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A NOTE ON THE OSCILLATION OF SOLUTIONS OF PERIODIC LINEAR DIFFERENTIAL EQUATIONS

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1. INTRODUCTION

During the past ten years, there has been considerable research by various authors into the problem of determining the frequency of zeros of solutions of second-order linear differential equations of the form,

$$w'' + A(z)w = 0, \tag{1.1}$$

where $A(z)$ is a transcendental entire function of finite order of growth (e.g. see [3]–[13], [16], [18] and [20]). In these results, the frequency of zeros of a solution $f \not\equiv 0$ is usually measured by the exponent of convergence (denoted $\lambda(f)$) of the zero-sequence of f. The results obtained have been mainly of two types. One type asserts that under certain conditions on $A(z)$, at least one of any two linearly independent solutions satisfies $\lambda(f) = \infty$, while the other type of result asserts that under more stringent conditions on $A(z)$, all solutions of (1.1) satisfy $\lambda(f) = \infty$. However, we remark that there are examples of (1.1) which possess two linearly independent solutions having no zeros (see [6; §5(b)]).

Recently, there has been a slightly different approach taken ([3; Theorem 1] and [4]), which is applicable in many cases where (1.1) is known to possess a solution $f_1 \not\equiv 0$ satisfying $\lambda(f_1) < \infty$. In these results, one can determine from the form of f_1 whether there can be a second linearly independent solution f_2 of (1.1) satisfying $\lambda(f_2) < \infty$. The result in [4] treats the case of an equation (1.1), where $A(z)$ is entire and which possesses a solution of the form, $f_1 = Ge^g$, where $g(z)$ and $G(z)$ are entire functions of finite order satisfying the following two conditions:

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(i) There exist two distinct rays, $\arg z = \theta_j$ for $j = 1, 2$, such that as $z \to \infty$ on these two rays, the function $g(z)$ is real and positive and satisfies $g(z)/|z|^\alpha \to +\infty$ for every real $\alpha > 0$;

(ii) There exist semi-infinite strips around these two rays on which $G(z)$ has no zeros.

Under these conditions, it is shown in [4] that $\lambda(f_2) = \infty$ for any solution f_2 of the equation (1.1) which is not a constant multiple of f_1. In fact, a stronger conclusion is proved in [4], namely that the integrated counting function $N(r, 1/f_2)$ for the zeros of f_2 satisfies the following condition for either $j = 1$ or $j = 2$: For any real number $a > 1$ there exist constants $r_0 > 0$ and $K_1 > 0$ such that,

$$N(r, 1/f_2) \geqslant K_1 g(re^{i\theta_j}/a) \text{ for all } r \geqslant r_0. \tag{1.2}$$

(We recall that $N(r, 1/f_2)$ is defined (see [17; p. 6]) as follows: If $n(t)$ denotes the number of zeros (counting multiplicity) of $f_2(z)$ in the disk $|z| \leqslant t$, then

$$N(r, 1/f_2) = \int_0^r \left(\frac{n(t) - n(0)}{t}\right) dt + n(0) \log r, \qquad \text{for } r > 0. \tag{1.3}$$

The counting function is related to the exponent of convergence by the following formula [14; p. 25]:

$$\lambda(f) = \limsup_{r \to +\infty}((\log N(r, 1/f))/\log r).) \tag{1.4}$$

We make two brief remarks about this result in [4]. First, there is no lack of examples to which this theorem will apply, since any function of the form $f_1 = e^g$ where $g(z)$ is entire, satisfies an equation of the form (1.1) with $A(z)$ entire, and, in addition, if $g(z)$ has the form $h(z^k)$, where k is a positive integer greater than one, and where $h(\zeta)$ is any transcendental entire function of finite order whose power series expansion around the origin has all nonnegative coefficients, then $g(z)$ satisfies the hypothesis (i) of the result for $\theta_1 = 0$ and $\theta_2 = 2\pi/k$. Of course, the hypothesis (ii) is automatically satisfied for $G \equiv 1$. The second remark we make is that the result in [4] is no longer true if $g(z)$ satisfies the condition in hypothesis (i) on only one ray. This is easily seen from the following example which is given in [3; p. 227]: The function $g(z) = e^z - \frac{1}{2}z$ satisfies (i) when $\theta_1 = 0$, but the equation (1.1) satisfied by $f_1 = e^g$ (namely where $A(z) = -(e^{2z} + \frac{1}{4})$), also possesses a second linearly independent solution f_2 which has no zeros, namely $e^{\varphi(z)}$ where $\varphi(z)$ equals $-(e^z + \frac{1}{2}z)$. We note that in this example the coeficient function $A(z)$ is periodic, and this example demonstrates that in many cases when $A(z)$ is periodic, the result in [4] will not be applicable. (Of course, there are also many examples where $A(z)$

is periodic and where the result in [4] would be applicable, namely those equations (1.1) which are satisfied by a function $f_1 = e^g$, where $g(z)$ has the form,

$$g(z) = \sum_{j=-m}^{n} \beta_j e^{jz}, \tag{1.5}$$

where m and n are positive integers, the β_j are real numbers, and $\beta_n > 0$ and $\beta_{-m} > 0$. (In this case, we take $\theta_1 = \pi$ and $\theta_2 = 0$.))

