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EIGENVALUE FREQUENCY AND CONSISTENT SIGN PATTERN MATRICES

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An $n \times n$ sign pattern matrix A is k-consistent if every real matrix whose sign pattern is indicated by A has k real eigenvalues and $n-k$ nonreal eigenvalues. First we establish several general properties of k-consistent patterns, for any integer k, where $0 \leqslant k \leqslant n$. We then characterize patterns that are permutation similar to an irreducible, sign symmetric, tridiagonal matrix. Further, we establish a graph theoretic necessary condition for irreducible, tridiagonal patterns to be consistent, and we relate this condition to the cycle structure of the matrix. Finally we provide other interesting classes of consistent sign patterns.

0. Introduction

A matrix whose entries consist of the symbols $+$, $-$, and 0 is called a *sign pattern matrix*. For a real matrix B, by $\operatorname{sgn} B$ we mean the sign pattern matrix in which each positive (respectively, negative, zero) entry is replaced by $+$ (respectively, $-$, 0). For each $n \times n$ sign pattern matrix A, there is a natural class of real matrices whose entries have the signs indicated by A. If $A = (a_{ij})$ is an $n \times n$ sign pattern matrix, then the *sign pattern class of A* is defined by

$$Q(A) = \{B \in M_n(R) \mid \operatorname{sgn} B = A\}.$$

To avoid repetition, we often use the word pattern to mean sign pattern matrix.

We shall be interested in the cycles in a sign pattern matrix, since every real matrix associated with it has the same qualitative cycle structure. If $A = (a_{ij})$ is an $n \times n$ sign pattern matrix, then a product of the form $\gamma = a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_k i_1}$, in

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which the index set $\{i_1, i_2, \ldots, i_k\}$ consists of distinct indices is called a *simple cycle of length* k. A cycle is said to be *negative* (respectively, *positive*) if it contains an *odd* (respectively, *even*) number of negative entries and no entries equal to zero. In the remainder of this paper, when we say simple cycle we mean a nonzero simple cycle.

A *matching of size* k in an $n \times n$ matrix $A = (a_{ij})$ corresponds to k entries in the matrix among whose collective initial indices there are no repetitions, and among whose collective terminal indices there are no repetitions. We call a matching *principal* if the set of initial indices is the same as the set of terminal indices. The product of entries in a principal matching of size k is either a simple k-cycle or a product of simple cycles whose total length is k and whose index sets are mutually disjoint. Henceforth, when we use the term principal matching, we mean the product of entries in the matching. Further, $\ell(\gamma)$ denotes the size (length) of the matching.

Suppose P is a property a real matrix may or may not have. Then A is said to *require* P if *every* real matrix in $Q(A)$ has property P, or to *allow* P if *some* real matrix in $Q(A)$ has property P. Let B be an $n \times n$ real matrix, and define the *eigenvalue frequency* of B to be the ordered pair $S_B = (k, n-k)$, where k is the number of real eigenvalues and $n - k$ is the number of nonreal eigenvalues of B. A sign pattern matrix A of order n is said to be *k-consistent*, for some fixed integer k such that $0 \leqslant k \leqslant n$, if $S_B = (k, n-k)$ for every B in $Q(A)$. Henceforth we write $S_A = (k, n-k)$ whenever A is k-consistent. When we say A is consistent we mean A is k-consistent for some integer k, such that $0 \leqslant k \leqslant n$.

If A is an $n \times n$ reducible sign pattern matrix, recall that A is permutation similar to a matrix in Frobenius normal form. Since the *spectrum* of a real matrix (the set of all eigenvalues) is a similarity invariant, we may assume that a reducible matrix is in Frobenius normal form. Clearly a sign pattern A is consistent if and only if each irreducible component A_{ii} is consistent. Thus the question of which sign patterns are consistent reduces to which irreducible sign patterns are consistent. The consistent sign patterns for which $S_A = (0, n)$ and the patterns for which $S_A = (n, 0)$ are characterized in [EJ1], and the patterns for which $S_A = (1, n-1)$ are characterized in [E]. All these patterns are nicely structured and have several common properties. Our purpose here is to establish some properties of consistent patterns for arbitrary k, and to provide several interesting examples.

In our proofs, we use the fact that the eigenvalues depend continuously upon the entries of a matrix. Suppose $A = (a_{ij})$ is an $n \times n$ sign pattern matrix that has a simple k-cycle γ. Define the $n \times n$ real matrix $B_\gamma(0) = (b_\gamma(0)_{ij})$ by

$$b\gamma(0)_{ij} = \begin{cases} 1 & \text{if } a_{ij} = + \text{ and is in } \gamma, \\ -1 & \text{if } a_{ij} = - \text{ and is in } \gamma, \\ 0 & \text{elsewhere,} \end{cases} \tag{0.1}$$

and define the perturbed matrix $B_\gamma(\varepsilon) = (b_\gamma(\varepsilon)_{ij})$ by

$$(0.2) \qquad b\gamma(\varepsilon)_{ij} = \begin{cases} b\gamma(0)_{ij} & \text{if } a_{ij} \text{ is in } \gamma, \\ \varepsilon & \text{if } a_{ij} = + \text{ and is not in } \gamma, \\ -\varepsilon & \text{if } a_{ij} = - \text{ and is not in } \gamma, \\ 0 & \text{elsewhere,} \end{cases}$$

for some $\varepsilon > 0$. Since the nonzero eigenvalues of $B_\gamma(0)$ are algebraically simple eigenvalues (eigenvalues of algebraic multiplicity one), for sufficiently small perturbations of the entries of $B_\gamma(0)$, the perturbed matrix $B\gamma(\varepsilon)$ has k algebraically simple eigenvalues close to the k distinct eigenvalues of $B\gamma(0)$. Hence $B\gamma(\varepsilon)$ is a matrix in the sign pattern class of A that has k distinct eigenvalues close to the k^{th} complex roots of 1 or -1 (depending upon whether γ is positive or negative). Therefore if γ is a negative even cycle of length k, then $B\gamma(0)$ has k distinct nonreal eigenvalues that are bounded away from the real axis. For sufficiently small $\varepsilon > 0$, the entries of $B_\gamma(0)$ can be perturbed and all the k nonreal eigenvalues of $B\gamma(\varepsilon)$ will remain bounded away from the real axis. Similarly if γ is a positive even cycle of length k, then $B_\gamma(0)$ has real eigenvalues $\lambda = 1$ and $\lambda = -1$. For sufficiently small $\varepsilon > 0$, the nonzero real eigenvalues of $B_\gamma(\varepsilon)$ remain algebraically simple, and cannot coalesce to form a complex conjugate pair.

