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FACTORABLE CONGRUENCES AND FACTORABLE CONGRUENCE BLOCKS ON POWERS OF A FINITE ALGEBRA

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1. INTRODUCTION

R. Willard has proved in [4] that any power A^n, $n \geqslant 2$, of a finite k-element algebra A, $k \geqslant 2$, has factorable congruences whenever the power $A^{k^3+k^2-k}$ has the same property. In this paper the exponent $k^3 + k^2 - k$ is reduced to $3k^2 - 2k$. Further, it is shown that the factorability of congruence blocks on the power A^{2k^2-k} ensures this property on any power A^n, $n \geqslant 2$.

2. FACTORABLE CONGRUENCES

Definition 1. Let $A_1, \ldots, A_n$, $n \geqslant 2$, be algebras of the same type. We say that the product $B = A_1 \times \ldots \times A_n$ has *factorable congruences* whenever $\Theta = \Theta_1 \times \ldots \times \Theta_n$ holds for any congruence Θ on B where $\Theta_1, \ldots, \Theta_n$ are congruences on $A_1, \ldots, A_n$, respectively.

Notation 1. *Let $A_1, \ldots, A_n$, $n \geqslant 2$, be algebras of the same type, $B = A_1 \times \ldots \times A_n$. Elements $\langle a_1, \ldots, a_n \rangle$, $\langle b_1, \ldots, b_n \rangle$, ... of B are denoted by $\bar{a}$, $\bar{b}$, ... Further, denote*

$$\sigma(B) = \{\langle \bar{a}, \bar{b}, \bar{c}, \bar{d} \rangle \in B^4;\ \text{for each } i \leqslant n \text{ either } \langle a_i, b_i \rangle = \langle c_i, d_i \rangle \text{ or } a_i = b_i\}$$

and

$$\gamma(B) = \left\{ \begin{array}{r} \langle \bar{a}, \bar{b}, \bar{c}, \bar{d} \rangle \in B^4;\ \text{for each } i \leqslant n \text{ either } \langle a_i, b_i \rangle = \langle c_i, d_i \rangle \\ \text{or } a_i = b_i,\ c_i = d_i \ \text{ or } a_i = b_i = d_i \end{array} \right\}.$$

Notation 2. *Let B be an algebra, $c, d \in B$. Then the symbol $\Theta_B(c, d)$ denotes the principal congruence on B generated by the pair $\langle c, d \rangle$.*

Lemma 1. *Let $A_1, \ldots, A_n$, $n \geqslant 2$, be algebras of the same type, $B = A_1 \times \ldots \times A_n$. The following conditions are equivalent:*

(1) B has factorable congruences;

(2) $\langle \bar{a}, \bar{b}, \bar{c}, \bar{d} \rangle \in \sigma(B)$ implies $\langle \bar{a}, \bar{b} \rangle \in \Theta_B(\bar{c}, \bar{d})$ for any elements $\bar{a}, \bar{b}, \bar{c}, \bar{d} \in B$;

(3) $\langle \bar{a}, \bar{b}, \bar{c}, \bar{d} \rangle \in \gamma(B)$ implies $\langle \bar{a}, \bar{b} \rangle \in \Theta_B(\bar{c}, \bar{d})$ for any elements $\bar{a}, \bar{b}, \bar{c}, \bar{d} \in B$.

Proof. (1) $\Leftrightarrow$ (2): See [4; Lemma 4.3, p. 339].

(2) $\Rightarrow$ (3) is trivial since $\gamma(B) \subseteq \sigma(B)$;

(3) $\Rightarrow$ (2): Let $\langle \bar{a}, \bar{b}, \bar{c}, \bar{d} \rangle \in \sigma(B)$. *Then*$\langle a_i, b_i \rangle = \langle c_i, d_i \rangle$, $i \in I$, and $a_i = b_i$, $i \in J$, for some disjoint index sets I, J, $I \cup J = \{1, \ldots, n\}$.