In the present paper, we use a different approach to investigate the case of equations (1.1) where $A(z)$ is periodic. (For convenience, we will assume that the period is $2\pi i$. The case of an arbitrary period can easily be transformed into this case by a linear change of independent variable.) Our approach is based on a representation theorem (Theorem A below) for solutions $f(z)$ of certain periodic equations (1.1), which satisfy the condition,

$$\log^+ N(r, 1/f) = o(r) \quad \text{as } r \to +\infty, \tag{1.6}$$

where $\log^+ x$ denotes $\max\{\ln x, 0\}$. This result was first proved in [7; Theorem 1] for the case $\lambda(f) < \infty$, and was extended in [3; Lemmas B, C] to solutions satisfying (1.6). Using Theorem A, we will prove our main results (Theorems 1 and 2 below) which sets forth a simple conditon to guarantee that if $A(z)$ is an entire function which is a rational function of e^z, and if (1.1) possesses a solution f_1 which satisfies condition (1.6), then no other solutions of (1.1) can satisfy (1.6) except for constant multiplies of f_1. We now state Theorem A from [7] and [3]:

Theorem A. *Let $A(z)$ be a nonconstant periodic entire function which is a rational function of e^z. Let $f_1(z) \not\equiv 0$ be a solution of (1.1) which satisfies (1.6). Then, the following are true:*

(A) *If the functions $f_1(z)$ and $f_1(z + 2\pi i)$ are linearly dependent, then $f_1(z)$ can be represented in the form,*

$$f_1(z) = \Phi(e^z) \exp\left(\sum_{j=q}^{m} d_j e^{jz} + dz \right), \tag{1.7}$$

where (1.8)-(1.12) below all hold:

$$\Phi(\zeta) \text{ is a polynomial all of whose roots are simple,} \tag{1.8}$$

$$\Phi(0) \neq 0, \tag{1.9}$$

$$m \text{ and } q \text{ are integers with } m \geqslant q, \tag{1.10}$$

$$d \text{ and } d_q, \ldots, d_m \text{ are complex constants,} \tag{1.11}$$

$$\text{for some } j \neq 0, \text{ we have } d_j \neq 0. \tag{1.12}$$

(B) *If the functions* $f_1(z)$ *and* $f_1(z+2\pi i)$ *are linearly independent, then the functions* $f_1(z)$ *and* $f_1(z+4\pi i)$ *are linearly dependent, and* $f_1(z)$ *can be represented in the form,*

$$f_1(z) = \Phi(e^{z/2}) \exp\left(\sum_{j=q}^{m} d_j e^{j(z/2)} + dz\right), \tag{1.13}$$

where (1.8)–(1.12) *all hold.*

We remark that the following result is an immediate corollary of Theorem A:

Theorem B. *Let* $A(z)$ *be a nonconstant periodic entire function which is a rational function of* e^z*. Then for any solution* $f \not\equiv 0$ *which satisfies* (1.6), *we have* $\lambda(f) \leqslant 1$.

From Theorem A, we now know the possible forms (1.7) and (1.13) of a solution satisfying (1.6). In our main results, which we now state, we show that if the equation (1.1) possesses a solution f_1 satisfying (1.6), and if we know the form of f_1 explicitely, then in many cases we can show that no other solutions (except for constant multiples of f_1) can satisfy (1.6). We remark that our results cover all three possibilities for m and q in (1.7) and (1.13), namely, $q \geqslant 0$, $m \leqslant 0$, and $q \leqslant 0 \leqslant m$. The proofs of our main results are given in §§4, 5 below.

Theorem 1. *Let* $f_1(z)$ *be a function of the form* (1.7) *where* (1.8)–(1.12) *all hold, and which satisfies an equation of the form* (1.1) *where* $A(z)$ *is a nonconstant entire function. Then:*

(a) *Assume that in* (1.7) *we have* $q \geqslant 0$, *and let* $\mathscr{P}(\zeta)$ *denote the polynomial* $\sum_{j=q}^{m} d_j \zeta^j$. *Assume that*

$$2d + \text{degree}(\mathscr{P}) + \text{degree}(\Phi) \notin \{0, -1, -2, \ldots\}. \tag{1.14}$$

Then, any solution $f_2 \not\equiv 0$ *of the same equation* (1.1), *which is not a constant multiple of* f_1 *satisfies the condition,*

$$\log^+ N(r, 1/f_2) \neq o(r) \text{ as } r \to +\infty. \tag{1.15}$$

(b) *Assume that in* (1.7) *we have* $m \leqslant 0$, *and let* $\mathscr{P}(\zeta)$ *denote the polynomial* $\sum_{j=-m}^{-q} d_{-j} \zeta^j$. *Then, if,*

$$-2d + \text{degree}(\mathscr{P}) - \text{degree}(\Phi) \notin \{0, -1, -2, \ldots\}, \tag{1.16}$$

any solution $f_2 \not\equiv 0$ *of the same equation* (1.1), *which is not a constant multiple of* f_1, *satisfies* (1.15).