Let $\gamma = \gamma_1\gamma_2\ldots\gamma_m$ be a principal matching, where each γ_i is a simple cycle of length $\ell(\gamma_i)$, and the index sets of the γ_i's are mutually disjoint. Of course, if $m = 1$, γ is a simple cycle of length $\ell(\gamma_1)$. Henceforth, we use the symbol γ to represent a principal matching. Define the matrix $B_\gamma(\varepsilon) = (b_\gamma(\varepsilon)_{ij}) \in Q(A)$ by

$$(0.3) \qquad |b_\gamma(\varepsilon)_{ij}| = \begin{cases} 1 & \text{if } a_{ij} \text{ is in } \gamma_1, \\ 2 & \text{if } a_{ij} \text{ is in } \gamma_2, \\ \vdots & \\ m & \text{if } a_{ij} \text{ is in } \gamma_m, \\ \varepsilon & \text{if } a_{ij} \neq 0 \text{ and is not in } \gamma, \\ 0 & \text{elsewhere,} \end{cases}$$

for some $\varepsilon > 0$. For $\varepsilon = 0$, we define $B_\gamma(0)$ by (0.3) with $\varepsilon = 0$. By permutation similarity (if necessary), the simple cycles γ_i, $i = 1, 2, \ldots, m$, are in successive diagonal blocks in the matrix. Thus, for $\varepsilon = 0$, the set of nonzero eigenvalues of $B\gamma(0)$ is given by

$$\bigcup_{j=1}^{m} W_j, \text{ where } W_j = \left\{ jx \,\middle|\, x^{\ell(\gamma_j)} = \begin{cases} 1 & \text{if } \gamma_j \text{ is positive} \\ -1 & \text{if } \gamma_j \text{ is negative} \end{cases} \right\}.$$

Clearly $B_\gamma(0)$ has $\sum_{i=1}^{m} \ell(\gamma_i)$ distinct nonzero eigenvalues. Consequently, for sufficiently small $\varepsilon > 0$, these nonzero eigenvalues remain distinct, that is, algebraically simple.

In several of our results, we use graph theoretic concepts. Therefore recall that the *undirected graph* $G(A)$ of an $n \times n$ sign pattern matrix A is the undirected graph on n vertices $1, 2, \ldots, n$ such that there is an undirected edge in $G(A)$ from i to j, denoted by the unordered pair $\{i, j\}$, if and only if $a_{ij} \neq 0$ or $a_{ji} \neq 0$. The *directed graph* $D(A)$ of A is the directed graph on n vertices $1, 2, \ldots, n$, such that there is a directed edge in $D(A)$, denoted by the ordered pair (i, j), if and only if $a_{ij} \neq 0$. The set of all vertices is called the *vertex set* V, and the set of all edges is called the *edge set* E. An *undirected path in* $G(A)$ from i to j is a sequence of undirected edges $\{i, i_1\}, \{i_1, i_2\}, \ldots, \{i_{k-1}, j\}$, where the indices $i, i_1, \ldots, i_{k-1}, j$ are distinct. Here the number of successive edges, k, is called the length of the undirected path. We note that a 1-cycle, that is, a nonzero diagonal entry a_{ii} in a matrix $A = (a_{ij})$ corresponds to an undirected (directed) loop at the vertex i in $G(A)$, respectively, $D(A)$.

1. Properties of consistent sign patterns

Let $\mathcal{C}$ be the class of all consistent sign pattern matrices. Since the results in our first lemma are clear, we state it without proof.

1.1 Lemma. *The class $\mathcal{C}$ is closed under the following operations:*

(i) *Permutation similarity;*

(ii) *Signature similarity SAS, where $A \in \mathcal{C}$ and S is any nonsingular diagonal matrix;*

(iii) *Transposition; and*

(iv) *Negation.*

In order to simplify some of our statements, we define several quantities associated with a consistent sign pattern matrix A. We denote the maximum length of the principal matchings in A by $C(A)$. For a (nonzero) principal matching γ in A, we let $\varrho(\gamma) =$ (number of odd simple cycles in γ) $+2$(number of positive even simple cycles in γ). Define $n_1(A) = \max_\gamma\{n - C(A) + \varrho(\gamma)\}$, and $n_2(A) = \max_\gamma\{\ell(\gamma) - \varrho(\gamma)\}$.

1.2 Theorem. *If A is an $n \times n$ k-consistent pattern, then*

(i) *For any principal matching γ with $\ell(\gamma) = C(A)$, $n_1(A) = n - C(A) + \varrho(\gamma)$ and $n_2(A) = \ell(\gamma) - \varrho(\gamma)$;*

(ii) *$k = n_1(A)$;*

(iii) *$n - k = n_2(A)$;*

(iv) *$n_1(A) + n_2(A) = n$; and*

(v) $k \geqslant$ *number of nonzero diagonal entries in* A.

Proof. Suppose A is an $n \times n$ k-consistent pattern. By the results in [EJ2], A requires at least $n - C(A)$ eigenvalues equal to 0. Let γ be a principal matching in A. If $B_\gamma(0)$ and $B_\gamma(\varepsilon)$ are defined as in (0.3), then, for sufficiently small $\varepsilon > 0, B_\gamma(\varepsilon)$ is a matrix in $Q(A)$ with at least $t_\gamma = n - C(A) + \varrho(\gamma)$ real eigenvalues.

By consistency, we know that every B in $Q(A)$ has at least t_γ real eigenvalues, and it follows that $k \geqslant t_\gamma$, for every principal matching γ in A. Since equality is achieved when $\ell(\gamma) = C(A)$, we conclude that statement (i) in the theorem follows. Further, $k \geqslant t_\gamma$ for all principal matchings γ in A implies that $k = \max_\gamma t_\gamma$, that is, $k = n_1(A)$, and statement (ii) follows. Since the proof of (iii) is similar, we omit the details. Statement (iv) follows from (ii) and (iii), and statement (v) is a special case of (ii). □

We state the next obvious corollary without proof.

1.3 Corollary. *If A is a k-consistent pattern of order n, and A has at least $n-1$ nonzero diagonal entries, then $k = n$.*

If $n_r(A)$ (respectively, $n_c(A)$) is the maximum number of real (respectively, nonreal) eigenvalues allowed by an $n \times n$ sign pattern matrix A, then A is consistent if and only if $n_r(A) + n_c(A) = n$. Clearly, $n_r(A) \geqslant n_1(A)$ and $n_c(A) \geqslant n_2(A)$, for *any* $n \times n$ sign pattern matrix A. However, if A is consistent, then $n_1(A) + n_2(A) = n, n_r(A) = n_1(A)$ and $n_c(A) = n_2(A)$, and the converse is obviously true. Further, we note that if n and $n_1(A)$ have different parity, then A is not consistent.