(a) Introduce a new quadruple $\langle \bar{a}', \bar{b}', \bar{c}', \bar{d}' \rangle \in B^4$ by the rule

$$\langle a_i', b_i', c_i', d_i' \rangle = \begin{cases} \langle a_i, b_i, c_i, d_i \rangle & \text{for } i \in I \\ \langle d_i, d_i, c_i, d_i \rangle & \text{for } i \in J. \end{cases}$$

Then $\langle \bar{a}', \bar{b}', \bar{c}', \bar{d}' \rangle \in \gamma(B)$ and so $\langle \bar{a}', \bar{b}' \rangle \in \Theta_B(\bar{c}', \bar{d}')$, by hypothesis (3).

(b) Further, introduce a quadruple $\langle \bar{a}'', \bar{b}'', \bar{c}'', \bar{d}'' \rangle \in B^4$ via

$$\langle a_i'', b_i'', c_i'', d_i'' \rangle = \begin{cases} \langle a_i, b_i, c_i, d_i \rangle & \text{for } i \in I \\ \langle a_i, a_i, d_i, d_i \rangle & \text{for } i \in J. \end{cases}$$

Since evidently $\langle \bar{a}'', \bar{b}'', \bar{c}'', \bar{d}'' \rangle \in \gamma(B)$ we have $\langle \bar{a}'', \bar{b}'' \rangle \in \Theta_B(\bar{c}'', \bar{d}'')$, by (3) again.

Moreover $\langle \bar{a}', \bar{b}' \rangle = \langle \bar{c}'', \bar{d}'' \rangle$, $\langle \bar{c}', \bar{d}' \rangle = \langle \bar{c}, \bar{d} \rangle$, $\langle \bar{a}'', \bar{b}'' \rangle = \langle \bar{a}, \bar{b} \rangle$, and thus $\langle \bar{a}, \bar{b} \rangle = \langle \bar{a}'', \bar{b}'' \rangle \in \Theta_B(\bar{c}'', \bar{d}'') = \Theta_B(\bar{a}', \bar{b}') \subseteq \Theta_B(\bar{c}', \bar{d}') = \Theta_B(\bar{c}, \bar{d})$, i.e. $\langle \bar{a}, \bar{b} \rangle \in \Theta_B(\bar{c}, \bar{d})$, as required. □

Lemma 2. *Let B, C be algebras of the same type, φ a homomorphism from B to C. Then $\langle a, b \rangle \in \Theta_B(c, d)$ implies $\langle \varphi(a), \varphi(b) \rangle \in \Theta_C(\varphi(c), \varphi(d))$ for any elements $a, b, c, d \in B$.*

Proof. Applying the binary scheme, see [2; Thm 1, p. 41], to the relation formula $\langle a, b \rangle \in \Theta_B(c, d)$ we obtain

$$\begin{aligned} a &= t_1(c, d, b_1, \ldots, b_m), \\ t_i(d, c, b_1, \ldots, b_m) &= t_{i+1}(c, d, b_1, \ldots, b_m), \ 1 \leqslant i < n, \\ b &= t_n(d, c, b_1, \ldots, b_m) \end{aligned}$$

for some elements $b_1, \ldots, b_m \in B$ and suitable terms $t_1 \ldots, t_n$. Then

$$\begin{aligned} \varphi(a) &= t_1(\varphi(c), \varphi(d), \varphi(b_1), \ldots, \varphi(b_m)), \\ t_i(\varphi(d), \varphi(c), \varphi(b_1), \ldots, \varphi(b_m)) &= t_{i+1}(\varphi(c), \varphi(d), \varphi(b_1), \ldots, \varphi(b_m)), \ 1 \leqslant i < n, \\ \varphi(b) &= t_n(\varphi(d), \varphi(c), \varphi(b_1), \ldots, \varphi(b_m)), \end{aligned}$$

which means that $\langle \varphi(a), \varphi(b) \rangle \in \Theta_C(\varphi(c), \varphi(d))$, see [2] again. □

Notation 3. *Let C be an algebra, $p_1, p_2, p_3, p_4\colon C^4 \to C$ canonical projections, and S a subset of C^4. Then p_1^S, p_2^S, p_3^S, p_4^S denote the restrictions of p_1, p_2, p_3, p_4, respectively to S.*

Theorem 1. *Let C be a finite algebra. The following conditions are equivalent:*
(1) C^n has factorable congruences for any $n \geqslant 2$;
(2) $C^{\gamma(C)}$ has factorable congruences.