(c) *Assume that in* (1.7) *there exist* $j > 0$ *and* $k < 0$ *such that* $d_j \neq 0$ *and* $d_k \neq 0$. *Then, any solution* $f_2 \not\equiv 0$ *of the same equation* (1.1), *which is not a constant multiple of* f_1, *satisfies* (1.15).

Theorem 2. *Let* $f_1(z)$ *be a function of the form* (1.13) *where* (1.8)–(1.12) *all hold, and which satisfies an equation of the form* (1.1) *where* $A(z)$ *is a nonconstant entire function. Then:*

(a) *Assume that in* (1.13), *we have* $q \geqslant 0$, *and let* $\mathscr{P}(\zeta)$ *denote* $\sum_{j=q}^{m} d_j \zeta^j$. *Then, if*

$$(1.17) \qquad 4d + \operatorname{degree}(\mathscr{P}) + \operatorname{degree}(\Phi) \notin \{0, -1, -2, \ldots\},$$

the conclusion (1.15) *holds for any solution* $f_2 \not\equiv 0$ *of the same equation* (1.1), *which is not a constant multiple of* f_1.

(b) *Assume that in* (1.13) *we have* $m \leqslant 0$, *and let* $\mathscr{P}(\zeta)$ *denote* $\sum_{j=-m}^{-q} d_{-j} \zeta^j$. *Then, if*

$$(1.18) \qquad -4d + \operatorname{degree}(\mathscr{P}) - \operatorname{degree}(\Phi) \notin \{0, -1, -2, \ldots\},$$

the conclusion (1.15) *holds for any solution* $f_2 \not\equiv 0$ *of the same equation* (1.1), *which is not a constant multiple of* f_1.

(c) *If in* (1.13), *there exist* $j > 0$ *and* $k < 0$ *such that* $d_j \neq 0$ *and* $d_k \neq 0$, *then any solution* $f_2 \not\equiv 0$ *of the same equation* (1.1), *which is not a constant multiple of* f_1, *satisfies* (1.15).

We make three brief remarks concerning these results. First, although the conditions (1.14), (1.16), (1.17) and (1.18), are sufficient conditions in their respective cases to guarantee that a solution f_2 which is linearly independent with f_1 satisfies (1.15), they are not necessary conditions for (1.15) to hold. In §6, we construct a simple example of a function $f_1(z)$ having the form (1.7), where (1.8)–(1.12) hold and where $q \geqslant 0$, which has the property that (1.14) is violated but (1.15) holds for all solutions f_2 of the same equation which are not constant multiplies of f_1. Second, from Part (a) of Theorem 1, we see that if $f_1(z)$ has the form (1.7), where the conditions (1.8)–(1.12) hold, and if the equation (1.1) satisfied by $f_1(z)$ has a nonconstant entire coefficient $A(z)$, and possesses a second linearly independent solution $f_2(z)$ satisfying the condition (1.6), then we must have,

$$(1.19) \qquad 2d + \operatorname{degree}(\mathscr{P}) + \operatorname{degree}(\Phi) \in \{0, -1, -2, \ldots\}.$$

A natural question is raised, namely, in this case are there any other possible restrictions on the value of the sum appearing in (1.19)? In §6, we answer this question in the negative by constructing examples which show that the sum in (1.19) can have any preassigned value in $\{0, -1, -2, \ldots\}$. (By simple changes of independent variable, one can construct similar examples for Part (b) of Theorem 1, and Parts (a) and (b) of Theorem 2.) Finally, we return to the example $f_1 = e^g$ where $g(z) = e^z - \frac{1}{2}z$, which was discussed earlier, and which has the property that the equation (1.1) satisfied by f_1 possesses a second linearly independent solution satisfying (1.6). If we consider the more general function,

$$f_1(z) = \exp(e^z + dz), \tag{1.20}$$

where d is an arbitrary complex number, then f_1 is a solution of the equation

$$w'' - (e^{2z} + (2d+1)e^z + d^2)w = 0. \tag{1.21}$$

It now follows from Part (a) of Theorem 1, that if $2d+1$ does not belong to the set $\{0, -1, -2, \ldots\}$, then the conclusion (1.15) holds for every solution of (1.21) which is not a constant multiple of f_1. For completeness, we show in §6, that if $2d+1$ is a nonpositive integer, then (1.21) does possess a second linearly independent solution which does not satisfy (1.15).

Finally, we remark that the actual location of the zeros of solutions of (1.1), when $A(z)$ is a rational function of e^z, is investigated in [2].

