

Vandana P. Bhamre; Madhukar M. Pawar
Covering energy of posets and its bounds

Mathematica Bohemica, Vol. 148 (2023), No. 4, 537–553

Persistent URL: <http://dml.cz/dmlcz/151973>

Terms of use:

© Institute of Mathematics AS CR, 2023

Institute of Mathematics of the Czech Academy of Sciences provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these *Terms of use*.



This document has been digitized, optimized for electronic delivery and stamped with digital signature within the project *DML-CZ: The Czech Digital Mathematics Library* <http://dml.cz>

COVERING ENERGY OF POSETS AND ITS BOUNDS

VANDANA P. BHAMRE, Amalner, MADHUKAR M. PAWAR, Dhule

Received February 28, 2022. Published online October 17, 2022.

Communicated by Radomír Halaš

Cordially dedicated to Professor N. K. Thakare on his 84th birthday

Abstract. The concept of covering energy of a poset is known and its McClelland type bounds are available in the literature. In this paper, we establish formulas for the covering energy of a crown with $2n$ elements and a fence with n elements. A lower bound for the largest eigenvalue of a poset is established. Using this lower bound, we improve the McClelland type bounds for the covering energy for some special classes of posets.

Keywords: covering energy of poset; eigenvalue; spectrum; upper bound; lower bound

MSC 2020: 06A07, 06A11, 06B05, 06B99, 05B20, 05C50.

1. INTRODUCTION AND PRELIMINARIES

The concept of the energy of graph was introduced by Gutman in 1978 (see [8]). This concept has its roots in the Hückel molecular orbital (HMO) theory, see Hückel [12]. The *energy of a graph* G is defined as the sum of the absolute values of all eigenvalues of the adjacency matrix $A(G)$ of G , denoted by $\mathcal{E}(G)$. We label the eigenvalues of $A(G)$ in the non-increasing order as $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \dots \geq \lambda_n$. This set of eigenvalues is called the *spectrum* of G and denoted by $\text{Spec}(G) = \{\lambda_1, \lambda_2, \dots, \lambda_n\}$. For details of the theory of graph energy, see Li, Shi and Gutman [15] and the survey article by Gutman and Ramane [11].

McClelland in [16] gave simple bounds for the energy of a graph. A number of researchers have improved these bounds, e.g. Altındağ and Bozkurt [2], Das et al. [5]. During the last four decades, many researchers have worked in this area. In the context of graphs, more than 200 different types of “energies” were proposed, see Gutman and Furtula [9] and [10]. Recently, Pawar and Bhangale [19] proposed one more graph energy.

Pawar and Bhamre (see [17] and [18]) extended the concept of the “energy” to posets by defining the covering energy. They obtained the McClelland type bounds for the covering energy of a poset. In the present paper, we give a lower bound for the largest eigenvalue of a poset. We improve the McClelland type bounds for the covering energy for some special classes of posets.

We recall some definitions. A nonempty set P , together with a binary relation \leq which is reflexive, antisymmetric, and transitive, is called a *partially ordered set*, in short, a *poset*. The *Hasse diagram of a poset* is a representation of a poset in the plane. For a poset (P, \leq) one represents each element of P as a vertex in the plane and draws a line segment that goes upward from x to y whenever y covers x (i.e., whenever $x < y$ and there is no z such that $x < z < y$, denoted by $x \prec y$). We call such a line as an *edge* and denote the set of all edges in P by $e(P)$. The Hasse diagram of a poset P considered as a graph is called the *covering graph*. We denote it by $G(P)$. Hasse diagrams of some posets are depicted in Figure 1.

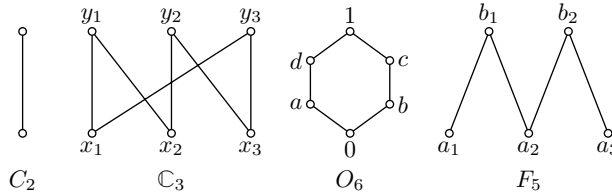


Figure 1. Hasse diagrams of some posets.

Two elements $a, b \in P$ are said to be *comparable* if either $a \leq b$ or $b \leq a$; otherwise they are said to be *incomparable*. A poset in which every pair of elements is comparable is called a *chain*, and if every pair of elements is incomparable, it is called an *antichain*.

A *lattice* is a poset in which every pair of elements has the supremum (called their *join*) and the infimum (called their *meet*). If a and b are elements in a lattice L , then their join and meet are denoted by $a \vee b$ and $a \wedge b$, respectively. An element x in a poset P is called *doubly-irreducible* if it covers and is covered by at most one element. The set of all doubly irreducible elements in P is denoted by $\text{Irr}(P)$. A *reducible element* is an element in P , which is not doubly irreducible. The set of all reducible elements in P is denoted by $R(P)$. If there is only one poset under discussion, then the notation $R(P)$ will be replaced by R .

A partially ordered set $F_n = \{x_1, x_2, \dots, x_n\}$ is called *fence*, if either $x_1 < x_2, x_2 > x_3, \dots, x_{2m-1} < x_{2m}, x_{2m} > x_{2m+1}, \dots, x_{n-1} < x_n$ if n is even ($x_{n-1} > x_n$ if n is odd) or $x_1 > x_2, x_2 < x_3, \dots, x_{2m-1} > x_{2m}, x_{2m} < x_{2m+1}, \dots, x_{n-1} > x_n$ if n is even ($x_{n-1} < x_n$, if n is odd) are the only comparability relations. A fence F_n is called a *lower fence* if $x_1 < x_2$ and an *upper fence* if $x_1 > x_2$, e.g. the fence F_5 as depicted in Figure 1 is a lower fence and its dual F_n^* is an upper fence.

For an integer $n \geq 3$, a *crown of order n* is a poset $P = \{x_1, x_2, x_3, \dots, x_n, y_1, y_2, y_3, \dots, y_n\}$ whose elements satisfy precisely the comparabilities $x_1 < y_1$, $y_1 > x_2$, $x_2 < y_2$, $y_2 > x_3$, $x_3 < y_3$, $y_3 > x_4, \dots, x_{n-1} < y_{n-1}$, $y_{n-1} > x_n$, $x_n < y_n$, $y_n > x_1$. We denote a crown of order n by \mathbb{C}_n . The crown \mathbb{C}_3 is as depicted in Figure 1. The concepts of a doubly irreducible element, crown, and fence are useful to study dismantlable lattices, see, e.g. Kelly and Rival [14], Rival [20], and Thakare et al. [21].

