr{S}(x)$ and $\mathscr{S}(y)$ are empty, the equality $\mathscr{S}(x) = \mathscr{S}(y)$ holds. Now suppose that at least one of $\mathscr{S}(x)$ and $\mathscr{S}(y)$ is nonempty. Without loss of generality let $\mathscr{S}(x) \neq \emptyset$.

Take $s \in \mathscr{S}(x)$, i.e. $x \circ s = x$. Then $(x, y) \in \Theta$ and $(s, s) \in \Theta$ imply $(x \circ s, y \circ s) = (x, y \circ s) \in \Theta$. By Definition 1.1, $y \circ s \in \{y, \infty\}$. In the case $y \circ s = \infty$ we get $(x, \infty) \in \Theta$, which contradicts the assumption $x \in A \setminus X$. Necessarily, $y \circ s = y$ which yields $s \in \mathscr{S}(y)$. We have obtained $\mathscr{S}(x) \subseteq \mathscr{S}(y)$. Analogously, we can infer $\mathscr{S}(y) \subseteq \mathscr{S}(x)$. Thus $\mathscr{S}(y) = \mathscr{S}(x)$.

(iii) First, consider an element $x \in X \cup \{\infty\}$. If $x = \infty$ then $\mathscr{P}(x) = \emptyset$ and thus $\mathscr{P}(x) \setminus X = \emptyset$. Now let $x \neq \infty$. Since $\mathscr{P}(x) \subseteq \mathscr{A}(x)$, in view of (i) we obtain $\mathscr{P}(x) \subseteq X$, thus for $x \in X \cup \{\infty\}$ we get $\mathscr{P}(x) \setminus X = \emptyset$. The assumption $(x, y) \in \Theta$ implies $y \in X \cup \{\infty\}$, thus $\mathscr{P}(y) \setminus X = \emptyset$. Therefore $\mathscr{P}(x) \setminus X = \mathscr{P}(y) \setminus X$.

Now take $x \in A \setminus X$. Clearly, $(x, y) \in \Theta$ implies $y \notin X$. If $\mathscr{P}(x) \setminus X = \emptyset$ then trivially $\mathscr{P}(x) \setminus X \subseteq \mathscr{P}(y) \setminus X$. Assume that $\mathscr{P}(x) \setminus X \neq \emptyset$. Then there exists an element $z \in A \setminus X$ such that $z \circ x = z$. From $(z, z) \in \Theta$ and $(x, y) \in \Theta$ we get $(z \circ x, z \circ y) = (z, z \circ y) \in \Theta$. The element z does not belong to X, therefore $z \circ y = z$. Consequently, $z \in \mathscr{P}(y) \setminus X$. We have shown that $\mathscr{P}(x) \setminus X \subseteq \mathscr{P}(y) \setminus X$. In an analogous way we can prove the converse inclusion $\mathscr{P}(y) \setminus X \subseteq \mathscr{P}(x) \setminus X$. Thus $\mathscr{P}(x) \setminus X = \mathscr{P}(y) \setminus X$.

Conversely, suppose that Θ is an equivalence satisfying the conditions (i)–(iii). Assume that $a, b, c, d \in A \cup \{\infty\}$ and $(a, b) \in \Theta$, $(c, d) \in \Theta$. Taking into account that $x \circ y \in \{x, \infty\}$, we obtain four possible combinations to verify.

First, let $a \circ c = a$ and $b \circ d = b$. In this case the result is trivial because $(a \circ c, b \circ d) = (a, b) \in \Theta$.

Next, let $a \circ c = \infty$ and $b \circ d = b \neq \infty$. Evidently, $d \neq \infty$. If $a = \infty$ then $b \in X \cup \{\infty\}$. Thus $(a \circ c, b \circ d) = (\infty, b) \in \Theta$. If $a \neq \infty$ and $c = \infty$ then $d \in X \cup \{\infty\}$. In view of (i) for $b \in \mathscr{P}(d)$ we obtain $b \in X$, thus again $(a \circ c, b \circ d) = (\infty, b) \in \Theta$. Now let $a \neq \infty \neq c$. The assumptions yield $a \notin \mathscr{P}(c)$ and $d \in \mathscr{S}(b)$. If $a \notin X$, then $b \notin X$. By (ii) we have $\mathscr{S}(a) = \mathscr{S}(b)$. Therefore $d \in \mathscr{S}(a)$, or inversely $a \in \mathscr{P}(d)$. In view of (iii) we obtain $\mathscr{P}(c) \setminus X = \mathscr{P}(d) \setminus X$. Since $a \notin X$ we get $a \in \mathscr{P}(c)$, a contradiction. Necessarily, $a \in X$ and consequently $b \in X$, which implies $(b, \infty) \in \Theta$. Therefore $(a \circ c, b \circ d) = (\infty, b) \in \Theta$.

The following possibility, $a\circ c = a \neq \infty$ and $b\circ d = \infty$, is analogous to the previous one because the relation Θ is symmetric.

Finally, let $a \circ c = \infty$ and $b \circ d = \infty$. Now $(a \circ c, b \circ d) = (\infty, \infty) \in \Theta$ because the relation Θ is reflexive. $\square$

3. Extensions of graph algebras

Let $\mathscr{A} = (A \cup \{\infty\}, \circ, \infty)$ be a graph algebra and Θ a congruence relation on $\mathscr{A}$. In what follows, we denote by $[x]_\Theta$ the set $\{y \in A \cup \{\infty\} \colon (x, y) \in \Theta\}$. We define a quotient graph algebra $\mathscr{A}/\Theta = (A \cup \{\infty\}/\Theta, \bullet, \mathrm{Id}(\mathscr{A}, \Theta))$ in a natural way, i.e. the binary operation $\bullet$ is defined as follows:

$$[x]_\Theta \bullet [y]_\Theta = [x \circ y]_\Theta \quad \text{whenever } [x]_\Theta, [y]_\Theta \in A \cup \{\infty\}/\Theta.$$

3.1 Definition. Let $\mathscr{X} = (X \cup \{\infty_{\mathscr{X}}\}, \odot, \infty_{\mathscr{X}})$, $\mathscr{Y} = (Y \cup \{\infty_{\mathscr{Y}}\}, \triangle, \infty_{\mathscr{Y}})$ be graph algebras. A graph algebra $\mathscr{A} = (A \cup \{\infty\}, \circ, \infty)$ is called an *ideal extension of the graph algebra $\mathscr{X}$ by the graph algebra $\mathscr{Y}$*, if $X \subseteq A$, $\infty = \infty_{\mathscr{X}}$ and there exists a congruence Θ on $\mathscr{A}$ such that the subalgebra $(X \cup \{\infty\}, \circ, \infty)$ of $\mathscr{A}$ is the ideal in $\mathscr{A}$ determined by the equivalence Θ and the quotient graph algebra $\mathscr{A}/\Theta$ is isomorphic to $\mathscr{Y}$.