1.4 Example. The 3×3 sign pattern matrix given below shows that the condition $n_1(A) + n_2(A) = n$ is not sufficient for A to be consistent, where

$$A = \begin{pmatrix} + & + & 0 \\ - & 0 & + \\ 0 & - & 0 \end{pmatrix}.$$

If $\gamma = a_{11}a_{23}a_{32}$, then the matrix $B_\gamma(\varepsilon) \in Q(A)$ has two nonreal eigenvalues for sufficiently small $\varepsilon > 0$. Now setting the entries a_{23} and a_{32} equal to zero, we obtain the subpattern

$$\hat{A} = \begin{pmatrix} + & + & 0 \\ - & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \text{with } \hat{B} = \begin{pmatrix} 3 & 2 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in Q(\hat{A}).$$

Since $\sigma(\hat{B}) = \{0, 1, 2\}$, if we let $\hat{B}(\varepsilon)$ be the matrix in $Q(A)$ obtained from $\hat{B}$ with $a_{23} = \varepsilon$ and $a_{32} = -\varepsilon$, then $\hat{B}(\varepsilon)$ has three real eigenvalues close to $0, 1$, and 2 for sufficiently small $\varepsilon > 0$. Hence A is *not* consistent. However, $n_1(A) = 1$, $n_2(A) = 2$ and $n_1(A) + n_2(A) = n$. Further, $n_r(A) = 3$ and $n_c(A) = 2$.

1.5 Lemma. *If an $n \times n$ sign pattern matrix A does not allow repeated real eigenvalues, then A is consistent.*

Proof. For any B_1 and B_2 in $Q(A)$, define

$$B(t) = B_1 + t(B_2 - B_1) = (1 - t)B_1 + tB_2,$$

and $S_{B(t)} = (k(t), n - k(t))$. Note that if $b_1 = b_2$, then $b_1 + t(b_2 - b_1) = b_1$ for any t; and if $b_1 \neq b_2$ and $\operatorname{sgn} b_1 = \operatorname{sgn} b_2$ then $b_1 + t(b_2 - b_1)$ has the same sign for any $t \in [0 - \delta, 1 + \delta]$, where

$$0 < \delta < \frac{1}{|b_2 - b_1|} \min\{|b_1|, |b_2|\}.$$

Consequently, for sufficiently small $\delta_1 > 0$, $B(t) \in Q(A)$ for any t in $[0 - \delta_1, 1 + \delta_1]$. Let c be in the interval $[0, 1]$. Then, by hypothesis, $B(c)$ has distinct real eigenvalues, and, without loss of generality, we may assume the real eigenvalues are λ_1, λ_2, ..., $\lambda_{k(c)}$, and the nonreal eigenvalues are $\lambda_{k(c)+1}$, $\lambda_{k(c)+2}$, ..., λ_n. Let

$$\begin{aligned}
\varepsilon_1 &= \min\left\{\frac{|\lambda_i - \lambda_j|}{2} \mid 1 \leqslant i < j \leqslant n \text{ and } \lambda_i \neq \lambda_j\right\}, \\
\varepsilon_2 &= \min\left\{|I_m(\lambda_j)| \mid k(c) + 1 \leqslant j \leqslant n\right\}, \\
\varepsilon_c &= \min\{\varepsilon_1, \varepsilon_2\}, \text{ and } D_i = \{x \in C \mid |x - \lambda_i| < \varepsilon_c\}.
\end{aligned}$$

Note that $D_i \cap D_j = \emptyset$ whenever $\lambda_i \neq \lambda_j$. At this point, we make use of the fact that the eigenvalues of $B(t)$ are continuous functions of t. Consequently, there exists a $\delta_c > 0$ such that for each $i, 1 \leqslant i \leqslant k(c)$, the disk D_i contains precisely one eigenvalue of $B(t)$ whenever $|t - c| < \delta_c$; and for each $j, k(c) + 1 \leqslant j \leqslant n$, if m_j is the algebraic multiplicity of λ_j, then the disk D_j contains exactly m_j eigenvalues of $B(t)$ whenever $|t - c| < \delta_c$. Since $B(t)$ is a real matrix for any real t, we know the nonreal eigenvalues occur in complex conjugate pairs, and it follows that each disk D_i, $1 \leqslant i \leqslant k(c)$, contains exactly one *real* eigenvalue of $B(t)$. The other $n - k(c)$ eigenvalues are contained in the disks D_j, $k(c) + 1 \leqslant j \leqslant n$. However, since $\varepsilon_c \leqslant \varepsilon_2$, $D_j \cap \mathcal{R} = \emptyset$, we conclude that the $n - k(c)$ eigenvalues contained in these disks are nonreal. Thus, $k(t) = k(c)$, for all $t \in (c - \delta_c, c + \delta_c)$. As c ranges over the interval $[0, 1]$, $\{(c - \delta_c, c + \delta_c)\}_c$ is an open cover of the compact set $[0, 1]$, and it follows that there is a finite subcover $\{(c - \delta_c, c + \delta_c)\}_c$ with $c \in \{c_1, c_2, \ldots, c_\ell\}$. Moreover, for any $c \in [0, 1]$, $k(t)$ is a constant function on $(c - \delta_c, c + \delta_c)$, and it follows that $k(t) \equiv k(c)$ on $[0, 1]$. Hence $S_{B_1} = S_{B_2}$ for any B_1 and B_2 in $Q(A)$, and we conclude that A is consistent. □

Before giving a counterexample to the converse of lemma 1.5, recall that a square sign pattern matrix A is a tree sign pattern matrix (t.s.p.) matrix if A is combinatorially symmetric, that is, $a_{ji} \neq 0$ if and only if $a_{ij} \neq 0$; and its undirected graph is connected, but acyclic.

1.6 Example. Let

$$A = \begin{pmatrix} 0 & + & + & + \\ + & 0 & 0 & 0 \\ + & 0 & 0 & 0 \\ + & 0 & 0 & 0 \end{pmatrix}.$$

Then A is a sign symmetric t.s.p. matrix, and by the results in [EJ1], it follows that $S_A = (n, 0)$. However if B is the real matrix in $Q(A)$ whose nonzero entries are one, then $\sigma(B) = \{0, 0, \pm\sqrt{3}\}$.