Throughout this paper, P denotes a finite poset. For undefined terms and notations from lattice theory, refer to Grätzer [7] or Davey and Priestley [6], and for graph theoretic terms, see Cvetković et al. [4] or Li, Shi and Gutman [15]. The following two results are used in the next sections.

Lemma 1.1 ([15]). *For a path $P_n, n \geq 2$,*

$$\text{Spec}(P_n) = \left\{ 2 \cos \frac{\pi r}{n+1} : r = 1, 2, \dots, n \right\}$$

and

$$\mathcal{E}(P_n) = \begin{cases} \frac{2}{\sin(\pi/2(n+1))} - 2 & \text{if } n \equiv 0 \pmod{2}, \\ \frac{2 \cos(\pi/2(n+1))}{\sin(\pi/2(n+1))} - 2 & \text{if } n \equiv 1 \pmod{2}. \end{cases}$$

Lemma 1.2 ([13]). *For $p \geq 3$, let \mathbf{C}_p and $\overline{\mathbf{C}}_p$ denote a p -cycle and the compliment of \mathbf{C}_p , respectively, then*

$$\text{Spec}(\mathbf{C}_p) = \left\{ 2, 2 \cos \frac{2\pi}{p}, 2 \cos \frac{4\pi}{p}, 2 \cos \frac{6\pi}{p}, \dots, 2 \cos \frac{2(p-1)\pi}{p} \right\}$$

and

$$\text{Spec}(\overline{\mathbf{C}}_p) = \left\{ p-3, -1 - 2 \cos \frac{2\pi}{p}, -1 - 2 \cos \frac{4\pi}{p}, \dots, -1 - 2 \cos \frac{2(p-1)\pi}{p} \right\}.$$

2. THE COVERING ENERGY OF A POSET

When the energy of a graph G was defined by Gutman [8], no loop edges were allowed. However, it is natural to ask what happens if some loops are present. It does not look promising to allow that vertices with loops form an *arbitrary subset* X of the vertex set of G . Much after Gutman [8], Adiga et al. [1] allowed X to be a minimal covering set of the graph. Now if G is the covering graph of a poset P , then we can use the language of posets to uniquely define a vertex set X in a natural way, and we can allow loops exactly at the vertices belonging to X . This is how Adiga et al. [1]

motivated Pawar and Bhamre [17] to introduce the concept of covering energy of a poset P , in which $R(P)$ plays the role of X . We recall the definition and some of the results proved in [17] and [18].

Definition 2.1 ([17]). Let $P = \{v_1, v_2, v_3, \dots, v_n\}$ be a poset. The *covering matrix* of P denoted by $C(P)$ is an $n \times n$ matrix $C(P) = (a_{ij})$, where

$$a_{ij} = \begin{cases} 1 & \text{if } i \neq j \text{ and } v_i \prec v_j \text{ or } v_j \prec v_i, \\ 1 & \text{if } i = j \text{ and } v_i \notin \text{Irr}(P), \\ 0 & \text{otherwise.} \end{cases}$$

The *characteristic polynomial* of P denoted by $\psi(P, \lambda)$ is the determinant $\det(\lambda I_n - C(P))$. The eigenvalues of $C(P)$ are called the *eigenvalues of the poset* P . The sum of absolute values of all these eigenvalues is called the *covering energy* of P and denoted by $E(P)$. If $E(P)$ is an integer, then P is called an *integral poset*.

Note that $C(P)$ depends on how (in which order) we list the elements of P , but this causes no problem since a different listing of elements gives a matrix similar to $C(P)$ and similar matrices have the same characteristic polynomial and eigenvalues. Hence, we can always fix a list v_1, v_2, \dots, v_n and work with $C(P)$ defined by this list. As $C(P)$ is a real and symmetric matrix, its all eigenvalues are real numbers. We label them in the non-increasing order as $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \dots \geq \lambda_n$. This set of eigenvalues is called the *spectrum* of P and denoted by $\text{Spec}(P)$. It is clear that

$$E(P) = \sum_{i=1}^n |\lambda_i|.$$

We compute covering energies of some simple posets depicted in Figure 1.

Example 2.2. The covering matrix of the chain C_2 is $C(C_2) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. This leads to $\psi(C_2, \lambda) = \lambda^2 - 1$, $\text{Spec}(C_2) = \{1, -1\}$ and $E(C_2) = 2$.

Example 2.3. For the lattice $O_6 = \{0, a, b, c, d, 1\}$ in Figure 1, the covering matrix, characteristic polynomial, spectrum and covering energy of O_6 are given by

$$C(O_6) = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix},$$

$$\psi(O_6, \lambda) = -3 - 6\lambda + 7\lambda^2 + 8\lambda^3 - 5\lambda^4 - 2\lambda^5 + \lambda^6,$$

$$\text{Spec}(O_6) = \{1 + \sqrt{2}, \sqrt{3}, 1, 1 - \sqrt{2}, -1, -\sqrt{3}\}, \quad E(O_6) = 2(1 + \sqrt{2} + \sqrt{3}).$$

Example 2.4. The crown $\mathbb{C}_3 = \{x_1, x_2, x_3, y_1, y_2, y_3\}$ in Figure 1 is an integral poset. The covering matrix, characteristic polynomial, spectrum and covering energy of \mathbb{C}_3 are

$$C(\mathbb{C}_3) = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{bmatrix},$$

$$\psi(\mathbb{C}_3, \lambda) = -12\lambda^2 + 4\lambda^3 + 9\lambda^4 - 6\lambda^5 + \lambda^6 = \lambda^2(\lambda + 1)(\lambda - 2)^2(\lambda - 3),$$

$$\text{Spec}(\mathbb{C}_3) = \{3, 2, 2, 0, 0, -1\}, \quad E(\mathbb{C}_3) = 8.$$

Example 2.5. Let $F_5 = \{a_1, b_1, a_2, b_2, a_3\}$ be the poset shown in Figure 1. The covering matrix, characteristic polynomial, spectrum and covering energy of F_5 are

$$C(F_5) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix},$$

$$\psi(F_5, \lambda) = 1 - \lambda - 5\lambda^2 + \lambda^3 + 3\lambda^4 - \lambda^5 = -(\lambda + 1)(\lambda^2 - 3\lambda + 1)(\lambda^2 - \lambda - 1),$$

$$\text{Spec}(F_5) = \left\{ \frac{1}{2}(3 + \sqrt{5}), \frac{1}{2}(1 + \sqrt{5}), \frac{1}{2}(3 - \sqrt{5}), \frac{1}{2}(1 - \sqrt{5}), -1 \right\}, \quad E(F_5) = 4 + \sqrt{5}.$$

Definition 2.6. Let P and Q be two disjoint posets. Then $P \cup Q$ is a poset under the partial order defined by $x \leq y$ in $P \cup Q$ if and only if either $x, y \in P$ and $x \leq y$ in P or $x, y \in Q$ and $x \leq y$ in Q .