Now our aim is to describe, for given graph algebras $\mathscr{X}$ and $\mathscr{Y}$, all possible ideal extensions $\mathscr{A}$, as well as to determine whether an ideal extension of $\mathscr{X}$ by $\mathscr{Y}$ always exists. Conforming with Definition 3.1, we shall take algebras $\mathscr{A}$ with $A \supseteq X$ and the nullary operation identical to that of $\mathscr{X}$. Therefore the index "$\mathscr{X}$" in $\infty_{\mathscr{X}}$ will not be necessary.

3.2 Definition. Let $\mathscr{X} = (X \cup \{\infty\}, \odot, \infty)$ and $\mathscr{Y} = (Y \cup \{\infty_{\mathscr{Y}}\}, \triangle, \infty_{\mathscr{Y}})$ be graph algebras such that $X \cap Y = \emptyset$.

Take an arbitrary set Z such that $Z \cap (X \cup \{\infty\}) = Z \cap (Y \cup \{\infty_{\mathscr{Y}}\}) = \emptyset$ and any mapping $\varphi\colon Z \to Y$.

We define a graph algebra $\mathscr{A} = (A \cup \{\infty\}, \circ, \infty)$. The base set of $\mathscr{A}$ is $X \cup Y \cup Z \cup \{\infty\}$, i.e. we put $A = X \cup Y \cup Z$. The operation $\circ$ is defined in the following way:

▷ if $a = \infty$ or $b = \infty$ then

$$a \circ b = \infty; \tag{3.2.1}$$

▷ if $a, b \in X$ then

$$a \circ b = a \odot b; \tag{3.2.2}$$

▷ if $a \in X$ and $b \in Y \cup Z$ then

$$a \circ b \in \{a, \infty\}; \tag{3.2.3}$$

▷ if $a \in Y \cup Z$ and $b \in X$ then

$$a \circ b = \infty; \tag{3.2.4}$$

▷ if $a, b \in Y$ then

$$a \circ b = \begin{cases} a & \text{if } a \bigtriangleup b = a, \\ \infty & \text{if } a \bigtriangleup b = \infty_{\mathscr{Y}}; \end{cases} \tag{3.2.5}$$

▷ if $a \in Y$ and $b \in Z$ then

$$a \circ b = \begin{cases} a & \text{if } \varphi(b) \in \mathscr{S}_{\mathscr{Y}}(a), \\ \infty & \text{if } \varphi(b) \notin \mathscr{S}_{\mathscr{Y}}(a); \end{cases} \tag{3.2.6}$$

▷ if $a \in Z$ and $b \in Y$ then

$$a \circ b = \begin{cases} a & \text{if } \varphi(a) \in \mathscr{P}_{\mathscr{Y}}(b), \\ \infty & \text{if } \varphi(a) \notin \mathscr{P}_{\mathscr{Y}}(b); \end{cases} \tag{3.2.7}$$

▷ if $a, b \in Z$ then

$$a \circ b = \begin{cases} a & \text{if } \varphi(a) \in \mathscr{P}_{\mathscr{Y}}(\varphi(b)), \\ \infty & \text{if } \varphi(a) \notin \mathscr{P}_{\mathscr{Y}}(\varphi(b)). \end{cases} \tag{3.2.8}$$

We denote the class of all graph algebras $\mathscr{A}$ constructed in this way by $\Gamma_{\mathscr{X},\mathscr{Y}}(Z, \varphi)$.

3.3 Lemma. *Let $\mathscr{X} = (X \cup \{\infty\}, \odot, \infty)$, $\mathscr{Y} = (Y \cup \{\infty_{\mathscr{Y}}\}, \bigtriangleup, \infty_{\mathscr{Y}})$, $\mathscr{A} = (A \cup \{\infty\}, \circ, \infty)$ be graph algebras such that $X \cap Y = \emptyset$, $\mathscr{A} \in \Gamma_{\mathscr{X},\mathscr{Y}}(Z, \varphi)$ and let Θ be an equivalence relation on $\mathscr{A}$ satisfying the following conditions:*
(a) *if $x \in X \cup \{\infty\}$ and $(x, y) \in \Theta$ then $y \in X \cup \{\infty\}$,*
(b) *if $x, y \in X \cup \{\infty\}$ then $(x, y) \in \Theta$,*
(c) *$(x, y) \in \Theta$ if and only if $x = y$, whenever $x, y \in Y$,*
(d) *$(x, y) \in \Theta$ if and only if $\varphi(y) = x$, whenever $x \in Y$, $y \in Z$,*
(e) *$(x, y) \in \Theta$ if and only if $\varphi(y) = \varphi(x)$, whenever $x, y \in Z$.*
Then Θ is a congruence relation on $\mathscr{A}$ and $\operatorname{Id}(\mathscr{A}, \Theta) = X \cup \{\infty\}$.

P r o o f. Let Θ be an equivalence relation on $\mathscr{A}$ satisfying the conditions (a)–(e). We shall verify the conditions (i)–(iii) of Theorem 2.1.

(i) For $x = \infty$ the condition (i) holds. Let $x \in X$ and $y \in A(x)$, i.e. there exist $n \in \mathbb{N}$ and a sequence $\{x_i\}_{i=0}^{n}$ such that $x_0 = x$, $x_n = y$, and $x_{i+1} \circ x_i = x_{i+1}$ for $i = 0, \dots, n-1$. If $x_1 \in Y \cup Z \cup \{\infty\}$ then by (3.2.1) and (3.2.4) we have $x_1 \circ x_0 = \infty$, a contradiction. Therefore $x_1 \in X$. By induction with respect to n we obtain that $y = x_n \in X$.