In our next two lemmas, we summarize some known results about sign patterns that require n real, respectively, n pure imaginary eigenvalues, where 0 is considered as both a real eigenvalue, and a pure imaginary eigenvalue.

1.7 Lemma. *If A is an $n \times n$ irreducible sign pattern matrix, then the following are equivalent:*
(i) *Every $B \in Q(A)$ is diagonally similar to a real symmetric matrix;*
(ii) *A requires n real eigenvalues; and*
(iii) *A is a sign symmetric t.s.p. matrix.*

1.8 Lemma. *If A is an $n \times n$ irreducible sign pattern matrix, then the following are equivalent:*
(i) *Every $B \in Q(A)$ is diagonally similar to a real skew-symmetric matrix;*
(ii) *A requires n pure imaginary eigenvalues; and*
(iii) *A is a sign skew symmetric t.s.p. matrix.*

Proof of 1.7 and 1.8. (ii) $\Leftrightarrow$ (iii) is proved in [EJ1]. (iii) $\Rightarrow$ (i) is proved in [M], and (i) $\Rightarrow$ (ii) is immediate. □

Using lemma 1.7, we are now prepared to state the following corollary to lemma 1.5.

1.9 Corollary. *If A is an irreducible $n \times n$ sign symmetric matrix that does not allow repeated real eigenvalues, then A is a t.s.p. matrix, and A requires n distinct real eigenvalues.*

Proof. Suppose A is an irreducible sign symmetric matrix that does not allow repeated real eigenvalues. Then by lemma 1.5, A is consistent. Define the real matrix $B = (b_{ij})$ in $Q(A)$ so that $b_{ij} = b_{ji}$ for all i and j in $\{1, 2, \ldots, n\}$. Then B is a real symmetrix matrix, and $S_B = (n, 0)$. Since A is consistent, we conclude that A requires n real eigenvalues. Further, A requires n distinct real eigenvalues follows from the assumption that A does not allow repeated real eigenvalues. Finally, lemma 1.7 implies that A is a t.s.p. matrix. □

2. Irreducible tridiagonal consistent sign patterns

2.1 Lemma. *If A is an $n \times n$ irreducible, sign skew-symmetric, tridiagonal pattern; or an irreducible, sign symmetric, tridiagonal pattern, then A requires n distinct pure imaginary eigenvalues, respectively, n distinct real eigenvalues.*

Proof. Since the proofs are identical for sign symmetric and sign skew- symmetrix tridiagonal matrices, we may assume, without loss of generality, that A is an irreducible, sign skew-symmetric, tridiagonal matrix. Let B be a matrix in $Q(A)$ defined by

$$B = \begin{pmatrix} 0 & a_1 & & & & \\ b_1 & 0 & a_2 & & & \\ & b_2 & 0 & a_3 & & \\ & & \ddots & \ddots & \ddots & \\ & & & & \ddots & a_{n-1} \\ & & & & b_{n-1} & 0 \end{pmatrix}.$$

Since B and $S^{-1}BS$ have the same spectrum for any signature matrix S, we may assume $a_i > 0$, for all $i = 1, 2, \ldots, n-1$. Let

$$D = \operatorname{diag}\left(1, \sqrt{\frac{|b_1|}{a_1}}, \sqrt{\frac{|b_1 b_2|}{a_1 a_2}}, \ldots, \sqrt{\frac{|b_1 b_2 \ldots b_{n-1}|}{a_1 a_2 \ldots a_{n-1}}}\right).$$

Then

$$B_1 = D^{-1}BD = \begin{pmatrix} 0 & \gamma_1 & & & & \\ -\gamma_1 & 0 & \gamma_2 & & & \\ & -\gamma_2 & 0 & \gamma_3 & & \\ & & \ddots & \ddots & \ddots & \\ & & & & 0 & \gamma_{n-1} \\ & & & & -\gamma_{n-1} & 0 \end{pmatrix},$$

where $\gamma_i = \sqrt{-a_i b_i}$. Since B_1 is a real skew-symmetric matrix, it is normal and diagonalizable. For any λ in $\sigma(B_1)$, $B_1 - \lambda I$ has a nonsingular submatrix of order $n-1$ in the upper right corner, so that $\operatorname{rank}(B_1 - \lambda I) = \operatorname{rank}(B - \lambda I) = n - 1$. Thus each eigenvalue of B is algebraically simple, and it follows that A requires distinct eigenvalues. Finally, we conclude from lemma 1.8 that A requires n distinct pure imaginary eigenvalues. □

Let $A = (a_{ij})$ be any $n \times n$ sign pattern matrix, and let $D = \operatorname{diag}(d_1, d_2, \ldots, d_n)$ be an $n \times n$ diagonal sign pattern. Define the matrix $A_D = ((a_d)_{ij})$ by

$$(a_d)_{ij} = \begin{cases} a_{ij} & \text{for } i \neq j, \\ d_i & \text{for } i = j. \end{cases}$$

Then for any $n \times n$ sign pattern matrix A, A_I is a pattern with a positive diagonal, where $I = \operatorname{diag}(+, +, \ldots, +)$ is the $n \times n$ qualitative identity matrix. We define A_{-I} similarly.

2.2 Theorem. *Let A be an $n \times n$ sign pattern matrix. Then the following are equivalent:*

(i) *A is permutation similar to an irreducible, sign symmetric, tridiagonal pattern;*

(ii) *A_I requires n distinct real eigenvalues;*

(ii)′ *A_{-I} requires n distinct real eigenvalues;*

(iii) *A_D requires n distinct real eigenvalues for any diagonal sign pattern matrix D.*

Proof. (i) $\Longrightarrow$ (ii). Suppose A is permutation similar to an irreducible, sign symmetric, tridiagonal pattern. Then lemmas 1.1 and 2.1 imply that A_I requires n distinct real eigenvalues. Similarly (i) $\Longrightarrow$ (ii)′.

(ii) $\Longrightarrow$ (iii). Suppose A_I requires n distinct real eigenvalues. Clearly each irreducible component of A_I requires all real eigenvalues. By lemma 1.7, each irreducible component of A_I is a sign symmetric t.s.p. matrix. Consequently, for any diagonal sign pattern matrix D, each irreducible component of A_D is a sign symmetric t.s.p. matrix; and, therefore, requires all real eigenvalues. We conclude that A_D requires all real eigenvalues, for any diagonal pattern D.