Note that, if $C(P)$, $C(Q)$ and $C(P \cup Q)$ are the covering matrices of P , Q and $P \cup Q$, respectively, then $C(P \cup Q) = \begin{bmatrix} C(P) & 0 \\ 0 & C(Q) \end{bmatrix}$. Hence, $\psi(P \cup Q, \lambda) = \psi(P, \lambda)\psi(Q, \lambda)$. This leads to next claim.

Proposition 2.7. Let P and Q be two disjoint posets. Then

$$E(P \cup Q) = E(P) + E(Q).$$

Corollary 2.8. Let P_1, P_2, \dots, P_k be k disjoint posets. Then

$$E\left(\bigcup_{i=1}^k P_i\right) = \sum_{i=1}^k E(P_i).$$

We note that if $P_1 \cong P_2$, then $E(P_1) = E(P_2)$. But, two non-isomorphic posets may have the same covering energy, e.g. a poset and its dual have the same covering matrix. This fact motivates us to define the known concepts of equienergetic and co-spectral graphs for posets as follows.

Definition 2.9. Two posets P and Q are said to be *co-spectral* if they have the same spectra and are said to be *equienergetic* if $E(P) = E(Q)$.

Naturally, dual posets are co-spectral. Moreover, two co-spectral posets are equienergetic but not conversely. Here is an example.

Example 2.10. Consider the poset P consisting of four disjoint copies of C_2 . From Corollary 2.8 and Example 2.2, we have

$$\text{Spec}(P) = \{1, 1, 1, 1, -1, -1, -1, -1\} \quad \text{and} \quad E(P) = 8.$$

Thus from Example 2.4, it is observed that \mathbb{C}_3 and P are equienergetic posets but not co-spectral. Moreover they are of different orders.

For an integer $n \geq 2$, the covering matrix of the chain C_n is the same as the adjacency matrix of the path P_n . This observation and Lemma 1.1 lead to the following results.

Observation 2.11. For a chain C_n , $n \geq 2$,

$$(1) \quad \text{Spec}(C_n) = \left\{ 2 \cos \frac{\pi r}{n+1} : r = 1, 2, 3, \dots, n \right\},$$

$$(2) \quad E(C_n) = \begin{cases} \frac{2}{\sin(\pi/2(n+1))} - 2 & \text{if } n \text{ is even,} \\ \frac{2 \cos(\pi/2(n+1))}{\sin(\pi/2(n+1))} - 2 & \text{if } n \text{ is odd.} \end{cases}$$

The crown \mathbb{C}_3 and lattices O_6 as depicted in Figure 1 have the same covering graph. Interestingly, these posets are not equienergetic. For an integer $n \geq 3$, let us write $\mathcal{P}_n = \{P : |P| = n, G(P) \cong P_n\}$. Then in the class of posets \mathcal{P}_n , the chain C_n and the fence F_n (maybe an upper fence or a lower fence) are posets with maximum and minimum number of doubly irreducible elements, respectively. Similarly, the crown \mathbb{C}_n is the poset with no doubly irreducible element and the covering graph $G(\mathbb{C}_n)$ is a cycle with $2n$ edges. These observations motivate us to compute the covering energy of the crown \mathbb{C}_n and the fence F_n in terms of n . One can easily derive the following result by using de Moivre's theorem.

Lemma 2.12. For an integer $m \geq 1$ and $\alpha \in \mathbb{R}$, we have

$$\sum_{r=1}^m \cos(\alpha r) = \frac{1}{2 \sin(\frac{1}{2}\alpha)} \left(\sin \frac{(2m+1)\alpha}{2} - \sin \frac{\alpha}{2} \right)$$

and

$$\sum_{r=1}^m \sin(\alpha r) = \frac{1}{2 \sin(\frac{1}{2}\alpha)} \left(\cos \frac{\alpha}{2} - \cos \frac{(2m+1)\alpha}{2} \right).$$

Using this result, we compute the covering energy of \mathbb{C}_n and F_n .

Theorem 2.13. For an integer $n \geq 3$, the spectrum and covering energy of the crown \mathbb{C}_n are

$$\text{Spec}(\mathbb{C}_n) = \left\{ 3, 1 + 2 \cos \frac{\pi}{n}, 1 + 2 \cos \frac{2\pi}{n}, \dots, 1 + 2 \cos \frac{(2n-1)\pi}{n} \right\},$$

$$E(\mathbb{C}_n) = \begin{cases} \frac{2n}{3} + 2\sqrt{3} \cot \frac{\pi}{2n} & \text{if } n \equiv 0 \pmod{3}, \\ \frac{2}{3}(n-1) + 4 \operatorname{cosec} \frac{\pi}{2n} \cos \left(\frac{\pi}{6} - \frac{\pi}{6n} \right) & \text{if } n \equiv 1 \pmod{3}, \\ \frac{2}{3}(n+1) + 4 \operatorname{cosec} \frac{\pi}{2n} \cos \left(\frac{\pi}{6} + \frac{\pi}{6n} \right) & \text{if } n \equiv 2 \pmod{3}. \end{cases}$$

Proof. In the crown \mathbb{C}_n , $n \geq 3$, every element is a reducible element and hence in the covering matrix $C(\mathbb{C}_n)$ all diagonal entries are 1. Also, if we denote the adjacency matrix of the graph $G(\mathbb{C}_n)$ by $A(\mathbb{C}_n)$, then $C(\mathbb{C}_n) = I + A(\mathbb{C}_n)$. Hence, the characteristic polynomial of \mathbb{C}_n can be expressed as

$$\psi(\mathbb{C}_n, \lambda) = |\lambda I - C(\mathbb{C}_n)| = |(\lambda - 1)I - A(\mathbb{C}_n)|.$$

