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MODULAR AND METRIC MULTILATTICES

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(*Communicated by Tibor Katriňák*)

ABSTRACT. A metric multilattice, which is shown to be a generalization of Birkhoff's metric lattice, is defined. A metric multilattice is modular, and a directed modular multilattice of locally finite length is metrizable.

1. Basic notions

Let $(P, \leq)$ be a partially ordered set. Given $a \in P$, denote by $[a) = \{x \in P : x \geq a\}$, $(a] = \{x \in P : x \leq a\}$. If $a, b \in P$, $a \leq b$, then the set $[a) \cap (b]$ will be denoted by $[a, b]$ and it will be referred to as an interval. P is said to be of locally finite length if all bounded chains in P are finite.

For $a, b \in P$ we denote by $a \vee b$ and $a \wedge b$ the set of all minimal elements of $[a) \cap [b)$ or all maximal elements of $(a] \cap (b]$, respectively. P is said to be a multilattice (cf. [1]) if for any $a, b, u, v \in P$ such that $u \in (a] \cap (b]$, $v \in [a) \cap [b)$, the sets $(a \wedge b) \cap [u)$, $(a \vee b) \cap (v]$ are not empty. A multilattice M is modular (cf. [1]) if, whenever $a, b, c \in M$, $(a \wedge b) \cap (a \wedge c) \neq \emptyset$, $(a \vee b) \cap (a \vee c) \neq \emptyset$, $b \leq c$, then $b = c$.

If L is a lattice and $a, b \in L$, then the symbols $a \vee b$ and $a \wedge b$ have the usual meaning.

2. Metric multilattice is modular

DEFINITION 2.1. By a *metric multilattice* we mean a multilattice M in which a metric d is given which fulfils the conditions:

M1. $a \leq b \leq c$ implies $d(a, b) + d(b, c) = d(a, c)$,
M2. if $u \in a \wedge b$, $v \in a \vee b$, then $d(a, b) = d(u, v)$.

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LEMMA 2.2. *Let a, b, u, v be elements of a metric multilattice M with a metric d, $u \in a \wedge b$ and $v \in a \vee b$. Then $d(u,a) = d(b,v)$.*

Proof. Suppose $d(u,a) < d(b,v)$. Using M2, the triangle inequality and M1 we get $d(u,v) = d(a,b) \leq d(a,u) + d(u,b) < d(b,v) + d(u,b) = d(u,v)$, a contradiction. The assumption $d(u,a) > d(b,v)$ leads to a contradiction analogously. □

THEOREM 2.3. *A metric multilattice is modular.*

Proof. Let a, b, c, u, v be elements of a metric multilattice such that

$$u \in (a \wedge b) \cap (a \wedge c), \quad v \in (a \vee b) \cap (a \vee c), \qquad b \leq c.$$

Using M1 and 2.2 we obtain

$$d(b,c) + d(c,v) = d(b,v) = d(u,a) = d(c,v),$$

which implies $d(b,c) = 0$. Hence $b = c$. □

3. Directed modular multilattice of locally finite length is metrizable

The following statement is a consequence of [1; 4.5].

THEOREM 3.1. *A modular multilattice of locally finite length fulfils the condition*

(JD) *if $a \leq b$, then all maximal chains in $[a,b]$ have the same length.*

In what follows, the symbols $l(a,b)$, $l(b,a)$ will be used to denote the length of maximal chains in the interval $[a,b]$ of a modular multilattice of locally finite length.

By [1; 4.741], we have:

THEOREM 3.2. *If a, b, u, v are elements of a modular multilattice of locally finite length such that $u \in a \wedge b$ and $v \in a \vee b$, then $l(u,a) = l(b,v)$.*

COROLLARY 3.3. *Let a, b, v_1, v_2 be elements of a modular multilattice of locally finite length such that $a \wedge b \neq \emptyset$, $v_1, v_2 \in a \vee b$. Then $l(a,v_1) + l(b,v_1) - l(a,v_2) + l(b,v_2)$.*