Next we show that A_D requires n distinct real eigenvalues for any diagonal sign pattern matrix D. To this end, suppose there is a diagonal matrix D such that A_D allows a repeated real eigenvalue. Then there is some $B \varepsilon Q(A_D)$ that has an eigenvalue λ of multiplicity k, $k \geqslant 2$. Let

$$r = \max_{1 \leqslant i \leqslant n} \{|b_{ii}|\}.$$

Then $B_1 = [(1 + r)I + B] \in Q(A_I)$, and $\lambda + (1 + r)$ is an eigenvalue of B_1 with multiplicity k. However this contradicts the assumption that A_I requires distinct eigenvalues. Similarly, (ii)′ $\Longrightarrow$ (iii).

(iii) $\Longrightarrow$ (i). From the results in [EJ1], A_D requires n real eigenvalues implies that each irreducible component of A is a sign symmetric, t.s.p. matrix. However, if A is reducible, then the diagonal entries in the irreducible components can be adjusted so that some $B_D \in Q(A_D)$ has a repeated eigenvalue. Thus A is irreducible. Now fix a real symmetric matrix $B \in Q(A)$. Since A_D requires n distinct real eigenvalues for any sign pattern diagonal matrix D, $B + \hat{D}$ has n distinct real eigenvalues for any real diagonal matrix $\hat{D}$. By theorem 2.8 in [F], B is permutation similar to an irreducible, tridiagonal matrix, and it follows that A is permutation similar to an irreducible, tridiagonal pattern. □

Before stating our corollary to theorem 2.2, we define the *rank of a sign pattern matrix* A to be the *minimum rank* over all matrices in the associated sign pattern class, denoted by $\mathrm{mr}(A)$, where

$$\mathrm{mr}(A) = \min_{B \in Q(A)} \operatorname{rank} B.$$

2.3 Corollary. *Let $A = A_I$ or $A = A_{-I}$ be an $n \times n$ sign pattern matrix. Then the following are equivalent:*

(i) *A requires n distinct real eigenvalues;*

(ii) *A requires n distinct eigenvalues;*

(iii) *A is permutation similar to a sign symmetric, irreducible, tridiagonal pattern; and*

(iv) *A is irreducible and requires n real eigenvalues, and $\mathrm{mr}(A_D) \geqslant n-1$, for any diagonal pattern D.*

Proof. From theorem 2.2, we know that (i) and (iii) of the corollary are equivalent, and (i) implies (ii) is immediate. Further, we know that condition (iii) implies that A_D requires n distinct real eigenvalues for any diagonal sign pattern matrix. Consequently $\mathrm{mr}(A_D) \geqslant n-1$, for any diagonal sign pattern matrix D, that is, (iii) implies (iv). Now suppose condition (iv) holds. Then by lemma 1.7, A is a sign symmetric t.s.p. matrix, and A requires diagonalizability. Finally, since $\mathrm{mr}(A_D) \geqslant n-1$ for any diagonal sign pattern matrix D, it follows that A requires n distinct real eigenvalues, that is, condition (i) holds.

To show that (ii) implies (iii), suppose A requires distinct eigenvalues. Then by lemma 1.5, A is consistent. Let $B = (b_{ij}) \in Q(A)$ be defined as follows: If $a_{ij} = 0$, then $b_{ij} = 0$, otherwise $|b_{ij}| = 1/n$ for $i \neq j$, and $|b_{ii}| = 2^i$, for all $i = 1, 2, \ldots, n$. From the Gersgorin Disc Theorem, we see that each disjoint disc

$$D_i = \{z \in C \mid |z - b_{ii}| \leqslant 1\}$$

contains precisely one eigenvalue of B. Consequently B has n distinct real eigenvalues. Since $B \in Q(A)$ and A is consistent, we conclude that A requires n distinct real eigenvalues. Thus, by condition (ii) of the theorem, we conclude that (ii) implies (iii). $\square$

A slight modification of the immediately preceeding proof in corollary 2.3 can be used to prove the following two propositions:

2.4 Proposition. *If A is an $n \times n$ sign pattern matrix that has, at most, one zero diagonal entry, then A requires n distinct eigenvalues only if A requires n real eigenvalues.*

2.5 Proposition. *If A is an $n \times n$ irreducible sign pattern matrix that has, at most, one zero diagonal entry, then A is consistent if and only if A is a sign symmetric t.s.p. matrix. (See corollary 1.3).*

A natural question now arises, namely, "If A is an $n \times n$ sign pattern matrix that requires n distinct real eigenvalues, then is A necessarily permutation similar to an irreducible, sign symmetric, tridiagonal matrix?" The following example illustrates that the answer to this question is **NO**.

2.6 Example. Let

$$A = \begin{pmatrix} 0 & + & & & \\ + & 0 & + & & \\ & + & 0 & + & + \\ & & + & 0 & 0 \\ & & + & 0 & 0 \end{pmatrix}, \quad \text{where } G(A)\colon \begin{array}{ccccccc} & & & & 4 & & \\ & & & & | & & \\ 1 & — & 2 & — & 3 & — & 5 \end{array}$$

Then A is a sign symmetric t.s.p. matrix that is *not* permutation similar to a tridiagonal matrix. However, we show that A requires five distinct real eigenvalues. This follows since any $B \in Q(A)$ is diagonally similar to a matrix of the form

$$\begin{pmatrix} 0 & a & & & \\ 1 & 0 & b & & \\ & 1 & 0 & c & d \\ & & 1 & 0 & 0 \\ & & 1 & 0 & 0 \end{pmatrix}, \quad \text{with } a, b, c, d > 0.$$

Hence the characteristic polynomial of B is given by

$$\begin{aligned} P_B(x) &= x^5 - (a+b+c+d)x^3 + (ac+ad)x \\ &= x\left[(x^2)^2 - (a+b+c+d)x^2 + (ac+ad)\right]. \end{aligned}$$

Since the discriminant of $t^2 - (a+b+c+d)t + (ac+ad)$ is

$$\begin{aligned}\Delta &= (a+b+c+d)^2 - 4(ac+ad)\\ &> (a+c+d)^2 - 4(ac+ad) = (a-c-d)^2 \geqslant 0,\end{aligned}$$

we conclude that any $B \in Q(A)$ has distinct eigenvalues.