2. Preliminaries

(a) As introduced in [8], we define an R-set to be a countable union of discs

$$B(z_n, r_n) = \{z : |z - z_n| < r_n\} \tag{2.1}$$

whose centers z_n converge to infinity, and whose radii r_n have finite sum. From [15], it follows that the set of θ for which the ray $\arg z = \theta$ meets infinitely many discs of a given R-set U has measure zero, while the set of r for which the circle $|z| = r$ meets U has finite linear measure. We shall restrict ourselves to the case where $r_n = |z_n|^{-d}$ for some positive constant d, and will make use of the fact that if $k \geqslant 1$ and $f(z)$ is a non-constant entire function of finite order, then there is a positive constant M such that for all large z outside an R-set of the above type we have

$$|f^{(n)}(z)/f(z)| \leqslant |z|^M \tag{2.2}$$

for $n = 1, \ldots, k$. (See [21; p. 74].)

(b) We will require the following two results:

Lemma 2.1. *Let $\Lambda(\zeta)$ be a polynomial in ζ with constant coefficients, and assume that $\Lambda(0) \neq 0$. Then, there exist an R-set U and a constant $M > 0$ such that*

$$|1/\Lambda(e^z)| \leqslant |z|^M \text{ for } z \notin U. \tag{2.3}$$

Proof. We may assume that $\Lambda(\zeta)$ has leading coefficient 1. The result is obvious if $\Lambda(\zeta)$ is a constant, so we may assume that the degree of Λ is positive. Then $\Lambda(\zeta)$ is the product of factors of the form $\zeta - a$, where by hypothesis, $a \neq 0$. Clearly, to prove (2.3), it sufficies to prove that for some $M > 0$, the inequality,

$$|1/(e^z - a)| \leqslant |z|^M \tag{2.4}$$

holds outside an R-set. Writing,

$$1/(e^z - a) = a^{-1}((e^z/(e^z - a)) - 1), \tag{2.5}$$

and noting that $e^z/(e^z - a)$ is $f'(z)/f(z)$, where $f(z)$ is $e^z - a$, it now follows from (2.2) and (2.5) that (2.4) holds outside an R-set for some $M > 0$. □

Lemma 2.2. *Let $A(z)$ be an entire function, and let $f_1(z)$ and $f_2(z)$ be linearly independent solutions of (1.1). Let $E(z) = f_1(z)f_2(z)$, and assume that $E(z)$ is of finite order of growth, and that there exist an R-set U and a constant $M > 0$ such that*

$$|1/E(z)| \leqslant |z|^M \text{ for } z \notin U. \tag{2.6}$$

Then $A(z)$ is a polynomial.

Proof. From [6; Formula (6)], the functions E and A are related by the equation,

$$-4A = (c/E)^2 - (E'/E)^2 + 2(E''/E), \tag{2.7}$$

for some constant $c \neq 0$. In view of (2.2) and (2.6), it follows that there exist an R-set U_1 and a constant $N > 0$ such that

$$|A(z)| \leqslant |z|^N \text{ for } z \notin U. \tag{2.8}$$

From, the definition of R-set (see Part (a) of §2), clearly (2.8) is valid on a sequence of circles $|z| = r_n$ where $r_n \to +\infty$ as $n \to \infty$. Using Cauchy's estimate, it now follows that $A(z)$ is a polynomial of degree at most N. □

3. Main Lemma

We now prove the following result:

Lemma 3.1. *Let $f_1(z)$ be a function of the form* (1.7), *where $q \geqslant 0$ and where* (1.8)–(1.12) *all hold, and assume that f_1 satisfies an equation of the form* (1.1) *where $A(z)$ is a nonconstant entire function. Assume that* (1.1) *possesses a solution $f_2(z)$ which is not a constant multiple of $f_1(z)$, and which satisfies* (1.6). *Then*

(A) *There exists a polynomial $\Psi(\zeta)$, all of whose roots are simple and nonzero, such that,*

$$f_2(z) = \Psi(e^z)\exp\left(-\sum_{j=q}^{m} d_j e^{jz} + dz\right), \tag{3.1}$$

where q, m, $d_1, \ldots, d_m$, and d are as in (1.7).

(B) *If $P(\zeta)$ denotes the polynomial $\sum\limits_{j=q}^{m} d_j\zeta^j$, then*

$$2d + \text{degree}(P) + \text{degree}(\Phi) = -\text{degree}\,\Psi. \tag{3.2}$$

Proof. Since $f_1(z)$ satisfies (1.1), it follows from routine calculation using (1.7) that $A(z)$ is a rational function of e^z. Thus from Theorem B (and the form (1.7)) it follows that,

$$\lambda(f_2) \leqslant 1 \qquad \text{and} \qquad \lambda(f_1) \leqslant 1. \tag{3.3}$$

We now prove the following assertion:

$$f_2(z) \text{ and } f_2(z + 2\pi i) \text{ are linearly dependent.} \tag{3.4}$$