Proof. We use the arguments from [4; Lemma 4.4, p. 339]: Let $\langle \bar{a}, \bar{b}, \bar{c}, \bar{d}\rangle$ be an arbitrary quadruple from $\gamma(C^n)$, $n \geqslant 2$. It is a routine to verify that
(a) $\langle p_1^{\gamma(C)}, p_2^{\gamma(C)}, p_3^{\gamma(C)}, p_4^{\gamma(C)}\rangle \in \gamma(C^{\gamma(C)})$;
(b) the correspondence $\varphi\colon g \mapsto \langle g(a_1, b_1, c_1, d_1), \dots, g(a_n, b_n, c_n, d_n)\rangle$ is homomorphism from $C^{\gamma(C)}$ to C^n which sends $p_1^{\gamma(C)}$, $p_2^{\gamma(C)}$, $p_3^{\gamma(C)}$, $p_4^{\gamma(C)}$ to $\bar{a}, \bar{b}, \bar{c}, \bar{d}$, respectively.
Now, by hypothesis (2) the algebra $C^{\gamma(C)}$ has factorable congruences and so (a) implies $\langle p_1^{\gamma(C)}, p_2^{\gamma(C)}\rangle \in \Theta_{C^{\gamma(C)}}(p_3^{\gamma(C)}, p_4^{\gamma(C)})$. Applying the homomorphism φ to this principal congruence formula we obtain $\langle \bar{a}, \bar{b}\rangle \in \Theta_{C^n}(\bar{c}, \bar{d})$ which proves (1), see Lemma 1 again. □

Corollary 1. *Let C be a finite k-element algebra, $k \geqslant 2$. The following conditions are equivalent:*
(1) C^n has factorable congruences for any $n \geqslant 2$;
(2) C^{3k^2-2k} has factorable congruences.

Proof. Evidently card $\gamma(C) = 3k^2 - 2k$ whenever card $C = k$. □

Factorable congruence blocks

Definition 2. Let $A_1, \dots, A_n$, $n \geqslant 2$, be algebras of tha same type. A subset S of $B = A_1 \times \dots \times A_n$ is said to be *factorable* whenever $S = S_1 \times \dots \times S_n$ for some subsets $S_i \subseteq A_i$, $i \leqslant n$.
Further, we say that B has factorable congruence blocks whenever any congruence block on B is factorable.

Lemma 3. *Let $A_1, \dots, A_n$, $n \geqslant 2$, be algebras of the same type, S a subset of $B = A_1 \times \dots \times A_n$. The following conditions are equivalent:*
(1) S is factorable;
(2) $\bar{c}, \bar{d} \in S$ implies $\bar{a} \in S$ where $a_i \in \{c_i, d_i\}$, $i \leqslant n$.

Proof. (1) $\Rightarrow$ (2): Let $\bar{c}, \bar{d} \in S = S_1 \times \dots \times S_n$. Then $c_i, d_i \in S_i$, $i \leqslant n$, and thus also $a_i \in S_i$, $i \leqslant n$, for $a_i \in \{c_i, d_i\}$, $i \leqslant n$. Altogether, $\bar{a} = \langle a_1, \dots, a_n\rangle \in S_1 \times \dots \times S_n = S$ as required.