For 1-consistent patterns, we use the characterization established in [E] for arbitrary sign patterns that require exactly one real eigenvalue. Applying the results in [E] to irreducible, tridiagonal patterns, we obtain the following:

2.7 Proposition. *An $n \times n$ (n odd, $n \geqslant 3$) irreducible, tridiagonal pattern A is 1-consistent if and only if all 2-cycles in A are nonpositive, and A has a 0-diagonal.*

To establish necessary conditions for k-consistent, irreducible, tridiagonal patterns A, we concentrate on the number and location of the positive 2-cycles in A. To this end, we assume A has a 0-diagonal. Further, we note that the nonreal eigenvalues of any real matrix occur in complex conjugate pairs. Consequently, if A is an $n \times n$ k-consistent matrix with n even, then k is necessarily even. Similarly, if n is odd, k is necessarily odd. Recall that the signed undirected graph of A is the graph $G(A)$, whose edges are signed, so that any edge $\{i,j\}$ is $+$ (respectively, $-$) if $a_{ij}a_{ji} = +$ (respectively, $-$). We define a *maximal signed positive* path in $G(A)$ to be a path in $G(A)$ that satisfies the following: (i) the path starts with the first positive edge, or with the first positive edge that follows a negative edge; (ii) contains successive positive edges; and (iii) ends at the last positive edge, or when a negative edge occurs. We define a maximal signed negative path similarly. For example, in the following graph of irreducible, tridiagonal 9-by-9 pattern, $2\overset{+}{—}3\overset{+}{—}4\overset{+}{—}5\overset{+}{—}6$ and $7\overset{+}{—}8\overset{+}{—}9$ are maximal signed positive paths, and $1\overset{-}{—}2$ and $6\overset{-}{—}7$ are maximal signed negative paths, where $G(A)$ is given by

$$1\overset{-}{—}2\overset{+}{—}3\overset{+}{—}4\overset{+}{—}5\overset{+}{—}6\overset{-}{—}7\overset{+}{—}8\overset{+}{—}9.$$

2.8 Theorem. *Let A be an irreducible, tridiagonal pattern with 0 diagonal. Then A is consistent only if the signed undirected graph of A has, at most, one maximal signed path with odd length.*

Proof. Suppose A is an $n \times n$ irreducible, tridiagonal, consistent pattern with 0 diagonal. For contradiction, assume $G(A)$ has two maximal signed paths of odd lengths. If the vertices of $G(A)$ are arranged from left to right in increasing order, denote the leftmost maximal signed path by p_ℓ and the rightmost path by p_r.

Case(i). Let n be odd. Without loss of generality, let $p_\ell = \{2p-1, 2p\}$, $\{2p, 2p+1\}, \ldots, \{2q-1, 2q\}$ and $p_r = \{2s, 2s+1\}$, $\{2s+1, 2s+2\}, \ldots, \{2t, 2t+1\}$ be maximal signed paths of odd length in $G(A)$. Let

$$S = \{\{2j-1, 2j\} \mid 1 \leqslant j \leqslant s\} \cup \{\{2i, 2i+1\} \mid s \leqslant i \leqslant \tfrac{1}{2}(n-1)\}.$$

Notice that the only two edges in S that are adjacent are $\{2s-1, 2s\}$ and $\{2s, 2s+1\}$, which have opposite signs. In the spirit of adjacency, the positive edges in S are independent, and the negative edges in S are independent. Let k_1 (respectively, k_2) denote the number of positive (respectively, negative) edges in S. Since every vertex occurs once in S, except the vertex $2s$, which occurs twice, it follows that $2k_1 + 2k_2 = n+1$. Let γ_1 be the matching consisting of the positive edges in S, and let γ_2 be the matching consisting of the negative edges in S. Then, for sufficiently small $\varepsilon > 0$, $B_{\gamma_1}(\varepsilon)$ is a matrix in $Q(A)$ that has at least $2k_1$ real eigenvalues, and $B_{\gamma_2}(\varepsilon)$ is a matrix in $Q(A)$ that has at least $2k_2$ nonreal eigenvalues. Thus A allows $2k_1$ real eigenvalues, and $2k_2$ nonreal eigenvalues. Since $2k_1 + 2k_2 = n+1$, it follows that A is not consistent.

Case (ii). Suppose n is even. Define p_ℓ as in case (i), and $p_r = \{2s-1, 2s\}, \ldots, \{2t-1, 2t\}$. Let

$$S = \{\{2i-1, 2i\} | 1 \leqslant i \leqslant q\} \cup \{\{2i, 2i+1\} \mid q \leqslant i \leqslant s-1\} \\ \cup \{\{2j-1, 2j\} \mid s \leqslant j \leqslant \tfrac{1}{2}n\}.$$

Then the vertices $2q$ and $2s-1$ occur twice in S, and all other vertices occur once. Defining k_1 and k_2 as in case (i), it follows that $2k_1 + 2k_2 = n+2$. Since the remainder of the argument is similar to the one used in case (i), we omit the details, and conclude that A is not consistent. Finally cases (i) and (ii) imply that A has, at most, one maximal signed path with odd length. □

In the following theorem, for a given sign pattern matrix A whose undirected graph is $G(A)$, we let $E_1 = \{\{1,2\}, \{3,4\}, \ldots\}$ and $E_2 = \{\{2,3\}, \{4,5\}, \ldots\}$ be subsets of the edge set E. In addition, we let α be the set of indices corresponding to the positive edges in E_1. Finally $A(\alpha)$ denotes the principal submatrix of A lying in the rows and columns indicated by the complement of the index set α.

2.9 Theorem. *If A is an $n \times n$ irreducible, tridiagonal pattern with 0 diagonal; and if $|\alpha| = 2k$, and $G(A)$ has, at most, one maximal signed path of odd length, then:*

(i) *the number of positive edges in E_2 is k or $k-1$; and*

(ii) *$A(\alpha)$ has no positive 2-cycles.*

Proof. Assume A has, at most, one maximal signed path in $G(A)$ of odd length.

Case (i). Assume n is odd. Since A is an irreducible, tridiagonal pattern, the length of the longest path in $G(A)$ is even and equal to $n-1$. If $G(A)$ has one maximal path of odd length, then all other maximal paths have even length. Since the length of the longest path in $G(A)$ is the sum of its maximal paths, it follows that the length of this path is even, which is a contradiction. Consequently all maximal paths are even, and we conclude that $\{2p-1, 2p\}$ and $\{2p, 2p+1\}$ always have the same sign for any p such that $1 \leqslant p \leqslant \frac{1}{2}(n-1)$. Since

$$E_1 = \left\{\{2p-1, 2p\} \mid 1 \leqslant p \leqslant \tfrac{1}{2}(n-1)\right\} \text{ and } E_2 = \left\{\{2p, 2p+1\} \mid 1 \leqslant p \leqslant \tfrac{1}{2}(n-1)\right\},$$

it follows that E_1 and E_2 have the same number of positive edges, namely, k.

Case (ii). Assume n is even. Then the length of the longest path in $G(A)$ is odd; and $G(A)$ contains exactly one maximal path of odd length.