To prove (3.4), we assume the contrary so that $\{f_2(z), f_2(z + 2\pi i)\}$ is a linearly independent set. Of course, $\{f_1(z), f_2(z)\}$ is also a linearly independent set, and the functions in both these sets have zero-sequences with exponent of convergence at most one by (3.3). It then follows from [1; Lemma 8.1] that the set $\{f_1(z), f_2(z+2\pi i\}$ must be linearly dependent, for otherwise [1; Lemma 8.1] would imply that f_1 is of finite order of growth, and consequently that $A(z)$ is a polynomial by [7; §4(B)] which must be constant since $A(z)$ is periodic. This violates the hypothesis and thus shows that $f_1(z)$ and $f_2(z + 2\pi i)$ are linearly dependent, so

$$f_1(z) \equiv K_1 f_2(z + 2\pi i) \text{ for some constant } K_1 \neq 0. \tag{3.5}$$

But from (1.7), clearly,

$$f_1(z+2\pi i) \equiv e^{2\pi i d} f_1(z), \tag{3.6}$$

and thus (3.5) and (3.6) would imply that $f_1(z+2\pi i)$ is a constant multiple of $f_2(z+2\pi i)$, which contradicts the hypothesis that $f_2(z)$ is not a constant multiple of $f_1(z)$. This contradiction establishes (3.4). Thus Part (A) of Theorem A can be applied to $f_2(z)$ and hence,

$$f_2(z) = \Psi_1(e^z) \exp\left(\sum_{j=s}^{t} c_j e^{jz} + cz\right), \tag{3.7}$$

where $\Psi_1(\zeta)$ is a polynomial all of whose roots are simple and nonzero, where s and t are integers with $s \leqslant t$, and where $c, c_s, \ldots, c_t$ are complex constants such that $c_j \neq 0$ for some $j \neq 0$. Let $Q(\zeta)$ denote the rational function $\sum_{j=s}^{t} c_j \zeta^j$, and set $E = f_1 f_2$. Since $A(z)$ is of finite order of growth, and since (3.3) holds, it follows from [7; Lemma B] that $E(z)$ is of finite order of growth. Since,

$$E(z) = \Phi(e^z)\Psi_1(e^z)e^{(d+c)z} \exp(P(e^z) + Q(e^z)), \tag{3.8}$$

it now follows (see [19; p. 337]) that the function

$$\varphi(z) = \exp(P(e^z) + Q(e^z)), \tag{3.9}$$

must be of finite order of growth. Since $\varphi(z)$ has no zeros, it follows from the Hadamard factorization theorem [19; p. 332], that $\varphi(z) = e^{R(z)}$, where $R(z)$ is a polynomial. Thus from (3.9), we have,

$$P(e^z) + Q(e^z) = R(z) + 2\pi i \beta, \tag{3.10}$$

where, by continuity, β is an integer constant. Thus $R(z)$ is a periodic polynomial, and hence is a constant, say $R(z) \equiv L$. Thus we may rewrite (3.7) as

$$f_2(z) = \Psi(e^z) \exp(-P(e^z) + cz), \quad \text{where } \Psi = e^L \Psi_1. \tag{3.11}$$

For ease of notation, we now write (1.7) as,

$$f_1(z) = G(z) e^{g(z)}, \text{ where} \tag{3.12}$$

$$G(z) = \Phi(e^z) \text{ and } g(z) = P(e^z) + dz. \tag{3.13}$$

Similarly, we write (3.11) in the form,

$$f_2(z) = H(z)e^{-g(z)}, \text{ where} \tag{3.14}$$

$$H(z) = \Psi(e^z)e^{(c+d)z}. \tag{3.15}$$

Since f_1 and f_2 are linearly independent solutions of (1.1), their Wronskian is a nonzero constant K. A simple computation using (3.12) and (3.14) yields,

$$(H'/H) - (G'/G) - 2g' = K/(HG), \tag{3.16}$$

at all points z such that $H(z)G(z) \neq 0$. From (3.13) and (3.15), it is easy to see that the left side of (3.16) is periodic of period $2\pi i$. Thus from (3.16), HG must also be periodic of period $2\pi i$ and hence $e^{(c+d)z}$ must be periodic of period $2\pi i$. It follows that $n = c + d$ is an integer. Using (3.13) and (3.15), we find that,

$$H'(z)/H(z) = n + (\Psi'(e^z)e^z/\Psi(e^z)), \tag{3.17}$$

$$G'(z)/G(z) = (\Phi'(e^z)e^z/\Phi(e^z)), \quad g'(z) = P'(e^z)e^z + d, \tag{3.18}$$

$$H(z)G(z) = \Phi(e^z)\Psi(e^z)e^{nz}. \tag{3.19}$$

Substituting (3.17)–(3.19) into (3.16), and noting that every complex number $\zeta \neq 0$ can be written as $\zeta = e^z$ for some z, we can write (3.16) in the form,

$$\alpha + \zeta((\Psi'(\zeta)/\Psi(\zeta)) - (\Phi'(\zeta)/\Phi(\zeta)) - 2P'(\zeta)) \equiv V(\zeta), \tag{3.20}$$

where

$$V(\zeta) = K\zeta^{-n}/(\Phi(\zeta)\Psi(\zeta)), \quad \text{and } \alpha = n - 2d, \tag{3.21}$$

at every point $\zeta \neq 0$ where $\Phi(\zeta) \neq 0$ and $\Psi(\zeta) \neq 0$.