(ii) Now take $x, y \in Y \cup Z$ such that $(x, y) \in \Theta$. If $x = y$ then obviously $S_{\mathscr{A}}(x) = S_{\mathscr{A}}(y)$. Let $x \neq y$. If both the sets $S_{\mathscr{A}}(x)$ and $S_{\mathscr{A}}(y)$ are empty, the equality $S_{\mathscr{A}}(x) = S_{\mathscr{A}}(y)$ holds. Now suppose that at least one of $S_{\mathscr{A}}(x)$ and $S_{\mathscr{A}}(y)$ is nonempty. Without loss of generality let $S_{\mathscr{A}}(x) \neq \emptyset$.

First, if $x, y \in Y$ then $(x, y) \in \Theta$ implies by (c) the equality $x = y$, a contradiction with the above assumption.

Next, let $x \in Y$, $y \in Z$. Then $(x, y) \in \Theta$ implies $x = \varphi(y)$ (see (d)). Let $v \in \mathscr{S}_{\mathscr{A}}(x)$, i.e. $x \circ v = x$. In view of Definition 3.2 we obtain for the element $v \in A$ that v is either in Y or in Z. If $v \in Y$ then (3.2.5) implies $x \bigtriangleup v = x$. Therefore $\varphi(y) = x \in \mathscr{P}_{\mathscr{Y}}(v)$ and in view of (3.2.7) we get $y \circ v = y$, i.e. $v \in \mathscr{S}_{\mathscr{A}}(y)$. Now let $v \in Z$. Then (3.2.6) yields $\varphi(v) \in \mathscr{S}_{\mathscr{Y}}(x)$, thus $x \in \mathscr{P}_{\mathscr{Y}}(\varphi(v))$. Since $x = \varphi(y)$, we get $\varphi(y) \in \mathscr{P}_{\mathscr{Y}}(\varphi(v))$ and therefore by (3.2.8), $y \circ v = y$. Consequently, $v \in \mathscr{S}_{\mathscr{A}}(y)$.

Finally, if $x, y \in Z$ then by (e) we have $\varphi(x) = \varphi(y)$. Again, let $v \in \mathscr{S}_{\mathscr{A}}(x)$. Then $x \circ v = x$. In view of Definition 3.2 we infer that $v \in Y \cup Z$. If $v \in Y$ then $\varphi(x) \in \mathscr{P}_{\mathscr{Y}}(v)$ by (3.2.7) and consequently $\varphi(y) \in \mathscr{P}_{\mathscr{Y}}(v)$. Thus $y \circ v = y$, i.e. $v \in \mathscr{S}_{\mathscr{A}}(y)$. If $v \in Z$ then by (3.2.8) we have $\varphi(x) \in \mathscr{P}_{\mathscr{Y}}(\varphi(v))$ and consequently $\varphi(y) \in \mathscr{P}_{\mathscr{Y}}(\varphi(v))$. Thus $y \circ v = y$, i.e. $v \in \mathscr{S}_{\mathscr{A}}(y)$.

In all cases we have shown that $S_{\mathscr{A}}(x) \subseteq \mathscr{S}_{\mathscr{A}}(y)$. In a similar way we can prove that $S_{\mathscr{A}}(y) \subseteq \mathscr{S}_{\mathscr{A}}(x)$. Thus $S_{\mathscr{A}}(y) = S_{\mathscr{A}}(x)$.

(iii) Let $x, y \in A \cup \{\infty\}$ and $(x, y) \in \Theta$. Assume that $x \in X \cup \{\infty\}$. In view of (a) and (b) we have $(x, y) \in \Theta$ if and only if $y \in X \cup \{\infty\}$. Notice that for $x = \infty$ we have $\mathscr{P}_{\mathscr{A}}(x) = \emptyset$ and obviously $\mathscr{P}_{\mathscr{A}}(x) \setminus X = \emptyset$. In the case $x \in X$, we get by (3.2.2) and (3.2.4) the inclusion $\mathscr{P}_{\mathscr{A}}(x) \subseteq X$. Consequently $\mathscr{P}_{\mathscr{A}}(x) \setminus X = \emptyset$. Therefore for $x, y \in X \cup \{\infty\}$ we obtain $\mathscr{P}_{\mathscr{A}}(x) \setminus X = \mathscr{P}_{\mathscr{A}}(y) \setminus X$.

Let $x, y \in Y$. Then $(x, y) \in \Theta$ implies $x = y$, thus $\mathscr{P}_{\mathscr{A}}(x) \setminus X = \mathscr{P}_{\mathscr{A}}(y) \setminus X$ is valid.

To verify all the remaining possibilities, take $v \in \mathscr{P}_{\mathscr{A}}(x) \setminus X$, i.e. $v \notin X$ and $v \circ x = v$. First, suppose that $x \in Y$ and $y \in Z$. The assumption $(x, y) \in \Theta$ implies $\varphi(y) = x$. Let $v \in Y$. Then in view of (3.2.5), $v \circ x = v$ implies $v \bigtriangleup x = v$ and $x \in \mathscr{S}_{\mathscr{Y}}(v)$. Since $\varphi(y) = x$, we obtain $\varphi(y) \in \mathscr{S}_{\mathscr{Y}}(v)$, i.e. $v \in \mathscr{P}_{\mathscr{Y}}(\varphi(y))$. Therefore by (3.2.7) we get $v \circ y = v$. If $v \in Z$ then $v \circ x = v$ yields $\varphi(v) \in \mathscr{P}_{\mathscr{Y}}(x) = \mathscr{P}_{\mathscr{Y}}(\varphi(y))$ (see (3.2.7)). Consequently, by (3.2.8) we get $v \circ y = v$.