Case (a). Suppose the maximal path is positive, and starts with the edge $(2p-1, 2p)$, for some integer p such that $1 \leqslant p \leqslant \frac{1}{2}(n-1)$. Since the first edge is positive, and belongs to E_1, we see that the maximal path of odd length contributes m positive edges to E_1 and $m-1$ to E_2, for some positive integer $m \leqslant k$. Since all other maximal paths are even, we conclude that they contribute the same number of positive edges to E_1 and E_2, and it follows that E_2 contains $k-1$ positive edges.

Case (b). Suppose $G(A)$ has a maximal negative path of odd length. Since the maximal positive paths have even lengths, the positive edges come in adjacent pairs, say, e_i and e_j. If e_i and e_j are to the left of the maximal odd path, then $e_i \in E_1$ and $e_j \in E_2$. However, if e_i and e_j are to the right of the maximal odd path, then $e_i \in E_2$ and $e_j \in E_1$. It is obvious that E_1 and E_2 contain the same number of positive edges, namely, k.

Finally we show that $A(\alpha)$ has no positive 2-cycles. To this end, suppose A has a positive 2-cycle, say, $a_{2p,2p+1}a_{2p+1,2p}$. Then $G(A)$ has a subgraph

$$(2p-1) \overset{-}{—} (2p) \overset{+}{—} (2p+1) \overset{-}{—} (2p+2), \text{ for some integer } p.$$

Since the length of the path $\{1,2\}$, $\{2,3\}$, ..., $\{2p-1, 2p\}$ is odd, it contains a maximal signed path of odd length. Consequently $G(A)$ contains two odd maximal signed paths, which contradicts the assumption that A has, at most, one maximal signed path of odd length. □

2.10 Example. Let

$$A = \begin{pmatrix} 0 & + & & & & \\ + & 0 & + & & & \\ & + & 0 & - & & \\ & & + & 0 & - & \\ & & & + & 0 & - \\ & & & & + & 0 \end{pmatrix}. \text{ Then } A \text{ is consistent, and } S_A = (2,4).$$

Proof. If $B \in Q(A)$, then B is similar to

$$\begin{pmatrix} 0 & a & & & & \\ 1 & 0 & b & & & \\ & 1 & 0 & -c & & \\ & & 1 & 0 & -d & \\ & & & 1 & 0 & -e \\ & & & & 1 & 0 \end{pmatrix}, \text{ where } a,b,c,d,e > 0, \text{ and}$$

$$P_B(x) = x^6 - (a+b-c-d-e)x^4 + [-a(c+d+e) - b(d+e) + ce]x^2 - ace.$$

We show that $P_B(x)$ has precisely one variation of sign. This is clearly true if the coefficient of x^4 is positive. If the coefficient of x^4 is negative, that is, $a+b > c+d+e$, then $-a(c+d+e) - b(d+e) + ce = -a(c+d) - bd - e(a+b-c) < 0$, (since $a+b-c > d+e > 0$). Hence $P_B(x)$ has one variation of sign. Similarly $P_B(-x) = P_B(x)$ has precisely one variation of sign. Clearly 0 is not a root of $P_B(x)$. Now we see that $P_B(x)$ has exactly two real roots, that is $S_A = (2,4)$. □

Remark. Similar proofs work for

$$A = \begin{pmatrix} 0 & + & & & \\ + & 0 & + & & \\ & + & 0 & - & \\ & & + & 0 & - \\ & & & + & 0 \end{pmatrix}, \text{ and for}$$

$$A = \begin{pmatrix} 0 & + & & & & \\ + & 0 & - & & & \\ & + & 0 & - & & \\ & & + & 0 & - & \\ & & & + & 0 & - \\ & & & & + & 0 \end{pmatrix}.$$

Each of the matrices in example 2.10 clearly satisfies the condition stated in theorem 2.8. In these examples, the necessary condition is also sufficient for consistency. An interesting open question is to determine, in general, if the condition stated in theorem 2.8 is sufficient.

An $n \times n$ matrix A (real or sign pattern) is said to be an *n-cycle matrix* if it has a simple n-cycle, and all entries of A not on this cycle are zero. Equivalently, A is an n-cycle matrix if A has exactly one principal matching of length n, and no principal matchings of length $\leqslant n-1$.

3.1 Proposition. *If A is an irreducible $n \times n$ (with $n \geqslant 2$) sign pattern matrix, then the following are equivalent:*

(i) *A requires n distinct eigenvalues of equal modulus;*

(ii) *For any $B \in Q(A), \lambda \in \sigma(B)$ implies that $|\lambda|$ is the spectral radius of B;*

(iii) *any (nonzero) principal matching of A is of length n; and*

(iv) *A is an n-cycle matrix.*

Proof. Recall that if $B \in Q(A)$, then the characteristic polynomial of B is given by

$$P_B(x) = x^n - E_1(B)x^{n-1} + E_2(B)x^{n-2} - \ldots \pm E_n(B),$$

where $E_k(B)$ is the sum of all properly signed principal matchings of length k in B. Now, (i) $\Longrightarrow$ (ii) and (iv) $\Longrightarrow$ (i) are clear. We complete the proof by showing (ii) $\Longrightarrow$ (iii) and (iii) $\Longrightarrow$ (iv). First assume that (ii) holds, and γ is a principal matching of A. If γ is not a simple cycle, then $\gamma = \gamma_1 \ldots \gamma_m$, where each γ_i is a simple cycle for $i = 1, \ldots, m$. For sufficiently small ε, $B_\gamma(\varepsilon)$, as defined in (0.3) does not have eigenvalues of equal modulus, contradicting (ii). Thus γ is a simple cycle. If γ is not an n-cycle, "emphasize" γ by defining $B_\gamma(\varepsilon)$ as in (0.2). Then $B_\gamma(\varepsilon)$ is a matrix in $Q(A)$ without all eigenvalues having equal modulus, contradicting (ii). Hence (ii) $\Longrightarrow$ (iii).

Finally, suppose (iii) holds. Since A is irreducible, A has a cycle γ, and (iii) implies γ is a simple n-cycle. Without loss of generality (perform a permutation similarity if necessary), we may assume $\gamma = a_{12}a_{23} \ldots a_{n-1,n}a_{n1}$. Suppose A has a nonzero entry a_{ij} not on γ. We consider three cases, each of which yields a cycle of length $\leqslant n-1$; and, hence, a contradiction:

(1) $i = j$. Here a_{ii} is a 1-cycle;

(2) $i < j$. In this case, $j \geqslant i + 2$ and $a_{12} \ldots a_{i-1,i}a_{ij}a_{j,j+1} \ldots a_{n1}$ has length $\leqslant n - 1$; and

(3) $i > j$. Here, $a_{ij}a_{j,j+1} \ldots a_{i-1,i}$ has length $\leqslant n - 1$.