Thus, if $\lambda_1, \lambda_2, \dots, \lambda_{2n}$ and $\beta_1, \beta_2, \dots, \beta_{2n}$ are the eigenvalues of $A(\mathbb{C}_n)$ and $C(\mathbb{C}_n)$, respectively, then $\beta_i = \lambda_i + 1$ for all $i = 1, 2, \dots, 2n$. Therefore Lemma 1.2 leads to

$$\text{Spec}(\mathbb{C}_n) = \left\{ 3, 1 + 2 \cos \frac{\pi}{n}, 1 + 2 \cos \frac{2\pi}{n}, \dots, 1 + 2 \cos \frac{(2n-1)\pi}{n} \right\},$$

$$E(\mathbb{C}_n) = 3 + \sum_{r=1}^{2n-1} \left| 1 + 2 \cos \frac{\pi r}{n} \right|$$

$$= 3 + \sum_{\substack{r=1 \\ 1+2 \cos(\pi r/n) \geq 0}}^{2n-1} \left(1 + 2 \cos \frac{\pi r}{n} \right) - \sum_{\substack{r=1 \\ 1+2 \cos(\pi r/n) < 0}}^{2n-1} \left(1 + 2 \cos \frac{\pi r}{n} \right).$$

In the interval $(0, 2\pi)$, it is $1 + 2 \cos(\pi r/n) < 0$ if and only if $\frac{2}{3}\pi < \pi r/n < \frac{4}{3}\pi$, that is, $\frac{2}{3} < r/n < \frac{4}{3}$.

Case (1): If $n \equiv 0 \pmod{3}$, that is, $n = 3k$, then $1 + 2 \cos(\pi r/n) < 0$ if and only if $r = 2k + 1, 2k + 2, \dots, 4k - 1$. Hence, by symmetry of the curve $1 + 2 \cos \alpha$, we have

$$\begin{aligned} E(\mathbb{C}_n) &= 3 + 2 \sum_{r=1}^{2k} \left(1 + 2 \cos \frac{\pi r}{n}\right) - \sum_{r=2k+1}^{4k-1} \left(1 + 2 \cos \frac{\pi r}{n}\right) \\ &= 3 + 4k - (2k - 1) + 4 \sum_{r=1}^{2k} \cos \frac{\pi r}{n} - 2 \sum_{r=2k+1}^{4k-1} \cos \frac{\pi r}{n} \\ &= 2k + 4 + 6 \sum_{r=1}^{2k} \cos \frac{\pi r}{n} - 2 \sum_{r=1}^{4k-1} \cos \frac{\pi r}{n}. \end{aligned}$$

Using Lemma 2.12, after simplification, we obtain

$$E(\mathbb{C}_n) = 2k + 2\sqrt{3} \cot \frac{\pi}{2n} = \frac{2n}{3} + 2\sqrt{3} \cot \frac{\pi}{2n}.$$

Case (2): If $n \equiv 1 \pmod{3}$, that is, $n = 3k + 1$, then $1 + 2 \cos(\pi r/n) < 0$ if and only if $\frac{2}{3} < r/n < \frac{4}{3}$ if and only if $2k + \frac{1}{3} < r < 4k + 1 + \frac{1}{3}$. That is, $r = 2k + 1, 2k + 2, \dots, 4k + 1$. Thus

$$\begin{aligned} E(\mathbb{C}_n) &= 3 + 2 \sum_{r=1}^{2k} \left(1 + 2 \cos \frac{\pi r}{n}\right) - \sum_{r=2k+1}^{4k+1} \left(1 + 2 \cos \frac{\pi r}{n}\right) \\ &= 3 + 4k - 2k - 1 + 6 \sum_{r=1}^{2k} \cos \frac{\pi r}{n} - 2 \sum_{r=1}^{4k+1} \cos \frac{\pi r}{n}. \end{aligned}$$

Using Lemma 2.12, after simplification, we obtain

$$E(\mathbb{C}_n) = \frac{2}{3}(n - 1) + 4 \cos\left(\frac{\pi}{6} - \frac{\pi}{6n}\right) \operatorname{cosec} \frac{\pi}{2n}.$$

Case (3): If $n \equiv 2 \pmod{3}$, that is, $n = 3k - 1$ then $1 + 2 \cos(\pi r/n) < 0$ if and only if $\frac{2}{3} < r/n < \frac{4}{3}$ if and only if $\frac{2}{3}n < r < \frac{4}{3}n$. That is, $r = 2k, 2k + 1, \dots, 4k - 2$. Therefore

$$E(\mathbb{C}_n) = 3 + 2 \sum_{r=1}^{2k-1} \left(1 + 2 \cos \frac{\pi r}{n}\right) - \sum_{r=2k}^{4k-2} \left(1 + 2 \cos \frac{\pi r}{n}\right).$$

Using Lemma 2.12, after simplification, we obtain

$$E(\mathbb{C}_n) = \frac{2}{3}(n + 1) + 4 \operatorname{cosec} \frac{\pi}{2n} \cos\left(\frac{\pi}{6} + \frac{\pi}{6n}\right).$$

□

Corollary 2.14. Among all crowns, \mathbb{C}_3 is the only integral crown.

Corollary 2.15. If λ is an eigenvalue of the crown \mathbb{C}_n , then $-1 \leq \lambda \leq 3$ and $\lambda^* = 2 - \lambda$ is another eigenvalue.