Further, if $x, y \in Z$, the assumption $(x, y) \in \Theta$ by (e) yields $\varphi(x) = \varphi(y)$. If $v \in Y$ then in view of (3.2.6) $v \circ x = v$ implies $\varphi(x) \in \mathscr{S}_{\mathscr{Y}}(v)$, thus $\varphi(y) \in \mathscr{S}_{\mathscr{Y}}(v)$ and again by (3.2.6) we get $v \circ y = v$. Finally, if $v \in Z$ then $v \circ x = v$ together with (3.2.8) yields the relation $\varphi(v) \in \mathscr{P}_{\mathscr{Y}}(\varphi(x))$. This implies $\varphi(v) \in \mathscr{P}_{\mathscr{Y}}(\varphi(y))$, thus again by (3.2.8) we obtain $v \circ y = v$.

To sum up, in all cases $v \in \mathscr{P}_{\mathscr{A}}(y) \setminus X$ therefore $\mathscr{P}_{\mathscr{A}}(x) \setminus X \subseteq \mathscr{P}_{\mathscr{A}}(y) \setminus X$. The converse inclusion can be proved analogously, thus $\mathscr{P}_{\mathscr{A}}(x) \setminus X = \mathscr{P}_{\mathscr{A}}(y) \setminus X$. $\square$

3.4 Theorem. *Let $\mathscr{X} = (X \cup \{\infty\}, \odot, \infty)$ and $\mathscr{Y} = (Y \cup \{\infty_{\mathscr{Y}}\}, \triangle, \infty_{\mathscr{Y}})$ be graph algebras such that $X \cap Y = \emptyset$. Let $\mathscr{A} = (A \cup \{\infty\}, \circ, \infty)$ be a graph algebra from the class $\Gamma_{\mathscr{X},\mathscr{Y}}(Z, \varphi)$ (for an arbitrary set Z and any mapping $\varphi\colon Z \to Y$). Then the graph algebra $\mathscr{A}$ is the ideal extension of $\mathscr{X}$ by $\mathscr{Y}$.*

P r o o f. Assume the equivalence relation Θ on $\mathscr{A}$ with one class $X \cup \{\infty\}$ and all other classes containing one element y of Y each and the elements of the set Z distributed to classes according to the mapping φ in the following way:

$$[y]_\Theta = \{y\} \cup \{z \in Z\colon \varphi(z) = y\} \quad \text{whenever } y \in Y.$$

In this way the equivalence Θ is uniquely determined and satisfies the conditions (a)–(e) of Lemma 3.3. Thus $\Theta \in \operatorname{Cong}(\mathscr{A})$.

In view of conditions (a) and (b), $\operatorname{Id}(\mathscr{A}, \Theta) = X \cup \{\infty\}$.

We shall prove that $\mathscr{A}/\Theta \cong \mathscr{Y}$. We recall that φ is an arbitrary mapping $Z \to Y$. Define a mapping $\Phi\colon A \cup \{\infty\}/\Theta \to Y \cup \{\infty_{\mathscr{Y}}\}$ such that

$$\Phi([x]_\Theta) = \begin{cases} \infty_{\mathscr{Y}} & \text{if } x \in X \cup \{\infty\}, \\ x & \text{if } x \in Y, \\ \varphi(x) & \text{if } x \in Z. \end{cases}$$

First, we shall look into the correctness of Φ by showing for $x, y \in A \cup \{\infty\}$ the implication

$$\text{if } [x]_\Theta = [y]_\Theta \quad \text{then} \quad \Phi([x]_\Theta) = \Phi([y]_\Theta).$$

Let $[x]_\Theta = [y]_\Theta$, i.e. $(x, y) \in \Theta$. By (a) we obtain that $x \in X \cup \{\infty\}$ and $(x, y) \in \Theta$ yield $y \in X \cup \{\infty\}$, thus $\Phi([x]_\Theta) = \infty_{\mathscr{Y}} = \Phi([y]_\Theta)$. If $x, y \in Y$ then $(x, y) \in \Theta$ implies $x = y$, thus $\Phi([x]_\Theta) = x = y = \Phi([y]_\Theta)$. Next, if $x \in Y$, $y \in Z$ and $(x, y) \in \Theta$ then $x = \varphi(y)$, therefore $\Phi([x]_\Theta) = x = \varphi(y) = \Phi([y]_\Theta)$. Finally, if $x, y \in Z$ then $(x, y) \in \Theta$ yields $\varphi(x) = \varphi(y)$, and so $\Phi([x]_\Theta) = \Phi([y]_\Theta)$.

Next, we shall prove injectivity of Φ, i.e. for $x, y \in A \cup \{\infty\}$ the implication

$$\text{if } \Phi([x]_\Theta) = \Phi([y]_\Theta) \quad \text{then} \quad [x]_\Theta = [y]_\Theta.$$

If $x \in X \cup \{\infty\}$ then $\Phi([y]_\Theta) = \Phi([x]_\Theta) = \infty_{\mathscr{Y}}$, which yields $y \in X \cup \{\infty\}$. Therefore by (b) $(x, y) \in \Theta$ and thus $[x]_\Theta = [y]_\Theta$. In view of (c)–(e) we obtain the following results. If $x, y \in Y$ then $x = \Phi([x]_\Theta) = \Phi([y]_\Theta) = y$, i.e. $[x]_\Theta = [y]_\Theta$. Next, if $x \in Y$, $y \in Z$ then $x = \Phi([x]_\Theta) = \Phi([y]_\Theta) = \varphi(y)$ again implies $(x, y) \in \Theta$. Finally, if $x, y \in Z$ then $\varphi(x) = \Phi([x]_\Theta) = \Phi([y]_\Theta) = \varphi(y)$, giving the same result as above: $[x]_\Theta = [y]_\Theta$.

Now we shall show that Φ is surjective. Take an arbitrary $y \in Y \cup \{\infty_{\mathscr{Y}}\}$. If $y = \infty_{\mathscr{Y}}$ then $\Phi([x]_\Theta) = y$ for any $x \in X \cup \{\infty\}$. In the case $y \neq \infty_{\mathscr{Y}}$ we take y as a pre-image of y, i.e. $\Phi([y]_\Theta) = y$.