Proof. Take any $u \in a \wedge b$. By 3.2, we have

$$l(a,v_1) + l(b,v_1) = l(u,b) + l(u,a) = l(a,v_2) + l(b,v_2).$$

□

THEOREM 3.4. *A directed modular multilattice of locally finite length is metrizable.*

Proof. Let M be a directed modular multilattice of locally finite length. Define a function d on $M \times M$ by

$$d(a,b) = l(a,v) + l(b,v),$$

where v is an element of $a \vee b$. According to 3.3, this function is well defined. It is easy to see that d is symmetric and that for elements $a,b \in M$, one has $d(a,b) = 0$ if and only if $a = b$. To prove the triangle inequality, take $a,b,c \in M$, $v_1 \in a \vee b$, $v_2 \in b \vee c$, $w \in v_1 \vee v_2$ and $v \in (a \vee c) \cap (w]$. We are going to show:

$$l(a,v) + l(c,v) \leq l(a,v_1) + l(b,v_1) + l(b,v_2) + l(c,v_2).$$

Choose $u \in (v_1 \wedge v_2) \cap [b)$, $p \in (v_1 \vee v) \cap (w]$, $q \in (v \vee v_2) \cap (w]$, $r \in (v_1 \wedge v) \cap [a)$ and $s \in (v \wedge v_2) \cap [c)$ (see Fig. 1). Using 3.2, we obtain $l(r,v) = l(v_1,p)$, $l(s,v) = l(v_2,q)$, $l(v_1,w) = l(u,v_2)$ and $l(v_2,w) = l(u,v_1)$.

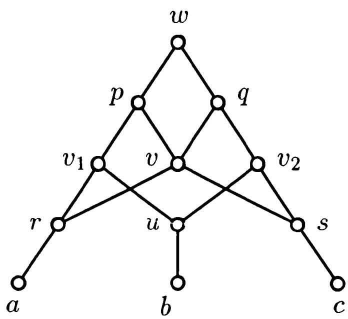


Figure 1.

Therefore

$$\begin{aligned} l(a,v) + l(c,v) &= l(a,r) + l(r,v) + l(c,s) + l(s,v) \\ &= l(a,r) + l(v_1,p) + l(c,s) + l(v_2,q) \\ &\leq l(a,r) + l(v_1,w) + l(c,s) + l(v_2,w) \\ &= l(a,r) + l(u,v_2) + l(c,s) + l(u,v_1) \\ &\leq l(a,v_1) + l(b,v_2) + l(c,v_2) + l(b,v_1) \\ &= l(a,v_1) + l(b,v_1) + l(b,v_2) + l(c,v_2). \end{aligned}$$

The condition M1 is trivially satisfied. As to M2, if $u \in a \wedge b$ and $v \in a \vee b$, then $d(a,b) = l(a,v) + l(b,v) = l(a,v) + l(u,a) = l(u,v) = d(u,v)$ by 3.2. This completes the proof. $\square$

4. Metric lattices

DEFINITION 4.1. (cf. [2]) By a *valuation* on a lattice L, we mean a real-valued function v defined on L, satisfying

V1. $v(x) + v(y) = v(x \vee y) + v(x \wedge y)$.

A valuation is *positive* if

V2. $x < y$ implies $v(x) < v(y)$.

A lattice with a positive valuation is called a *metric lattice.*

Applying Definition 2.1 to the case of lattices we obtain another definition of a metric lattice:

DEFINITION 4.2. *Metric lattice* is a lattice with a metric d satisfying:

L1. $a \leq b \leq c$ implies $d(a,b) + d(b,c) = d(a,c)$,
L2. $d(a,b) = d(a \wedge b, a \vee b)$.

We are going to make clear the relation between these two definitions. In [2], it is proved that, if v is a positive valuation on a lattice L, then the distance function d defined by $d(x,y) = v(x \vee y) - v(x \wedge y)$ is a metric. It is easy to see that this metric satisfies also L1 and L2. Hence, if a lattice is metric in the sense of 4.1, then it is metrizable in the sense of 4.2, too. Now we are going to prove the converse.

In 4.3–4.13, L will be a lattice, and d a metric on L satisfying L1 and L2. We will construct positive valuations on L.