Proof. The first part is trivial. If $\lambda = 1 + 2 \cos(\pi r/n)$ is an eigenvalue of \mathbb{C}_n , then $\lambda^* = 2 - \lambda = 1 + 2 \cos(\pi(n+r)/n)$ is also an eigenvalue of \mathbb{C}_n . \square

Theorem 2.16. Let $F_n = \{v_1, v_2, \dots, v_n\}$ and (F_n, \leq) be a fence. Then

$$(1) \quad \text{Spec}(F_n) = \left\{ -1, 1 + 2 \cos \frac{\pi}{n}, 1 + 2 \cos \frac{2\pi}{n}, \dots, 1 + 2 \cos \frac{(n-1)\pi}{n} \right\}.$$

(2) The covering energy of the fence F_n is given by

$$E(F_n) = \begin{cases} \frac{n-3}{3} + \sqrt{3} \cot \frac{\pi}{2n} & \text{if } n \equiv 0 \pmod{3}, \\ \frac{n-4}{3} + \left(\sqrt{3} \cos \frac{\pi}{6n} + \sin \frac{\pi}{6n} \right) \text{cosec} \frac{\pi}{2n} & \text{if } n \equiv 1 \pmod{3}, \\ \frac{n-2}{3} + \left(\sqrt{3} \cos \frac{\pi}{6n} - \sin \frac{\pi}{6n} \right) \text{cosec} \frac{\pi}{2n} & \text{if } n \equiv 2 \pmod{3}. \end{cases}$$

Proof. For an integer $n \geq 3$, the covering graph $G(F_n)$ is the path P_n . Let $\psi(F_n, \lambda)$ and $\varphi(P_n, \lambda)$ denote the characteristic polynomials of F_n and P_n , respectively. Then the covering matrix $C(F_n)$, the adjacency matrix $A(P_n)$ and the characteristic polynomial of F_n can be expressed as

$$C(F_n) = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 & 0 & 0 \\ 1 & 1 & 1 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 1 & 1 \\ 0 & 0 & 0 & \dots & 0 & 1 & 0 \end{bmatrix}, \quad A(P_n) = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 & 0 & 0 \\ 1 & 0 & 1 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 & 1 \\ 0 & 0 & 0 & \dots & 0 & 1 & 0 \end{bmatrix}$$

and

$$\psi(F_n, \lambda) = \begin{vmatrix} \lambda & -1 & 0 & \dots & 0 & 0 & 0 \\ -1 & \lambda - 1 & -1 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & -1 & \lambda - 1 & -1 \\ 0 & 0 & 0 & \dots & 0 & -1 & \lambda \end{vmatrix}.$$

If we write $\lambda' = \lambda - 1$ and split the determinant using the first row, then

$$\begin{aligned} \psi(F_n, \lambda) &= \begin{vmatrix} \lambda' & -1 & 0 & \dots & 0 & 0 \\ -1 & \lambda' & -1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \lambda' & -1 \\ 0 & 0 & 0 & \dots & -1 & \lambda' + 1 \end{vmatrix} + \begin{vmatrix} 1 & 0 & 0 & \dots & 0 & 0 \\ -1 & \lambda' & -1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \lambda' & -1 \\ 0 & 0 & 0 & \dots & -1 & \lambda' + 1 \end{vmatrix} \\ &= \begin{vmatrix} \lambda' & -1 & 0 & \dots & 0 & 0 \\ -1 & \lambda' & -1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \lambda' & -1 \\ 0 & 0 & 0 & \dots & -1 & \lambda' + 1 \end{vmatrix} + \begin{vmatrix} \lambda' & -1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \lambda' & -1 \\ 0 & 0 & \dots & -1 & \lambda' + 1 \end{vmatrix}. \end{aligned}$$

Applying the same technique to split these two determinants with respect to the last row, we obtain

$$\psi(F_n, \lambda) = \varphi(P_n, \lambda') + 2\varphi(P_{n-1}, \lambda') + \varphi(P_{n-2}, \lambda').$$

It is well known that, for $n \geq 3$, $\varphi(P_n, \lambda) = \lambda\varphi(P_{n-1}, \lambda) - \varphi(P_{n-2}, \lambda)$. Using this recurrence relation, we have

$$\begin{aligned} \psi(F_n, \lambda) &= \lambda'\varphi(P_{n-1}, \lambda') - \varphi(P_{n-2}, \lambda') + 2\varphi(P_{n-1}, \lambda') + \varphi(P_{n-2}, \lambda'), \\ \psi(F_n, \lambda) &= (\lambda + 1)\varphi(P_{n-1}, \lambda') = (\lambda + 1) \prod (\lambda' - \alpha_i), \end{aligned}$$

where α_i , $i = 1, 2, \dots, n - 1$, are eigenvalues of the matrix $A(P_{n-1})$. Hence,

$$\psi(F_n, \lambda) = (\lambda + 1) \prod_{r=1}^{n-1} \left(\lambda - 1 - 2 \cos \frac{\pi r}{n} \right).$$

Thus,

$$\text{Spec}(F_n) = \left\{ -1, 1 + 2 \cos \frac{\pi}{n}, 1 + 2 \cos \frac{2\pi}{n}, \dots, 1 + 2 \cos \frac{(n-1)\pi}{n} \right\},$$

and

$$\begin{aligned} E(F_n) &= 1 + \sum_{r=1}^{n-1} \left| 1 + 2 \cos \frac{\pi r}{n} \right| \\ &= 1 + \sum_{\substack{r=1 \\ \frac{1}{2} < \cos(\pi r/n) < 1}}^{n-1} \left(1 + 2 \cos \frac{\pi r}{n} \right) + \sum_{\substack{r=1 \\ -\frac{1}{2} \leq \cos(\pi r/n) \leq \frac{1}{2}}}^{n-1} \left(1 + 2 \cos \frac{\pi r}{n} \right) \\ &\quad - \sum_{\substack{r=1 \\ -1 < \cos(\pi r/n) < -\frac{1}{2}}}^{n-1} \left(1 + 2 \cos \frac{\pi r}{n} \right). \end{aligned}$$

Using the symmetry of the curve $1 + 2 \cos \theta$, $0 < \theta < \pi$, we have

$$\begin{aligned} E(F_n) &= 1 + 4 \sum_{r=1}^{k=\max\{r: 3r < n\}} \left(\cos \frac{\pi r}{n} \right) + (n - 1 - 2k) \\ &= n - 2k + \frac{2}{\sin(\pi/2n)} \left(\sin \frac{(2k+1)\pi}{2n} - \sin \frac{\pi}{2n} \right) \quad (\text{using Lemma 2.12}) \\ &= n - 2k - 2 + 2 \operatorname{cosec} \frac{\pi}{2n} \sin \frac{(2k+1)\pi}{2n}. \end{aligned}$$