We now assert that,

$$n \neq 0. \tag{3.22}$$

To prove (3.22), we note first that by (3.12) and (3.14), the product $E = f_1 f_2$ is simply HG. Thus, if we assume that (3.22) fails, then $n = 0$ and so by (3.19), we would have that $E(z)$ is the product $\Phi(e^z)\Psi(e^z)$. But $\Phi(\zeta)\Psi(\zeta)$ is a polynomial all of whose roots are nonzero, and hence by (2.3) of Lemma 2.1, we see that $E(z)$ would then satisfy an equality (2.6) for some R-set U and some constant $M > 0$. Since $E(z)$ is of finite order, Lemma 2.2 would yield the conclusion that $A(z)$ is a polynomial, and hence must be a constant since it is a rational function of e^z (by

(1.7)) and thus periodic. This contradicts the hypothesis of this lemma. This proves (3.22).

Since the polynomials $\Phi(\zeta)$ and $\Psi(\zeta)$ do not have zero as a root, clearly (3.20) shows that $V(\zeta) \to \alpha$ as $\zeta \to 0$. Thus (3.21) shows that $n \leqslant 0$ or otherwise $V(\zeta)$ would have a pole at $\zeta = 0$. In view of (3.22), we must have $n < 0$, and so (3.21) shows that $V(\zeta) \to 0$ as $\zeta \to 0$. Thus from (3.20), clearly $\alpha = 0$, and hence $n = 2d$. Since $n = c + d$, we thus have

$$c = d, \qquad \text{and} \qquad n = 2d. \tag{3.23}$$

From (3.11), we now obtain (3.1) proving Part (A) of the lemma.

To prove Part (B), let a, b, and r denote respectively the degrees of $\Phi(\zeta)$, $\Psi(\zeta)$, and $P(\zeta)$, so that for nonzero constants σ_1, σ_2, and σ_3 we have as $\zeta \to \infty$,

$$\Phi(\zeta) = \sigma_1\zeta^a(1 + o(1)), \quad \Psi(\zeta) = \sigma_2\zeta^b(1 + o(1)). \tag{3.24}$$

and

$$\zeta P'(\zeta) = \sigma_3\zeta^r(1 + o(1)), \text{ where } r \geqslant 1 \text{ by hypothesis.} \tag{3.25}$$

Since $\Phi(\zeta)$ and $\Psi(\zeta)$ are polynomials, clearly both $\zeta\Phi'(\zeta)/\Psi(\zeta)$ and $\zeta\Psi'(\zeta)/\Psi(\zeta)$ approach finite limits as $\zeta \to \infty$, and hence from (3.20) and (3.21), we have

$$2\zeta P'(\zeta) + K\zeta^{-n}/(\Phi(\zeta)\Psi(\zeta)) \to L_1 \text{ as } \zeta \to \infty, \tag{3.26}$$

where L_1 is a complex number. In view of (3.24), (3.25), and (3.26), we thus have as $\zeta \to \infty$,

$$2\sigma_3\zeta^r(1 + o(1)) + L_2\zeta^{-(n+a+b)}(1 + o(1)) \to L_1. \tag{3.27}$$

where $L_2 = K/\sigma_1\sigma_2 \neq 0$. If $r > -(n + a + b)$, then the left side of (3.27) tends to ∞ as $\zeta \to \infty$ (since $r \geqslant 1$), which would contradict (3.27). If $r < -(n + a + b)$, then since $r \geqslant 1$, again the left side of (3.27) tends to ∞ contradicting (3.27). The only possibility left is, $r = -(n + a + b)$, which is precisely (3.2) since $n = 2d$ by (3.23). This proves Part (B). □

4. Proof of Theorem 1

Part (a). If $q \geqslant 0$ and (1.14) holds, then clearly (3.2) cannot hold for any polynomial Ψ. Hence by Lemma 3.1, $f_2(z)$ cannot satisfy (1.6) and so it satisfies (1.15), proving Part (a).

Part (b). Let $f_1(z)$ have the form (1.7) where $m \leqslant 0$. Set $h_1(t) = f_1(-t)$ for all complex t, so that $u = h_1(t)$ solves the equation,

$$u'' + A(-t)u = 0, \tag{4.1}$$

and $h_1(t)$ has the form,

$$h_1(t) = \Phi(\mathrm{e}^{-t})\exp(P(\mathrm{e}^t) - \mathrm{d}t), \tag{4.2}$$

where $P(\zeta)$ is as in the statement of Part (b). Let n be the degree of $\Phi(\zeta)$, so we may write (by (1.8) and (1.9)),

$$\Phi(\zeta) = b(\zeta - a_1)\cdots(\zeta - a_n), \text{ where } b \neq 0, a_j \neq 0, \tag{4.3}$$

and where the a_j are distinct. Thus $\Phi(\mathrm{e}^{-t})$ can be written $\mathrm{e}^{-nt}\Phi_1(\mathrm{e}^t)$, where,