Take any fixed element of L and denote it by 0. We will use the symbols a^+ and a^- for $a \vee 0$ and $a \wedge 0$, respectively. Define a function v_0 on L by

$$v_0(a) = d(0, a^+) - d(a^-, 0).$$

We are going to show that v_0 is a positive valuation on L. Let us remind that, by 2.3, the lattice L is modular, and 2.2 ensures that, if $[u,a]$, $[b,v]$ are transposed intervals, then $d(u,a) = d(b,v)$. We will use these facts.

LEMMA 4.3. *The function v_0 satisfies* V2.

P r o o f. Let $x < y$. Then $x^- \leq y^- \leq 0 \leq x^+ \leq y^+$, and the modularity of L yields that either $x^- < y^-$ or $x^+ < y^+$. Hence $d(y^-,0) + d(0,x^+)$ $d(y^-,x^+) < d(x^-,y^+) = d(x^-,0) + d(0,y^+)$, which implies

$$v_0(x) = d(0,x^+) - d(x^-,0) < d(0,y^+) - d(y^-,0) = v_0(y).$$

□

LEMMA 4.4. *If $x, y \in L$, then*

$$d\big((x \wedge y)^-, x^-\big) = d(y^-, x^- \vee y^-) \quad \text{and} \quad d\big(x^+, (x \vee y)^+\big) = d(x^+ \wedge y^+, y^+).$$

Proof. Evidently, $\big[(x \wedge y)^-, x^-\big]$, $\big[y^-, x^- \vee y^-\big]$ and $\big[x^+ \wedge y^+, y^+\big]$, $\big[x^+, (x \vee y)^+\big]$ are couples of transposed intervals. □

LEMMA 4.5. *If $x, y \in L$, then*

$$d\big(x^- \vee y^-, (x \vee y)^-\big) = d\big((x \wedge y) \vee x^- \vee y^-, (x \wedge y) \vee (x \vee y)^-\big).$$

Proof. We will show that the intervals $\big[x^- \vee y^-, (x \vee y)^-\big]$ and $\big[(x \wedge y) \vee x^- \vee y^-, (x \wedge y) \vee (x \vee y)^-\big]$ are transposed. It is clear that $(x \wedge y) \vee x^- \vee y^- \vee (x \vee y)^- = (x \wedge y) \vee (x \vee y)^-$. Further, using the modularity of L, we get $\big((x \wedge y) \vee x^- \vee y^-\big) \wedge (x \vee y)^- = x^- \vee y^- \vee \big(x \wedge y \wedge (x \vee y)^-\big) = x^- \vee y^- \vee \big((x \wedge y) \wedge (x \vee y) \wedge 0\big) = (x^- \vee y^-) \vee \big((x \wedge y) \wedge 0\big) = (x^- \vee y^-) \vee (x^- \wedge y^-) = x^- \vee y^-$. □

Analogously, it can be proved:

LEMMA 4.6. *If $x, y \in L$, then*

$$d\big((x \vee y) \wedge (x \wedge y)^+, (x \vee y) \wedge x^+ \wedge y^+\big) = d\big((x \wedge y)^+, x^+ \wedge y^+\big).$$

LEMMA 4.7. *If $x, y \in L$, then*

$$d\big((x \wedge y) \vee x^- \vee y^-, (x \wedge y) \vee (x \vee y)^-\big) = d\big(x^- \vee (y \wedge x^+), (x \vee y) \wedge x^+ \wedge y^+\big).$$

Proof. It is sufficient to show that the intervals

$$\big[(x \wedge y) \vee x^- \vee y^-, (x \wedge y) \vee (x \vee y)^-\big], \qquad \big[x^- \vee (y \wedge x^+), (x \vee y) \wedge x^+ \wedge y^+\big]$$