Case (1): If $n \equiv 0 \pmod{3}$, then $k = \frac{1}{3}n - 1$ and after routine calculations we have

$$E(F_n) = n - \frac{2n}{3} + 2 \sin \left(\frac{\pi}{3} - \frac{\pi}{2n} \right) \operatorname{cosec} \frac{\pi}{2n} = \frac{n-3}{3} + \sqrt{3} \cot \frac{\pi}{2n}.$$

Case (2): If $n \equiv 1 \pmod{3}$, then $k = \frac{1}{3}(n-1)$ and after simple calculations we obtain

$$E(F_n) = \frac{n-4}{3} + \left(\sqrt{3} \cos \frac{\pi}{6n} + \sin \frac{\pi}{6n} \right) \operatorname{cosec} \frac{\pi}{2n}.$$

Case (3): If $n \equiv 2 \pmod{3}$, then $k = \frac{1}{3}(n-2)$, which leads to

$$E(F_n) = \frac{n-2}{3} + \left(\sqrt{3} \cos \frac{\pi}{6n} - \sin \frac{\pi}{6n} \right) \operatorname{cosec} \frac{\pi}{2n}.$$

□

To compute the covering energy of other poset classes whose covering graphs are isomorphic to P_n is a separate subject. We do not go into those details. Now, we turn our attention to the bounds of the covering energy of posets.

Pawar and Bhamre (see [17] and [18]) have studied the covering energy of some special classes of posets. They have obtained formulas for coefficients of λ^n , λ^{n-1} , λ^{n-2} and λ^{n-3} in the characteristic polynomial $\psi(P, \lambda)$ of a poset P in terms of its number of vertices, edges, reducible and doubly irreducible elements.

The following results are from Pawar and Bhamre (see [17]).

Theorem 2.17 ([17]). *If $\lambda_1, \lambda_2, \dots, \lambda_n$ are eigenvalues of a poset P , then*

$$(1) \quad \sum_{i=1}^n \lambda_i = n - |\operatorname{Irr}(P)| = |R|,$$

$$(2) \quad \sum_{i=1}^n \lambda_i^2 = n + 2|e(P)| - |\operatorname{Irr}(P)| = 2|e(P)| + |R|.$$

Pawar and Bhamre (see [17]) have also obtained McClelland type bounds for the covering energy of a poset in terms of its number of vertices, number of edges and the determinant of the covering matrix $C(P)$.

Theorem 2.18 ([17]). *If a poset P has n elements, m edges and $D = \det(C(P))$, then*

$$(2.1) \quad \sqrt{2m + n - |\text{Irr}(P)| + n(n-1)D^{2/n}} \leq E(P) \leq \sqrt{n(2m + n - |\text{Irr}(P)|)}.$$

We improve these bounds for a particular class of posets in the next section.

3. IMPROVEMENT OF BOUNDS FOR THE COVERING ENERGY OF A POSET

In the literature, we can find definitions of singular and non-singular graphs. We extend these concepts to posets as follows.

Definition 3.1. Let P be a poset with $|P| = n$ and $\lambda_1, \lambda_2, \dots, \lambda_n$ be the eigenvalues of P . Then P is called a singular poset if $\lambda_i = 0$ for some $1 \leq i \leq n$. Otherwise, if $\lambda_i \neq 0$ for each i , then P is called a non-singular poset.

We recall that a *Hermitian matrix* (or a *self adjoint matrix*) is a complex square matrix which equals its conjugate transpose. Every real symmetric matrix is a Hermitian matrix. In the next result, we obtain a lower bound for the largest eigenvalue of a poset.

Theorem 3.2. *Let P be a poset with n elements and m edges. Let r be the greatest eigenvalue of P and $R = \{v \in P : v \notin \text{Irr}(P)\}$. Then*

$$(3.1) \quad \frac{2m + |R|}{n} \leq r.$$

Moreover, the equality holds if and only if P is C_2 or more copies of C_2 than one or $\text{Irr}(P) = \emptyset$ and the covering graph $G(P)$ is regular.

Proof. Since the covering matrix $C(P) = (a_{ij})_{n \times n}$ is symmetric and real, it is a Hermitian matrix. It is well known that the problem of finding the maximal value of Rayleigh's quotient

$$(3.2) \quad Rq = \frac{\sum_{i=1}^n \sum_{j=1}^n a_{ij} x_i x_j}{\sum_{i=1}^n x_i^2}$$

(the x_i being arbitrary real numbers not all equal to zero) has the solution $Rq = r$. The maximum is attained if and only if the $x_i, i = 1, 2, \dots, n$, are the components of the eigenvector of $C(P)$ belonging to r . If we put $x_i = 1, i = 1, 2, \dots, n$, in equation (3.2), we have $Rq = \bar{d} = (1/n) \sum_{i=1}^n d_i$, where

$$(3.3) \quad d_i = \sum_{j=1}^n a_{ij} = \begin{cases} d(v_i) & \text{if } v_i \in \text{Irr}(P), \\ d(v_i) + 1 & \text{if } v_i \notin \text{Irr}(P) \end{cases}$$

(here $d(v_i)$ denotes degree of the vertex v_i in the covering graph $G(P)$). So, \bar{d} is a particular value of Rayleigh's quotient. This leads to

$$(3.4) \quad \bar{d} = \frac{2m + |R|}{n} \leq r.$$

Thus the first part of the theorem is proved.

In Case-I, we prove the second part of the theorem for a connected poset and in Case-II, as a consequence of Case-I, we can prove it for disconnected posets.

Case-I: For the chain C_2 , it is trivial to show that the equality holds in (3.1). If P is a connected poset, for which $\text{Irr}(P) = \varphi$, and $G(P)$ is regular of order k , then

$$\frac{2m + |R|}{n} = k + 1 = r,$$

i.e., equality holds in the inequality (3.1).

Conversely, if the equality holds in (3.1), then the values $x_i = 1, i = 1, 2, \dots, n$, constitute an eigenvector for $C(P)$ belonging to r and $\sum_{j=1}^n a_{ij}x_j = rx_i, i = 1, 2, \dots, n$, implies $d_i = \sum_{j=1}^n a_{ij} = r, i = 1, 2, \dots, n$. As d_i is an integer for each i , hence $|R| = 0$ or $|R| = n$. In both the cases $G(P)$ is regular. As $G(P)$ is connected, $|R| = 0$ if and only if P is a chain C_2 , and $|R| = n$ if and only if $\text{Irr}(P) = \emptyset$. This proves the result for connected posets.