Finally, we have to prove that Φ is a homomorphism, i.e. for $x, y \in A \cup \{\infty\}$

$$\Phi([x]_\Theta \bullet [y]_\Theta) = \Phi([x]_\Theta) \vartriangle \Phi([y]_\Theta).$$

Since $x \circ y \in \{x, \infty\}$, we shall consider two cases. If $x \circ y = \infty$ then

$$\Phi([x]_\Theta \bullet [y]_\Theta) = \Phi([x \circ y]_\Theta) = \Phi([\infty]_\Theta) = \infty_{\mathscr{Y}}.$$

To determine $\Phi([x]_\Theta) \vartriangle \Phi([y]_\Theta)$ we distinguish:

- ▷ if $x \in X \cup \{\infty\}$ then $\Phi([x]_\Theta) \vartriangle \Phi([y]_\Theta) = \infty_{\mathscr{Y}} \vartriangle \Phi([y]_\Theta) = \infty_{\mathscr{Y}}$ (the assumption $y \in X \cup \{\infty\}$ leads to the same result);
- ▷ if $x, y \in Y$ then by (3.2.5)

$$\Phi([x]_\Theta) \vartriangle \Phi([y]_\Theta) = \infty_{\mathscr{Y}};$$

- ▷ if $x \in Y$ and $y \in Z$ then $\varphi(y) \notin \mathscr{S}_{\mathscr{Y}}(x)$ by (3.2.6), which implies

$$\Phi([x]_\Theta) \vartriangle \Phi([y]_\Theta) = x \vartriangle \varphi(y) = \infty_{\mathscr{Y}};$$

- ▷ if $x \in Z$ and $y \in Y$ then $\varphi(x) \notin \mathscr{P}_{\mathscr{Y}}(y)$ by (3.2.7), thus

$$\Phi([x]_\Theta) \vartriangle \Phi([y]_\Theta) = \varphi(x) \vartriangle y = \infty_{\mathscr{Y}};$$

- ▷ if $x, y \in Z$ then $\varphi(x) \notin \mathscr{P}_{\mathscr{Y}}(\varphi(y))$ by (3.2.8), therefore

$$\Phi([x]_\Theta) \vartriangle \Phi([y]_\Theta) = \varphi(x) \vartriangle \varphi(y) = \infty_{\mathscr{Y}}.$$

Next, let $x \circ y = x$, hence $\Phi([x]_\Theta \bullet [y]_\Theta) = \Phi([x \circ y]_\Theta) = \Phi([x]_\Theta)$. Again, we distinguish the following possibilities:

- ▷ if $x \in X \cup \{\infty\}$ then

$$\Phi([x]_\Theta) \vartriangle \Phi([y]_\Theta) = \infty_{\mathscr{Y}} \vartriangle \Phi([y]_\Theta) = \infty_{\mathscr{Y}} = \Phi([x]_\Theta);$$

- ▷ the assumptions $y \in X \cup \{\infty\}$ and $x \notin X \cup \{\infty\}$ lead to a contradiction $x \circ y = \infty$;

▷ if $x, y \in Y$ then we get by (3.2.5)

$$\Phi([x]_\Theta) \vartriangle \Phi([y]_\Theta) = x \vartriangle y = x \circ y = x = \Phi([x]_\Theta);$$

▷ if $x \in Y$ and $y \in Z$ then $\varphi(y) \in \mathscr{S}_{\mathscr{Y}}(x)$ by (3.2.6), therefore

$$\Phi([x]_\Theta) \vartriangle \Phi([y]_\Theta) = x \vartriangle \varphi(y) = x = \Phi([x]_\Theta);$$

▷ if $x \in Z$ and $y \in Y$ then $\varphi(x) \in \mathscr{P}_{\mathscr{Y}}(y)$ by (3.2.7), thus

$$\Phi([x]_\Theta) \vartriangle \Phi([y]_\Theta) = \varphi(x) \vartriangle y = \varphi(x) = \Phi([x]_\Theta);$$

▷ if $x, y \in Z$ then $\varphi(x) \in \mathscr{P}_{\mathscr{Y}}(\varphi(y))$ by (3.2.8), which yields

$$\Phi([x]_\Theta) \vartriangle \Phi([y]_\Theta) = \varphi(x) \vartriangle \varphi(y) = \varphi(x) = \Phi([x]_\Theta).$$

For completeness we have to note that $\Phi(\mathrm{Id}(\mathscr{A}, \Theta)) = \Phi([\infty]_\Theta) = \infty_{\mathscr{Y}}$.

Now we can conclude that $\Phi\colon A \cup \{\infty\}/\Theta \to Y \cup \{\infty_{\mathscr{Y}}\}$ is a bijective homomorphism, therefore $\mathscr{A}$ is an ideal extension of $\mathscr{X}$ by $\mathscr{Y}$. □

In this way we can construct graph algebras $\mathscr{A}$ which are ideal extensions of $\mathscr{X}$ by $\mathscr{Y}$. Therefore the answer to the question (Q1), whether the ideal extension is always possible, is affirmative. In what follows we shall show that the class $\Gamma_{\mathscr{X},\mathscr{Y}}(Z, \varphi)$ contains all ideal extensions. In other words: if $\mathscr{B}$ is an ideal extension of $\mathscr{X}$ by $\mathscr{Y}$ then there exists a graph algebra $\mathscr{A} \in \Gamma_{\mathscr{X},\mathscr{Y}}(Z, \varphi)$ which is isomorphic to $\mathscr{B}$. Thus in the end we shall be able to claim that the reply to the question (Q2) is affirmative as well.

3.5 Theorem. *Let $\mathscr{X} = (X \cup \{\infty\}, \odot, \infty)$, $\mathscr{Y} = (Y \cup \{\infty_{\mathscr{Y}}\}, \vartriangle, \infty_{\mathscr{Y}})$, $\mathscr{B} = (B \cup \{\infty\}, \lozenge, \infty)$ be graph algebras, $X \cap Y = \emptyset$ and let $\mathscr{B}$ be the ideal extension of $\mathscr{X}$ by $\mathscr{Y}$. Then there exists a graph algebra $\mathscr{A} = (A \cup \{\infty\}, \circ, \infty)$ such that $\mathscr{A} \in \Gamma_{\mathscr{X},\mathscr{Y}}(Z, \varphi)$ and $\mathscr{A} \cong \mathscr{B}$.*

Proof. Let $\mathscr{B}$ be the ideal extension of $\mathscr{X}$ by $\mathscr{Y}$. Thus there exist $\Sigma \in \mathrm{Cong}(\mathscr{B})$ with $\mathrm{Id}(\mathscr{B}, \Sigma) = X \cup \{\infty\}$ and an isomorphism $\Omega\colon B \cup \{\infty\}/\Sigma \to Y$ of the quotient graph algebra $\mathscr{B}/\Sigma = (B \cup \{\infty\}/\Sigma, \lozenge, \mathrm{Id}(\mathscr{B}, \Sigma))$ onto $\mathscr{Y} = (Y \cup \{\infty_{\mathscr{Y}}\}, \vartriangle, \infty_{\mathscr{Y}})$.