are transposed. Due to the modularity of L, we have $\big((x \wedge y) \vee (x \vee y)^-\big) \vee \big(x^- \vee (y \wedge x^+)\big) = (x \wedge y) \vee \big((x \vee y) \wedge 0\big) \vee (x \wedge 0) \vee \big(y \wedge (x \wedge 0)\big) = \big((x \vee y) \wedge 0\big) \vee \big(y \wedge (x \vee 0)\big) = \big(\big(y \wedge (x \vee 0)\big) \vee 0\big) \wedge (x \vee y) = \big((0 \vee y) \wedge (x \vee 0)\big) \wedge (x \vee y) = (x \vee y) \wedge x^+ \wedge y^+$. Using again the modularity of L several times we obtain $\big((x \wedge y) \vee (x \vee y)^-\big) \wedge \big(x^- \vee (y \wedge x^+)\big) = \big((x \wedge y) \vee \big((x \vee y) \wedge 0\big)\big) \wedge \big((x \wedge 0) \vee \big(y \wedge (x \vee 0)\big)\big) = \big(\big((x \wedge y) \vee 0\big) \wedge (x \vee y)\big) \wedge \big(\big((x \wedge 0) \vee y\big) \wedge (x \vee 0)\big) = \big((x \wedge y) \vee 0\big) \wedge (x \vee y) \wedge \big((x \wedge 0) \vee y\big) \wedge (x \vee 0) = \big((x \wedge y) \vee 0\big) \wedge \big((x \wedge 0) \vee y\big) = (x \wedge y) \vee \big(0 \wedge \big((x \wedge 0) \vee y\big)\big) = (x \wedge y) \vee \big((x \wedge 0) \vee (y \wedge 0)\big) = (x \wedge y) \vee x^- \vee y^-$. □

LEMMA 4.8. *If $x, y \in L$, then*

$$d\big((x \wedge y) \vee x^- \vee y^-, y^- \vee (x \wedge y^+)\big) = d\big((x \wedge y) \vee (x \vee y)^-, (x \vee y) \wedge x^+ \wedge y^+\big).$$

The Proof is obtained by interchanging the roles of x and y in the previous proof.

LEMMA 4.9. *If $x, y \in L$, then*

$$d\big(x^- \vee (y \wedge x^+),\ (x \vee y) \wedge x^+ \wedge y^+\big) = d\big((x \wedge y) \vee x^- \vee y^-,\ y^- \vee (x \wedge y^+)\big)\,.$$

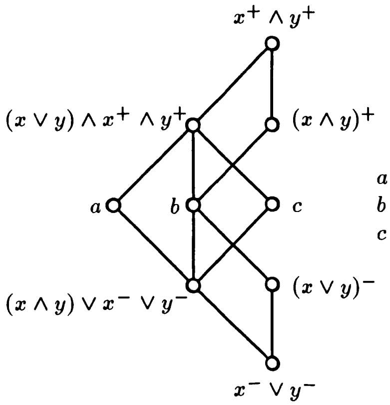
Proof. It is clear that

$$x \wedge y^+ \le (y^- \vee x) \wedge y^+ = y^- \vee (x \wedge y^+) \le x \vee y^-\,,$$
$$x^+ \wedge y \le (x^- \vee y) \wedge x^+ = x^- \vee (y \wedge x^+) \le x^- \vee y\,.$$

Further $(x \wedge y^+) \vee (x^+ \wedge y) = \big((x \wedge y^+) \vee y\big) \wedge x^+ = \big((y \vee x) \wedge y^+\big) \wedge x^+ = (x \vee y) \wedge x^+ \wedge y^+$, which implies $\big(y^- \vee (x \wedge y^+)\big) \vee \big(x^- \vee (y \wedge x^+)\big) = (x \vee y) \wedge x^+ \wedge y^+$ and analogously $(x \vee y^-) \wedge (x^- \vee y) = x^- \vee \big(y \wedge (x \vee y^-)\big) = x^- \vee \big(y^- \vee (x \wedge y)\big) = (x \wedge y) \vee x^- \vee y^-$ implies $\big(y^- \vee (x \wedge y^+)\big) \wedge \big(x^- \vee (y \wedge x^+)\big) = (x \wedge y) \vee x^- \vee y^-$. Hence the intervals $\big[(x \wedge y) \vee x^- \vee y^-,\ y^- \vee (x \wedge y^+)\big]$, $\big[x^- \vee (y \wedge x^+),\ (x \vee y) \wedge x^+ \wedge y^+\big]$ are transposed and the proof is complete. □

Using successively 4.5, 4.7, 4.9, 4.8, 4.6 and taking into consideration the fact that $(x \wedge y) \vee (x \vee y)^- = (x \vee y) \wedge (x \wedge y)^+$ we obtain (see Fig. 2):

LEMMA 4.10. *If $x, y \in L$, then* $d\big(x^- \vee y^-, (x \vee y)^-\big) = d\big((x \wedge y)^+, x^+ \wedge y^+\big)$.