Case-II: Let us consider a disconnected poset P with s connected components P_1, P_2, \dots, P_s with n_1, n_2, \dots, n_s elements and m_1, m_2, \dots, m_s edges, respectively.

Case (a): P is regular of order one if and only if each component P_i is regular of order one, i.e., $P_i = C_2, i = 1, 2, \dots, s$. Hence $m = s, n = 2s, |R| = 0$ and $r = 1 = (2m + |R|)/n$.

Case (b): The poset P is regular of order $k > 1$ and no element of P is doubly irreducible if and only if for each $i = 1, 2, \dots, s, P_i$ is regular of order k and no element of P_i is doubly irreducible. Using Case-I for each $P_i, 1 \leq i \leq s$, the largest eigenvalues of P_i ,

$$r_i = \frac{2m_i + |R(P_i)|}{n_i} = \frac{kn_i + n_i}{n_i} = k + 1.$$

Hence

$$r = \max\{r_1, r_2, \dots, r_s\} = k + 1 = \frac{kn + n}{n} = \frac{2m + |R|}{n}.$$

□

The Theorem 3.2 is a generalization of a result which is originally due to Collatz and Sinogowitz [3] (for the English version of this result, see [4]). The following two results are applications of Theorem 3.2.

Theorem 3.3. *Let P be a non-singular poset with n elements, m edges and $R = \{v \in P: v \notin \text{Irr}(P)\}$. Then*

$$(3.5) \quad \frac{2m + |R|}{n} + n - 1 + \ln \frac{n|\det(C(P))|}{2m + |R|} \leq E(P).$$

Proof. As P is a non-singular poset, $|\lambda_i| > 0$ for each $1 \leq i \leq n$. Consider the function

$$f(x) = x - 1 - \ln(x) \quad \text{for } x > 0.$$

It is easy to show that $f(x)$ is decreasing for $x \in (0, 1]$ and increasing for $x \geq 1$. Hence, $f(x) \geq f(1) = 0$ implies that

$$(3.6) \quad x \geq 1 + \ln(x) \quad \text{for } x > 0.$$

Moreover, equality holds if and only if $x = 1$. Also, using the inequality (3.6), we have

$$(3.7) \quad E(P) = \lambda_1 + \sum_{i=2}^n |\lambda_i| \geq \lambda_1 + n - 1 + \sum_{i=2}^n \ln |\lambda_i| = \lambda_1 + n - 1 + \ln \prod_{i=2}^n |\lambda_i|.$$

Thus,

$$(3.8) \quad E(P) \geq \lambda_1 + n - 1 + \ln |\det(C(P))| - \ln(\lambda_1).$$

By Theorem 3.2, we have $\lambda_1 \geq (2m + |R|)/n$. The function $g(x) = x + n - 1 + \ln |\det(C(P))| - \ln x$ is increasing for $x \in [1, n]$ and we conclude that

$$(3.9) \quad g(\lambda_1) \geq \frac{2m + |R|}{n} + n - 1 + \ln |\det(C(P))| - \ln \frac{2m + |R|}{n}$$

for $x \geq 2m/n$. Combining (3.9) with (3.8), we arrive at (3.5). \square

Suppose that equality holds in (3.5). Then all the inequalities considered in the Theorem 3.3 must be equalities. From the equality (3.7), we obtain $|\lambda_2| = |\lambda_3| = \dots = |\lambda_n| = 1$. Since P is connected, the condition $|\lambda_i| = 1$, $i = 2, 3, \dots, n$, is satisfied if and only if $P = C_2$. This discussion leads to the following claim.

Remark 3.4. Equality holds in (3.5) if and only if P is the chain C_2 .

For crowns \mathbb{C}_k , $k = 4, 5, 7, 8$, one can numerically verify that (3.5) is superior to (2.1). But for $k > 9$, the situation is the opposite. (If $3|k$, then $\det(C(\mathbb{C}_k)) = 0$ and the inequality (3.5) is not applicable). That is, in general, (3.5) may not be superior to (2.1).

The next result gives a sufficient condition for (3.5) to be superior to (2.1).

Observation 3.5. Let Ω be the class of connected posets with n elements and m edges, and for which the following conditions are satisfied:

$$(3.10) \quad 16 \leq \frac{n}{2} \leq \frac{2m + |R|}{n} \leq |\det(C(P))| \leq n - 1.$$

Then for each poset $P \in \Omega$, the inequality (3.5) is better than the inequality (2.1).

Proof. As $P \in \Omega$, by the inequality (3.10), we have

$$\ln |\det(C(P))| - \ln \frac{2m + |R|}{n} = \ln \frac{n|\det(C(P))|}{2m + |R|} \geq 0 \quad \text{and} \quad n \geq 32,$$

which leads to $(n - 1)^{2/n} \leq \frac{5}{4}$. Thus, using Theorem 3.3, we obtain

$$(3.11) \quad E(P) \geq \frac{2m + |R|}{n} + n - 1 + \ln \frac{n|\det(C(P))|}{2m + |R|} \geq \frac{2m + |R|}{n} + n - 1.$$

Also,

$$(3.12) \quad 2m + |R| + n(n - 1)|\det(C(P))|^{2/n} \leq 2m + |R| + n(n - 1)(n - 1)^{2/n} \leq 2m + |R| + n(n - 1)\frac{5}{4}.$$

In the light of inequalities (3.11) and (3.12), to prove the result it is sufficient to show that

$$2m + |R| + n(n - 1)\frac{5}{4} \leq \left(\frac{2m + |R|}{n} + n - 1\right)^2.$$

That is, to show that,

$$2m + |R| + n(n - 1)\frac{5}{4} \leq \left(\frac{2m + |R|}{n} - 1\right)^2 + n^2 + 4m + 2|R| - 2n,$$

i.e.,

$$(3.13) \quad \frac{n^2 + 3n}{4} \leq \left(\frac{2m + |R|}{n} - 1\right)^2 + 2m + |R|.$$

As $(2m + |R|)/n \geq \frac{1}{2}n$, we have

$$(3.14) \quad \left(\frac{2m + |R|}{n} - 1\right)^2 + 2m + |R| \geq \left(\frac{n}{2} - 1\right)^2 + \frac{n^2}{2} = \frac{3n^2 - 4n - 4}{4}.$$

But for $n \geq 16$, the inequality

$$\frac{3n^2 - 4n - 4}{4} \geq \frac{n^2 + 3n}{4}$$

is always true. Thus, using (3.13) and (3.14), the proof is complete. \square

Now, we improve the McClelland type upper bound for the covering energy of a poset whose covering graph $G(P)$ contains no isolated vertex.