$$\Phi_1(\zeta) = b(1 - a_1\zeta)\cdots(1 - a_n\zeta), \tag{4.4}$$

so that $\Phi_1(\zeta)$ also satisfies (1.8) and (1.9) and, has degree n. Then (4.2) can be written,

$$h_1(t) = \Phi_1(\mathrm{e}^t)\exp(P(\mathrm{e}^t) - (d+n)t), \tag{4.5}$$

which is now of the form (1.7) where the corresponding q is nonnegative, and so Part (a) can be applied to $h_1(t)$. Now, if $f_2(z)$ is a solution of (1.1) and is not a constant multiple of f_1, then $h_2(t) = f_2(-t)$ satisfies (4.1), and is not a constant multiple of h_1. Thus by Part (a), applied to $h_1(t)$, we see that if,

$$-2(d+n) + \mathrm{degree}(P) + \mathrm{degree}(\Phi_1) \notin \{0, -1, \ldots\}, \tag{4.6}$$

then $h_2(t)$ satisfies (1.15). But (4.6) is precisely the same statement as (1.16), and clearly, $N(r, 1/f_2)$ is the same as $N(r, 1/h_2)$, so that if (1.16) holds, then $f_2(z)$ satisfies (1.15). This proves Part (b).

Part (c). In this case, we may clearly assume that $d_m \neq 0$, $d_q \neq 0$ and $q < 0 < m$ in (1.7). Writing $f_1 = Ge^g$, where

$$(4.7) \qquad g(z) = \sum_{j=q}^{m} d_j e^{jz}, \qquad \text{and} \qquad G(z) = \Phi(e^z)e^{dz},$$

we have by (1.1) that,

$$(4.8) \qquad -A = (g')^2 + g'' + 2g'(G'/G) + (G''/G).$$

Since $G(z)$ and $g(z)$ are of finite order it follows from (2.2) and §2(a) that there exist a set B of real numbers having measure zero, and a constant $M > 0$, such that if $\theta \notin B$, then on $\arg z = \theta$, we have for $z = re^{i\theta}$,

$$(4.9) \qquad |G'/G| + |G''/G| + |g''/g| \leqslant r^M,$$

for all sufficiently large r, say $r \geqslant r_0(\theta)$. We note that by (4.7) we have,

$$(4.10) \qquad g'(z) = md_m e^{mz}(1 + R_1(e^{-z})),$$

where $R_1(\zeta)$ is a polynomial in ζ, vanishing for $\zeta = 0$, and we note also that $m > 0$. It easily forllows from (4.8)-(4.10), that if $\arg z = \theta$ is a ray in the right half plane, and $\theta \notin B$, then

$$(4.11) \qquad |A(re^{i\theta})| \geqslant (m^2|d_m|^2/2)e^{2mr(\cos\theta)},$$

for all sufficiently large r, say $r \geqslant r_1(\theta)$, and thus,

$$(4.12) \qquad r^{-N}|A(re^{i\theta}| \to +\infty \quad \text{as } r \to +\infty \text{ for each } N > 0.$$

Similarly from (4.7), we have,

$$(4.13) \qquad g'(z) = qd_q e^{qz}(1 + R_2(e^z)),$$

where R_2 is a polynomial vanishing at $\zeta = 0$, and noting that $q < 0$, we obtain again from (4.8) and (4.9) that (4.12) holds on all rays $\arg z = \theta$ lying in the left half-plane for which $\theta \notin B$. Thus (4.12) holds on all rays $\arg z = \theta$ with the exception of a set of θ of finite measure, and so by [8; Theorem 1], we must have $\lambda(f_2) = \infty$ for any solution f_2 of (1.1) which is not a constant multiple of f_1. By Theorem B in §1, the solution f_2 cannot satisfy (1.6) and hence it satisfies (1.15). This proves Theorem 1 completely.

5. Proof of Theorem 2

If $f_1(z)$ has the form (1.13), then clearly the function $h_1(t) = f_1(2t)$, for complex t, has the form,

$$h_1(t) = \Phi(e^t)\exp\left(\sum_{j=q}^{m} d_j e^{jt} + 2dt\right), \tag{5.1}$$

and clearly $v = h_1(t)$ satisfies the equation,

$$v'' + 4A(2t)v = 0, \tag{5.2}$$

if $f_1(z)$ satisfies (1.1). We note that (5.1) has the form (1.7) with $2d$ replacing d, and that if $f_2(z)$ is any solution of (1.1) which is not a constant multiple of f_1, then $h_2(t) = f_2(2t)$ is a solution of (5.2) which is not a constant multiple of $h_1(t)$. Thus, in Part (a) of Theorem 2, if $q \geqslant 0$ and (1.17) holds for $f_1(z)$, then (1.14) holds for $h_1(t)$, and so by Part (a) of Theorem 1, we can conclude that $h_2(t)$ satisfies (1.15). But it is easy to verify from (1.3) that for all $r > 0$,

$$N(r, 1/f_2) = N(r/2, 1/h_2) + O(1), \tag{5.3}$$

and so (1.15) also holds for f_2. This proves Part (a).