Thus there are no nonzero entries of A not on γ, and it follows that (iii) $\Longrightarrow$ (iv). $\square$

The n-cycle sign pattern matrices A provide another class of sign patterns that are consistent and require distinct eigenvalues. Since $E_n(B) = \det B$, for $B \in Q(A)$ the roots of $P_B(x)$ are the n distinct n^{th} complex roots of $\pm\det(B)$. We also note that if A is an $n \times n$ sign pattern matrix whose only principal matchings have length $n-1$, then A requires distinct eigenvalues. Obviously, in general, A is consistent does not imply A requires distinct eigenvalues. In fact, A consistent does not imply that A requires diagonalizability (see example 3.2).

It is shown in [EJ1] that an $n \times n$ (n even) sign pattern matrix A requires n nonreal eigenvalues if and only if each irreducible component of A satisfies the following: **(i)** is bipartite; **(ii)** has all negative simple cycles; and **(iii)** is sign nonsingular. These patterns clearly are consistent with $S_A = (0, n)$. However, the following example shows that they do not require diagonalizability.

3.2 Example. Let

$$A = \begin{pmatrix} 0 & + & 0 & 0 \\ - & 0 & + & 0 \\ 0 & 0 & 0 & + \\ - & 0 & - & 0 \end{pmatrix}.$$

Then A is irreducible, satisfies (i–iii) above, and, hence, is consistent. However,

$$B = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -4 & 0 & -6 & 0 \end{pmatrix}$$

is in $Q(A)$, and $\sigma(B) = \{2\mathrm{i}, 2\mathrm{i}, -2\mathrm{i}, -2\mathrm{i}\}$. Further, the dimension of the eigenspace for $\lambda = 2\mathrm{i}$ is 1. Thus B is not diagonalizable, and A does not require diagonalizability.

On the other hand, it is an open question as to whether A requires diagonalizability implies A is consistent. In this regard, we point out that A requires diagonalizability does not imply that A requires distinct eigenvalues.

3.3 Example. Let $n \geqslant 4$, and consider the $n \times n$ sign pattern matrix

$$A = \begin{pmatrix} + & + & \cdots & + \\ + & + & & \\ \vdots & & \ddots & \\ + & & & + \end{pmatrix}.$$

Then A requires diagonalizability, since A is a sign symmetric t.s.p. matrix. Let

$$B = \begin{pmatrix} 3 & 1 & 1 & \dots & 1 \\ 1 & 1 & & & \\ 1 & & 1 & & \\ \vdots & & & \ddots & \\ 1 & & & & 1 \end{pmatrix}.$$

Then $B \in Q(A)$, and since $P_{B-I}(x) = x^{n-2}(x^2 - 2x - (n-1)) = x^{n-2}((x-1)^2 - n)$, we have $\sigma(B) = \sigma(I + (B - I)) = \{1, 1, \dots, 1, 2 + \sqrt{n}, 2 - \sqrt{n}\}$. Since 1 is repeated $n - 2$ times, A does not require distinct eigenvalues.

There is an interesting class of $n \times n$ sign pattern matrices C for which C requires diagonalizability implies C requires distinct eigenvalues, therefore, C is consistent in this case.

First consider a qualitative polynomial $p(x) = x^n + a_{n-1}x^{n-1} + \dots + a_1x + a_0$, where $a_i \in \{0, +, -\}, i = 0, \dots, n-1$. Let $Q(p(x))$ denote the set of real polynomials $q(x) = x^n + c_{n-1}x^{n-1} + \dots + c_1x + c_0$, where $\operatorname{sgn} c_i = a_i$, $i = 0, \dots, n-1$. We say $p(x)$ requires distinct roots if $q(x)$ has n distinct roots for every $q(x) \in Q(p(x))$. Associated with $p(x)$ is the *companion* sign pattern matrix given by

$$C = \begin{pmatrix} 0 & & & & & -a_0 \\ + & 0 & & & & -a_1 \\ & + & 0 & & & -a_2 \\ & & \ddots & \ddots & & \vdots \\ & & & & 0 & -a_{n-2} \\ & & & & + & -a_{n-1} \end{pmatrix}.$$

(Note that C is irreducible if and only if $a_0 \neq 0$.) For $B \in Q(C)$, there exists a diagonal matrix S such that

$$S^{-1}BS = \begin{pmatrix} 0 & & & & -b_0 \\ 1 & 0 & & & -b_1 \\ & \ddots & \ddots & & \vdots \\ & & & 0 & \\ & & & 1 & -b_{n-1} \end{pmatrix} = B_1, \text{ with } \operatorname{sgn} b_i = a_i,\ i = 0, \dots, n-1.$$

Consequently, the characteristic polynomial $P_B(x) \in Q(p(x))$. Further, each $q(x) \in Q(p(x))$ is the characteristic polynomial of some companion matrix $B_1 \in Q(C)$. Hence, $Q(p(x))$ is the set of characteristic polynomials for real matrices in $Q(C)$.

Now, C requires diagonalizability

$\iff$ every $B \in Q(C)$ is diagonalizable;
$\iff$ for every $B \in Q(C)$, the companion matrix B_1 is diagonalizable;
$\iff$ for every $B \in Q(C)$, the companion matrix B_1 has distinct eigenvalues;
$\iff$ every $B \in Q(C)$ has distinct eigenvalues;
$\iff$ C requires distinct eigenvalues; also, every $B \in Q(C)$ has distinct eigenvalues;
$\iff$ $P_B(x)$ has n distinct roots for every $B \in Q(C)$;
$\iff$ $q(x)$ has n distinct roots for every $q(x) \in Q(p(x))$; or
$\iff$ $p(x)$ requires distinct roots.

We thus have the following equivalent statements.

3.4 Proposition. *Let C be the companion sign pattern matrix of the qualitative polynomial $p(x) = x^n + a_{n-1}x^{n-1} + \ldots + a_1x + a_0$. Then, the following are equivalent:*

(i) *C requires diagonalizability;*

(ii) *C requires distinct eigenvalues; and*

(iii) *$p(x)$ requires distinct roots.*

In this case, $p(x)$ requires consistent roots and $p(x)$ has at most one variation in sign.

More generally, if A is in the class of irreducible $n \times n$ upper-Hessenberg sign patterns, then A requires diagonalizability if and only if A requires distinct eigenvalues. In this case, A is consistent. We note that this class includes the $n \times n$ irreducible, tridiagonal matrices.

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