By Axiom of Choice there is a mapping $\nu\colon B \cup \{\infty\}/\Sigma \to B$ in which to each equivalence class of Σ there corresponds a representant of the class.

Since Ω is bijective we can put

$$Z = B \setminus (X \cup \{\nu(\Omega^{-1}(y))\colon\ y \in Y\} \cup \{\infty\})$$

where Ω^{-1} is the inverse mapping to Ω and thus $\Omega^{-1}(y) \in B \cup \{\infty\}/\Sigma$. Notice that the sets X, $\{\nu(\Omega^{-1}(y))\colon\ y \in Y\}$ and $\{\infty\}$ are mutually disjoint because if

$$(X \cup \{\infty\}) \cap \{\nu(\Omega^{-1}(y))\colon\ y \in Y\} \neq \emptyset$$

then there is an element $x \in X \cup \{\infty\}$ satisfying $x = \nu(\Omega^{-1}(y))$ for some $y \in Y$. Therefore $\Omega^{-1}(y) = \mathrm{Id}(\mathscr{B}, \Sigma) = X \cup \{\infty\}$ and thus $y = \infty_{\mathscr{Y}}$, a contradiction to $y \in Y$.

Now, we shall define an algebra $\mathscr{A}$. Denote $A = X \cup Y \cup Z \cup \{\infty\}$ and take a mapping $\varphi\colon\ Z \to Y$ defined by the formula $\varphi(x) = \Omega([x]_\Sigma)$.

Now consider a mapping $\omega\colon\ A \to B$ defined in the following way:

$$\omega(x) = \begin{cases} x & \text{if } x \in X \cup Z \cup \{\infty\}, \\ \nu(\Omega^{-1}(x)) & \text{if } x \in Y. \end{cases}$$

Notice that for $x \in Y$ the image is $\omega(x) = \nu(\Omega^{-1}(x)) \in B \setminus (X \cup Z)$.

We shall show that ω is injective. Take $x, y \in A \cup \{\infty\}$ such that $\omega(x) = \omega(y)$. If $x, y \in X \cup Z \cup \{\infty\}$ then $x = \omega(x) = \omega(y) = y$. If both $x, y \in Y$, we apply injectivity of ν and Ω^{-1} to obtain

$$\nu(\Omega^{-1}(x)) = \nu(\Omega^{-1}(y)) \Longrightarrow \Omega^{-1}(x) = \Omega^{-1}(y) \Longrightarrow x = y.$$

Further, if $x \in X \cup Z \cup \{\infty\}$ and $y \in Y$ (or conversely) then the assumption $\omega(x) = \omega(y)$ leads to a contradiction because $\omega(x) \in X \cup Z \cup \{\infty\}$ and $\omega(y) \in B \setminus (X \cup Z \cup \{\infty\})$.

Let us prove that ω is surjective. If we take $x \in X \cup Z \cup \{\infty\}$ then the pre-image in ω is x itself, i.e. $\omega(x) = x$. Next, if $x \in B \setminus (X \cup Z \cup \{\infty\}) = \{\nu(\Omega^{-1}(y))\colon\ y \in Y\}$ then there exists an element $y \in Y$ such that $x = \nu(\Omega^{-1}(y))$. This very element y will be the pre-image of x in ω, i.e. $\omega(y) = \nu(\Omega^{-1}(y)) = x$.

So far, we have proved that ω is bijective.

Now, we shall define an operation $\circ$ on the set $A \cup \{\infty\}$. For elements $x, y \in A \cup \{\infty\}$ we put

$$x \circ y = \omega^{-1}(\omega(x) \lozenge \omega(y)).$$

In what follows, we verify that the operation $\circ$ satisfies the rules (3.2.1)–(3.2.8).

[3.2.1] Let $x = \infty$ or $y = \infty$. Then $x \circ y = \omega^{-1}(\omega(x) \lozenge \omega(y)) = \omega^{-1}(\infty) = \infty$.

[3.2.2] If $x, y \in X$ then $x \lozenge y = x \odot y$ ($\mathscr{B}$ is the ideal extension of $\mathscr{X}$) and $x \odot y \in \{x, \infty\}$. Therefore

$$x \circ y = \omega^{-1}(\omega(x) \lozenge \omega(y)) = \omega^{-1}(x \lozenge y) = \omega^{-1}(x \odot y) = x \odot y.$$

[3.2.3] Let $x \in X$ and $y \in Y \cup Z$, then $x \circ y \in \{x, \infty\}$ is always valid regardless of the result of $\omega^{-1}(\omega(x) \lozenge \omega(y))$.

[3.2.4] Let $x \in Y \cup Z$ and $y \in X$. Suppose that $\omega(x) \lozenge \omega(y) = \omega(x)$, i.e. $\omega(x) \in \mathscr{P}_{\mathscr{B}}(\omega(y)) = \mathscr{P}_{\mathscr{B}}(y)$. In view of the assumption $\Sigma \in \mathrm{Cong}(\mathscr{B})$ and Theorem 2.1 (i) we obtain that $y \in X$ implies $\omega(x) \in \mathscr{P}_{\mathscr{B}}(y) \subseteq \mathscr{B}(y) \subseteq X$. In the case $x \in Y$ we get $\nu(\Omega^{-1}(x)) = \omega(x) \in X$, then $\Omega^{-1}(x) = \mathrm{Id}(\mathscr{B}, \Sigma)$ and thus $x = \infty_{\mathscr{Y}}$, a contradiction to $x \in Y$. In the case $x \in Z$ we get $x = \omega(x) \in X$, again a contradiction to $x \in Y$. Therefore $\omega(x) \lozenge \omega(y) = \infty$ and

$$x \circ y = \omega^{-1}(\omega(x) \lozenge \omega(y)) = \omega^{-1}(\infty) = \infty.$$