$a = y^- \vee (x \wedge y^+) = (y^- \vee x) \wedge y^+$
$b = (x \wedge y) \vee (x \vee y)^- = (x \vee y) \wedge (x \wedge y)^+$
$c = x^- \vee (y \wedge x^+) = (x^- \vee y) \wedge x^+$

Figure 2.

LEMMA 4.11. *The function v_0 satisfies the condition* V1.

Proof. Take any $x, y \in L$, and let us calculate $v_0(x \vee y) + v_0(x \wedge y)$. By the definition of v_0, we have

$$v_0(x \vee y) + v_0(x \wedge y) = d\big(0, (x \vee y)^+\big) - d\big((x \vee y)^-, 0\big) + d\big(0, (x \wedge y)^+\big) - d\big((x \wedge y)^-, 0\big)\,.$$

In view of L1, we have

$$\begin{aligned} d\big(0,(x\vee y)^+\big) &= d\big(x^-,\,(x\vee y)^+\big) - d(x^-,0)\,,\\ d\big((x\vee y)^-,\,0\big) &= d\big((x\vee y)^-,\,y^+\big) - d(0,y^+)\,,\\ d\big(0,\,(x\wedge y)^+\big) &= d\big(y^-,\,(x\wedge y)^+\big) - d(y^-,0)\,,\\ d\big((x\wedge y)^-,\,0\big) &= d\big((x\wedge y)^-,\,x^+\big) - d(0,x^+)\,. \end{aligned}$$

Consequently

$$\begin{aligned} &v_0(x\vee y)+v_0(x\wedge y)\\ &= d\big(x^-,\,(x\vee y)^+\big) - d(x^-,0) - d\big((x\vee y)^-,\,y^+\big) + d(0,y^+) + d\big(y^-,\,(x\wedge y)^+\big)\\ &\qquad - d(y^-,0) - d\big((x\wedge y)^-,\,x^+\big) + d(0,x^+)\,. \end{aligned} \tag{$*$}$$

But $d\big(x^-,\,(x\vee y)^+\big) = d\big((x\wedge y)^-,\,(x\vee y)^+\big) - d\big((x\wedge y)^-,\,x^-\big) = d\big((x\wedge y)^-,\,(x\vee y)^+\big) - d\big(y^-,\,x^-\vee y^-\big)$, and analogously $d\big((x\wedge y)^-,\,x^+\big) = d\big((x\wedge y)^-,\,(x\vee y)^+\big) - d\big(x^+,\,(x\vee y)^+\big) = d\big((x\wedge y)^-,\,(x\vee y)^+\big) - d(x^+\wedge y^+,\,y^+)$ by L1 and 4.4. Further $d\big((x\vee y)^-,\,y^+\big) = d(y^-,y^+) - d\big(y^-,\,(x\vee y)^-\big)$ and $d\big(y^-,\,(x\wedge y)^+\big) = d(y^-,\,y^+) - d\big((x\wedge y)^+,\,y^+\big)$, again by L1. Substituting into $(*)$ and arranging we obtain $v_0(x\vee y)+v_0(x\wedge y) = d(0,x^+) - d(x^-,0) + d(0,y^+) - d(y^-,0) + d\big(y^-,\,(x\vee y)^-\big) - d(y^-,\,x^-\vee y^-) - d\big((x\wedge y)^+,\,y^+\big) + d(x^+\wedge y^+,\,y^+)$. Now, since $d\big(y^-,\,(x\vee y)^-\big) - d(y^-,\,x^-\vee y^-) = d\big(x^-\vee y^-,\,(x\vee y)^-\big)$, $d\big((x\wedge y)^+,\,y^+\big) - d(x^+\wedge y^+,\,y^+) = d\big((x\wedge y)^+,\,x^+\wedge y^+\big)$ by L1, using the definition of v_0 and 4.10, we get $v_0(x\vee y)+v_0(x\wedge y) = v_0(x)+v_0(y)$. This completes the proof. □

In view of 4.11 and 4.3, we have

COROLLARY 4.12. *The function v_0 is a positive valuation on L.*

COROLLARY 4.13. *Let v_0 be as above, and let t be any real number. The function v_t defined on L by $v_t(a) = v_0(a)+t$ is a positive valuation on L.*

The proof is straightforward.