(2) $\Rightarrow$ (1): Denote $S_i = pr_i S$, $i \leqslant n$. Evidently the inclusion $S \subseteq S_1 \times \ldots \times S_n$ holds. Conversely, let $\bar{s} = \langle s_1, \ldots, s_n \rangle \in S_1 \times \ldots \times S_n$. Then there are elements $\langle a_{i1}, \ldots, a_{in} \rangle \in S$, $i \leqslant n$, such that $a_{ii} = s_i$, $i \leqslant n$, by the definition of subsets S_i, $i \leqslant n$. Now from $\langle a_{11}, \ldots, a_{1n} \rangle$, $\langle a_{21}, \ldots, a_{2n} \rangle \in S$ we obtain $\langle s_1, s_2, a_{23}, \ldots, a_{2n} \rangle = \langle a_{11}, a_{22}, a_{23}, \ldots, a_{2n} \rangle \in S$, by hypothesis (2). Repeating this process we find that $\bar{s} = \langle s_1, \ldots, s_n \rangle \in S$, which proves the factorability of ^{C} $\square$

Notation 4. *Let $A_1, \ldots, A_n$, $n \geqslant 2$, be algebras of the same type, $B = A_1 \times \ldots \times A_n$. Denote by*

$$\beta(B) = \{\langle \bar{a}, \bar{b}, \bar{c}, \bar{d} \rangle \in B^4, \text{ for each } i \leqslant n \text{ either } \langle a_i, b_i \rangle = \langle c_i, d_i \rangle \text{ or } a_i = b_i = d_i\}.$$

Lemma 4. *Let $A_1, \ldots, A_n, n \geqslant 2$, be algebras of the same type, $B = A_1 \times \ldots \times A_n$. The following conditions are equivalent:*
(1) B has factorable congruence blocks;
(2) $\langle \bar{a}, \bar{b}, \bar{c}, \bar{d} \rangle \in \beta(B)$ implies $\langle \bar{a}, \bar{b} \rangle \in \Theta_B(\bar{c}, \bar{d})$ for any elements $\bar{a}, \bar{b}, \bar{c}, \bar{d} \in B$.

P r o o f. (1) $\Rightarrow$ (2): Let $\langle \bar{a}, \bar{b}, \bar{c}, \bar{d} \rangle \in \beta(B)$. Then $\bar{b} = \bar{d}$ and $a_i \in \{c_i, d_i\}$, $i \leqslant n$. Evidently $\bar{c}, \bar{d} \in [\bar{d}]\Theta_B(\bar{c}, \bar{d})$ and thus also $\bar{a} \in [\bar{d}]\Theta_B(\bar{c}, \bar{d})$, by Lemma 3. In other words, we have $\langle \bar{a}, \bar{b} \rangle = \langle \bar{a}, \bar{d} \rangle \in \Theta_B(\bar{c}, \bar{d})$.

(2) $\Rightarrow$ (1): Let S be an arbitrary congruence block on B and let $\bar{c}, \bar{d} \in S$. Consider an element $\bar{a} = \langle a_1, \ldots, a_n \rangle$ such that $a_i \in \{c_i, d_i\}$, $i \leqslant n$. Then $\langle \bar{a}, \bar{d}, \bar{c}, \bar{d} \rangle \in \beta(B)$ and so $\langle \bar{a}, \bar{d} \rangle \in \Theta_B(\bar{c}, \bar{d})$, by hypothesis (2). This means that $\bar{a} \in [\bar{d}]\Theta_B(\bar{c}, \bar{d}) \subseteq S$ and so S is factorable, see Lemma 3. $\square$

Theorem 2. *Let C be a finite algebra. The following conditions are equivalent:*
(1) C^n has factorable congruence blocks for any $n \geqslant 2$;
(2) $C^{\beta(C)}$ has factorable congruence blocks.

P r o o f goes along the same lines as in Theorem 1 and hence can be omitted. $\square$

Corollary 2. *Let C be a finite k-element algebra, $k \geqslant 2$. The following conditions are equivalent:*
(1) C^n has factorable congruence blocks;
(2) C^{2k^2-k} has factorable congruence blocks.

P r o o f. We have card $\beta(C) = 2k^2 - k$ whenever card $C = k$. $\square$

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