Parts (b) and (c) of Theorem 2 are proved exactly the same way using, respectively, Parts (b) and (c) of Theorem 1 applied to $h_1(t)$.

6. Examples from §1

(a) In this example, we construct a function $f_1(z)$ of the form (1.7), where $q \geqslant 0$, and where (1.8)–(1.12) hold, and for which (1.14) is violated, but the conclusion (1.15) holds. We define $f_1(z)$ by,

$$f_1(z) = \exp(e^{2z} + ae^z - z), \qquad \text{where } a \neq 0. \tag{6.1}$$

It is easy to verify that $f_1(z)$ satisfies (1.1) where,

$$-A(z) = 4e^{4z} + 4ae^{3z} + a^2e^{2z} - ae^z + 1. \tag{6.2}$$

Clearly, f_1 has the form (1.7) where $d = -1$, and the degrees of $\Phi(\zeta)$ and $P(\zeta)$ are respectively 0 and 2. Thus (1.14) is violated. If the equation (1.1) possessed a solution f_2, which is not a constant multiple of f_1 and which violates (1.15), then by Lemma 3.1, $f_2(z)$ would have the form (3.1), namely,

$$f_2(z) = \Psi(e^z)\exp(-e^{2z} - ae^z - z), \tag{6.3}$$

where by (3.2), the degree of $\Psi(\zeta)$ would be 0. Thus $\Psi(\zeta)$ is a constant, and the resulting function $f_2(z)$ solves (1.1) where,

$$-A(z) = 4e^{4z} + 4ae^{3z} + a^2e^{2z} + ae^z + 1, \tag{6.4}$$

which by (6.2) is not the same equation as f_1 solves since $a \neq 0$. This contradiction shows that in this case (1.15) does hold even though (1.14) is violated.

(b) In this example, we show that the sum in (1.19) can have any preassigned value in $\{0, -1, -2, \ldots\}$ in the case where $f_1(z)$ has the form (1.7), where $q \geqslant 0$ and (1.8)-(1.12) hold, and where $f_1(z)$ satisfies an equation (1.1) where $A(z)$ is a nonconstant entire function. Let s be an arbitrary element of $\{0, -1, -2, \ldots\}$, and let $n = -s$. It is proved in [8; p. 23], that there are polynomials $R_1(\zeta)$ and $R_2(\zeta)$, both of degree n and having simple, nonzero roots, such that the functions, (for $j = 1, 2$),

$$h_j(\zeta) = R_j(e^{-\zeta/2})\exp(2i(-1)^j e^{\zeta/2} - (\zeta/4)), \tag{6.5}$$

both solve the equation,

$$h'' + (e^\zeta - (2n+1)^2/16)h = 0. \tag{6.6}$$

Now set $f_j(z) = h_j(2z)$ for $j = 1, 2$, so that the $f_j(z)$ solve (1.1), where

$$A(z) = -4(e^{2z} - (2n+1)^2/16). \tag{6.7}$$

Clearly (as in (4.3) and (4.4)), we have from (6.5),

$$f_j(z) = T_j(e^z)\exp(2i(-1)^j e^z - (n + (1/2))z), \tag{6.8}$$

where the $T_j(\zeta)$ are polynomials of degree n, having simple nonzero roots. For $j = 1$, the sum in (1.19) is $-n$ which was the preassigned number s.

(c) In this example, we show that if $2d+1$ is a nonpositive integer, then the equation (1.21) satisfied by $f_1(z)$ in (1.20) possesses a second linearly independent solution which does not satisfy (1.15). Let $n = -(2d+1)$, so n is a nonnegative integer. Lemma 3.1 suggests that we seek a second solution of the form (3.1), namely,

$$f_2(z) = \Psi(e^z)\exp(-e^z + dz), \tag{6.9}$$

where (by (3.2)), the degree of $\Psi(\zeta)$ should be n. Using this suggestion as a starting point, we seek a polynomial $\Psi(\zeta)$ such that $f_2(z)$ in (6.9) and $f_1(z)$ in (1.20) will solve the same equation (1.21). It obviously suffices to find $\Psi(\zeta)$ so that the Wronskian of $f_1(z)$ and $f_2(z)$ will be a nonzero constant, say 1, for then f_1''/f_1 equals f_2''/f_2 by differentiation. With f_1 and f_2 in (1.20) and (6.9), we directly compute the Wronskian $W(f_1, f_2)$ and we find,

$$W(f_1, f_2) \equiv e^{-nz}(\Psi'(e^z) - 2\Psi(e^z)), \tag{6.10}$$

since $n = -(2d+1)$. Now the procedure is obvious. We use the classical elementary method of undetermined coefficients to produce a polynomial $\Psi(\zeta)$ such that,

$$\Psi'(\zeta) - 2\Psi(\zeta) \equiv \zeta^n. \tag{6.11}$$

For this polynomial $\Psi(\zeta)$, the resulting function $f_2(z)$ in (6.9) obviously satisfies $W(f_1, f_2) \equiv 1$ by (6.10), and the assertion is proved.

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