[3.2.5] Let $x, y \in Y$. Since Ω (and thus Ω^{-1} as well) is an isomorphism, we obtain

$$\begin{aligned}
[\omega(x) \lozenge \omega(y)]_\Sigma &= [\nu(\Omega^{-1}(x)) \lozenge \nu(\Omega^{-1}(y))]_\Sigma \\
&= [\nu(\Omega^{-1}(x))]_\Sigma \blacklozenge [\nu(\Omega^{-1}(y))]_\Sigma \\
&= \Omega^{-1}(x) \blacklozenge \Omega^{-1}(y) = \Omega^{-1}(x \vartriangle y).
\end{aligned}$$

If $x \vartriangle y = x$ then $[\omega(x) \lozenge \omega(y)]_\Sigma = \Omega^{-1}(x) = [\nu(\Omega^{-1}(x))]_\Sigma = [\omega(x)]_\Sigma$. Suppose that $\omega(x) \lozenge \omega(y) = \infty$. Then $[\omega(x)]_\Sigma = \mathrm{Id}(\mathscr{B}, \Sigma)$ and $\omega(x) \in \mathrm{Id}(\mathscr{B}, \Sigma)$, a contradiction to $\omega(x) \in B \setminus (X \cup Z)$. Necessarily $\omega(x) \lozenge \omega(y) = \omega(x)$ and thus

$$x \circ y = \omega^{-1}(\omega(x) \lozenge \omega(y)) = \omega^{-1}(\omega(x)) = x.$$

Conversely, if $x \vartriangle y = \infty$ then $[\omega(x) \lozenge \omega(y)]_\Sigma = \Omega^{-1}(\infty_{\mathscr{Y}}) = \mathrm{Id}(\mathscr{B}, \Sigma)$. Thus $\omega(x) \lozenge \omega(y) \in \mathrm{Id}(\mathscr{B}, \Sigma)$. Since $\omega(x) \in B \setminus (X \cup Z)$, we can conclude that $\omega(x) \lozenge \omega(y) = \infty$ and consequently

$$x \circ y = \omega^{-1}(\omega(x) \lozenge \omega(y)) = \omega^{-1}(\infty) = \infty.$$

[3.2.6] Let $x \in Y$ and $y \in Z$. Then we have

$$\begin{aligned}
[\omega(x) \lozenge \omega(y)]_\Sigma &= [\nu(\Omega^{-1}(x)) \lozenge y]_\Sigma = [\nu(\Omega^{-1}(x))]_\Sigma \blacklozenge [y]_\Sigma \\
&= \Omega^{-1}(x) \blacklozenge \Omega^{-1}(\Omega([y]_\Sigma)) = \Omega^{-1}(x \vartriangle \Omega([y]_\Sigma)) \\
&= \Omega^{-1}(x \vartriangle \varphi(y)).
\end{aligned}$$

If $\varphi(y) \in \mathscr{S}_{\mathscr{Y}}(x)$ then $[\omega(x) \lozenge \omega(y)]_\Sigma = \Omega^{-1}(x) = [\nu(\Omega^{-1}(x))]_\Sigma = [\omega(x)]_\Sigma$. Suppose that $\omega(x) \lozenge \omega(y) = \infty$. Then $[\omega(x)]_\Sigma = \mathrm{Id}(\mathscr{B}, \Sigma)$ and $\omega(x) \in \mathrm{Id}(\mathscr{B}, \Sigma)$, a contradiction to $\omega(x) \in B \setminus (X \cup Z)$. Necessarily $\omega(x) \lozenge \omega(y) = \omega(x)$ and thus

$$x \circ y = \omega^{-1}(\omega(x) \lozenge \omega(y)) = \omega^{-1}(\omega(x)) = x.$$

Conversely, if $\varphi(y) \notin \mathscr{S}_{\mathscr{Y}}(x)$ then $[\omega(x) \Diamond \omega(y)]_{\Sigma} = \Omega^{-1}(\infty_{\mathscr{Y}}) = \mathrm{Id}(\mathscr{B}, \Sigma)$. Thus $\omega(x) \Diamond \omega(y) \in \mathrm{Id}(\mathscr{B}, \Sigma)$. Since $\omega(x) \in B \setminus (X \cup Z)$, we can conclude that $\omega(x) \Diamond \omega(y) = \infty$ and consequently

$$x \circ y = \omega^{-1}(\omega(x) \Diamond \omega(y)) = \omega^{-1}(\infty) = \infty.$$

[3.2.7] Let $x \in Z$ and $y \in Y$. In a similar way we obtain

$$\begin{aligned}[\omega(x) \Diamond \omega(y)]_{\Sigma} &= [x \Diamond \nu(\Omega^{-1}(y))]_{\Sigma} = [x]_{\Sigma} \blacklozenge [\nu(\Omega^{-1}(y))]_{\Sigma} \\ &= \Omega^{-1}(\Omega([x]_{\Sigma})) \blacklozenge \Omega^{-1}(y) = \Omega^{-1}(\Omega([x]_{\Sigma}) \vartriangle y) \\ &= \Omega^{-1}(\varphi(x) \vartriangle y).\end{aligned}$$

If $\varphi(x) \in \mathscr{P}_{\mathscr{Y}}(y)$ then $[\omega(x) \Diamond \omega(y)]_{\Sigma} = \Omega^{-1}(\varphi(x)) = \Omega^{-1}(\Omega([x]_{\Sigma})) = [x]_{\Sigma}$. Suppose that $\omega(x) \Diamond \omega(y) = \infty$. Then $[x]_{\Sigma} = \mathrm{Id}(\mathscr{B}, \Sigma)$ and $x \in \mathrm{Id}(\mathscr{B}, \Sigma)$, a contradiction to $x \in Z$. Necessarily $\omega(x) \Diamond \omega(y) = \omega(x)$ and thus

$$x \circ y = \omega^{-1}(\omega(x) \Diamond \omega(y)) = \omega^{-1}(\omega(x)) = x.$$