We have obtained:

THEOREM 4.14. *Let L be a metric lattice in the sense of Definition 4.2. Then L is a metrizable in the sense of Definition 4.1, too. Moreover, for any $x\in L$ and $t\in R$ there exists a positive valuation v on L with $v(x)=t$.*

P r o o f. Denote by 0 the chosen element x and take v_t as above. Then v_t is as we need. □

COROLLARY 4.15. *Definitions* 4.1 *and* 4.2 *are equivalent.*

Given a metric d on L satisfying L1 and L2, we can take some of the above constructed positive valuations and the corresponding distance function. What can be said about the relation between these two metrics? And what about the relation between a positive valuation v on L and a positive valuation v_1 corresponding to the distance function of v? The answers are given in the following theorems.

THEOREM 4.16. *Let d be a metric on a lattice L satisfying* L1 *and* L2. *Then d is the distance function of any of the above valuations.*

Proof. Take any one of the above valuations, say v_t. Denote by d_t the distance function of v_t. Then for any $x, y \in L$ we have

$$\begin{aligned} d_t(x,y) &= v_t(x \vee y) - v_t(x \wedge y) \\ &= v_0(x \vee y) + t - v_0(x \wedge y) - t = v_0(x \vee y) - v_0(x \wedge y) \\ &= d\big(0,\, (x \vee y)^+\big) - d\big((x \vee y)^-, 0\big) - d\big(0,\, (x \wedge y)^+\big) + d\big((x \wedge y)^-,\, 0\big) \\ &= \Big(d\big(0,\, (x \vee y)^+\big) - d\big(0,\, (x \wedge y)^+\big)\Big) \\ &\qquad\qquad + \Big(d\big((x \wedge y)^-,\, 0\big) - d\big((x \vee y)^-,\, 0\big)\Big) \\ &= d\big((x \wedge y)^+,\, (x \vee y)^+\big) + d\big((x \wedge y)^-,\, (x \vee y)^-\big)\,. \end{aligned}$$

Due to the modularity of L, we have $(x \wedge y) \vee (x \vee y)^- = (x \wedge y)^+ \wedge (x \vee y)$ and denoting this element by z, we can see that the intervals $\big[(x \wedge y)^+,\, (x \vee y)^+\big]$, $[z,\, x \vee y]$ are transposed, and so are $\big[(x \wedge y)^-,\, (x \vee y)^-\big]$, $[x \wedge y,\, z]$. Hence $d\big((x \wedge y)^+,\, (x \vee y)^+\big) = d(z,\, x \vee y)$ and $d\big((x \wedge y)^-,\, (x \vee y)^-\big) = d(x \wedge y,\, z)$. Therefore we have $d_t(x,y) = d(z,\, x \vee y) + d(x \wedge y,\, z) = d(x \wedge y,\, x \vee y) = d(x,y)$. □

THEOREM 4.17. *Let v be any positive valuation on a lattice L, and d its distance function. Let v_t correspond to d as above. Then v and v_t differ at most by a constant, i.e. there is a real number ϱ such that $v_t(a) = v(a) + \varrho$ for every $a \in L$.*

Proof. Let $a \in L$. Then $v_t(a) = v_0(a) + t = d(0, a^+) - d(a^-, 0) + t = v(a^+) - v(0) - v(0) + v(a^-) + t = v(a \vee 0) + v(a \wedge 0) - 2v(0) + t = v(a) + v(0) - 2v(0) + t = v(a) + t - v(0) = v(a) + v_t(0) - v(0)$. It is sufficient to set $\varrho = v_t(0) - v(0)$. □

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