Conversely, if $\varphi(x) \notin \mathscr{P}_{\mathscr{Y}}(y)$ then $[\omega(x) \Diamond \omega(y)]_{\Sigma} = \Omega^{-1}(\infty_{\mathscr{Y}}) = \mathrm{Id}(\mathscr{B}, \Sigma)$. Thus $\omega(x) \Diamond \omega(y) \in \mathrm{Id}(\mathscr{B}, \Sigma)$. Since $\omega(x) = x \in Z$, we can conclude that $\omega(x) \Diamond \omega(y) = \infty$ and consequently

$$x \circ y = \omega^{-1}(\omega(x) \Diamond \omega(y)) = \omega^{-1}(\infty) = \infty.$$

[3.2.8] Finally, take $x, y \in Z$. In this case

$$\begin{aligned}[\omega(x) \Diamond \omega(y)]_{\Sigma} &= [x \Diamond y]_{\Sigma} = \Omega^{-1}(\Omega([x \Diamond y]_{\Sigma})) = \Omega^{-1}(\Omega([x]_{\Sigma} \blacklozenge [y]_{\Sigma})) \\ &= \Omega^{-1}(\Omega([x]_{\Sigma}) \vartriangle \Omega([y]_{\Sigma})) = \Omega^{-1}(\varphi(x) \vartriangle \varphi(y)).\end{aligned}$$

If $\varphi(x) \in \mathscr{P}_{\mathscr{Y}}(\varphi(y))$ then $[\omega(x) \Diamond \omega(y)]_{\Sigma} = \Omega^{-1}(\varphi(x)) = \Omega^{-1}(\Omega([x]_{\Sigma})) = [x]_{\Sigma}$. Since $x \in Z$ we can conclude that $\omega(x) \Diamond \omega(y)$ cannot be ∞. Therefore

$$x \circ y = \omega^{-1}(\omega(x) \Diamond \omega(y)) = \omega^{-1}(\omega(x)) = x.$$

Conversely, if $\varphi(x) \notin \mathscr{P}_{\mathscr{Y}}(\varphi(y))$ then we get $[\omega(x) \Diamond \omega(y)]_{\Sigma} = \Omega^{-1}(\infty_{\mathscr{Y}}) = \mathrm{Id}(\mathscr{B}, \Sigma)$ and this time $\omega(x) \Diamond \omega(y)$ cannot be $\omega(x) = x \in Z$. Thus

$$x \circ y = \omega^{-1}(\omega(x) \Diamond \omega(y)) = \omega^{-1}(\infty) = \infty.$$

In this way we have verified that $\mathscr{A} \in \Gamma_{\mathscr{X}, \mathscr{Y}}(Z, \varphi)$. In view of Theorem 3.4 the algebra $\mathscr{A}$ is an ideal extension of $\mathscr{X}$ by $\mathscr{Y}$.

Finally, we shall show that the graph algebras $\mathscr{A} = (A \cup \{\infty\}, \circ, \infty)$ and $\mathscr{B} = (B \cup \{\infty\}, \Diamond, \infty)$ are isomorphic. Recall that the mapping ω is bijective. Furthermore, this mapping preserves both operations, i.e. for any $x, y \in A \cup \{\infty\}$ we have

$$\omega(x \circ y) = \omega(\omega^{-1}(\omega(x) \Diamond \omega(y))) = \omega(x) \Diamond \omega(y)$$

and for the nullary operation we get $\omega(\infty) = \infty$. Thus ω is an isomorphism of $\mathscr{A}$ onto $\mathscr{B}$. $\square$

Remark. For the graph algebra $\mathscr{A}$ defined in the proof of the previous theorem we could consider the binary relation Θ on $\mathscr{A}$ defined by

$$(x, y) \in \Theta \Longleftrightarrow (\omega(x), \omega(y)) \in \Sigma \quad \text{whenever } x, y \in A \cup \{\infty\}.$$

Since $\Sigma \in \mathrm{Eq}(\mathscr{B})$ we immediately get $\Theta \in \mathrm{Eq}(\mathscr{A})$. Now take $a, b, c, d \in A \cup \{\infty\}$ such that $(a, b) \in \Theta$ and $(c, d) \in \Theta$. Then $(\omega(a), \omega(b)) \in \Sigma$ and $(\omega(c), \omega(d)) \in \Sigma$. Since Σ is a congruence, we obtain

$$\begin{aligned}(\omega(a) \Diamond \omega(c), \omega(b) \Diamond \omega(d)) &= (\omega(\omega^{-1}(\omega(a) \Diamond \omega(c))), \omega(\omega^{-1}(\omega(b) \Diamond \omega(d)))) \\ &= (\omega(a \circ c), \omega(b \circ d)) \in \Sigma,\end{aligned}$$

which implies that $(a \circ c, b \circ d) \in \Theta$. Thus $\Theta \in \mathrm{Cong}(\mathscr{A})$. Therefore

$$\mathscr{A}/\Theta \cong \mathscr{Y} \cong \mathscr{B}/\Sigma.$$

3.6 Remark. The class $\Gamma_{\mathscr{X},\mathscr{Y}}(Z, \varphi)$ contains all ideal extensions (up to isomorphism) of the graph algebra $\mathscr{X}$ by the graph algebra $\mathscr{Y}$.

3.7 Remark. In Definition 3.2 we define a new algebra $\mathscr{A}$ to be really an "extension" of the algebra $\mathscr{X}$. Therefore for the elements $a, b \in X$ we define the result of $a \circ b$ to be $a \odot b$ (see (3.2.2)). Nevertheless, the quotient graph algebra $\mathscr{A}/\Theta$ does not depend on the operation "$\odot$" in $\mathscr{X}$. Thus instead of (3.2.2) and (3.2.3) we could put one common condition

$$\text{if } a \in X \text{ and } b \in A \text{ then } a \circ b \in \{a, \infty\}.$$

In this case we do not require $\mathscr{X}$ to be a subalgebra of $\mathscr{A}$. Under this assumption we would construct a larger class $\Gamma_{\mathscr{X},\mathscr{Y}}(Z, \varphi)$.

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