Theorem 3.6. *Let P be a poset with n elements and m edges. If $G(P)$ contains no isolated vertex and $R = \{v \in P: v \notin \text{Irr}(P)\}$, then*

$$(3.15) \quad E(P) \leq \frac{2m + |R|}{n} + \sqrt{(n-1)\left(2m + |R| - \left(\frac{2m + |R|}{n}\right)^2\right)}.$$

Proof. As $G(P)$ does not contain any isolated vertex, $2m \geq n$. Suppose $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$ are the eigenvalues of P , then by the inequality (3.1), we have

$$(3.16) \quad 1 \leq \frac{2m + |R|}{n} \leq \lambda_1.$$

Moreover, by Theorem 2.17 (2), we have

$$(3.17) \quad \sum_{i=2}^n \lambda_i^2 = 2m + |R| - \lambda_1^2.$$

Using (3.17) together with the Cauchy-Schwarz inequality applied to the vectors $(|\lambda_2|, |\lambda_3|, \dots, |\lambda_n|)$ and $(1, 1, \dots, 1)$ with $n-1$ entries, we obtain the inequality

$$(3.18) \quad \sum_{i=2}^n |\lambda_i| \leq \sqrt{(n-1)(2m + |R| - \lambda_1^2)}.$$

Thus, we have

$$(3.19) \quad E(P) \leq \lambda_1 + \sqrt{(n-1)(2m + |R| - \lambda_1^2)}.$$

Now, as the function $F(x) = x + \sqrt{(n-1)(2m + |R| - x^2)}$ is decreasing on the interval $(\sqrt{(2m + |R|)/n}, \sqrt{2m + |R|}]$, in view of the fact $2m + |R| \geq n$, we observe that $\sqrt{(2m + |R|)/n} \leq (2m + |R|)/n \leq \lambda_1$ holds. Hence,

$$(3.20) \quad F(\lambda_1) \leq F\left(\frac{2m + |R|}{n}\right) \leq F\left(\sqrt{\frac{2m + |R|}{n}}\right).$$

The inequalities (3.19) and (3.20) lead to (3.15). □

It is easy to verify that $F(\sqrt{(2m + |R|)/n}) = \sqrt{n(2m + |R|)}$. Hence in the light of the inequality (3.20), we observe that the bound given in Theorem 3.6 is an improved bound for the given class of posets. Also, if P is $\frac{1}{2}n$ copies of C_2 , then the eigenvalues for P are ± 1 (both with multiplicity $\frac{1}{2}n$) and $|R| = 0$. It is easy to check that for the poset P , equality holds in the relation (3.15).

Acknowledgment. The authors express their deep gratitude to the referee and communicator for their valuable suggestions and comments for the improvement of this work.

References

- [1] *C. Adiga, A. Bayad, I. Gutman, S. A. Srinivas*: The minimum covering energy of a graph. *Kragujevac J. Sci.* *34* (2012), 39–56.
- [2] *Š. B. Altındağ, D. Bozkurt*: Lower bounds for the energy of (bipartite) graphs. *MATCH Commun. Math. Comput. Chem.* *77* (2017), 9–14. [zbl](#) [MR](#)
- [3] *L. Collatz, U. Sinogowitz*: Spektren endlicher Grafen. *Abh. Math. Semin. Univ. Hamb.* *21* (1957), 63–77. (In German.) [zbl](#) [MR](#) [doi](#)
- [4] *D. M. Cvetković, M. Doob, H. Sachs*: Spectra of Graphs: Theory and Application. Academic Press, New York, 1980. [zbl](#) [MR](#)
- [5] *K. C. Das, S. A. Mojallal, I. Gutman*: Improving McClelland’s lower bounds for energy. *MATCH Commun. Math. Comput. Chem.* *70* (2013), 663–668. [zbl](#) [MR](#)
- [6] *B. A. Davey, H. A. Priestley*: Introduction to Lattices and Order. Cambridge University Press, Cambridge, 1990. [zbl](#) [MR](#) [doi](#)
- [7] *G. Grätzer*: General Lattice Theory. Pure and Applied Mathematics 75. Academic Press, New York, 1978. [zbl](#) [MR](#) [doi](#)
- [8] *I. Gutman*: The energy of a graph. *Ber. Math.-Stat. Sect. Forschungszent. Graz* *103* (1978), 22 pages. [zbl](#) [MR](#)
- [9] *I. Gutman, B. Furtula*: The total π -electron energy saga. *Croat. Chem. Acta* *90* (2017), 359–368. [doi](#)
- [10] *I. Gutman, B. Furtula*: Energies of Graphs: Survey, Census, Bibliography. Center for Scientific Research, Kragujevac, 2019.
- [11] *I. Gutman, H. S. Ramane*: Research on graph energies in 2019. *MATCH Commun. Math. Comput. Chem.* *84* (2020), 277–292.
- [12] *E. Hückel*: Quantentheoretische Beiträge zum Benzolproblem I. Die Elektronenkonfiguration des Benzols und verwandter Verbindungen. *Z. Phys.* *70* (1931), 204–286. (In German.) [zbl](#) [doi](#)
- [13] *G. Indulal, A. Vijayakumar*: A note on energy of some graphs. *MATCH Commun. Math. Comput. Chem.* *59* (2008), 269–274. [zbl](#) [MR](#)
- [14] *D. Kelly, I. Rival*: Crowns, fences, and dismantlable lattices. *Can. J. Math.* *27* (1974), 1257–1271. [zbl](#) [MR](#) [doi](#)
- [15] *X. Li, Y. Shi, I. Gutman*: Graph Energy. Springer, Berlin, 2012. [zbl](#) [MR](#) [doi](#)
- [16] *B. J. McClelland*: Properties of the latent roots of a matrix: The estimation of π -electron energies. *J. Chem. Phys.* *54* (1971), 640–643. [doi](#)
- [17] *M. M. Pawar, V. P. Bhamare*: On covering energy of posets. *Math. Sci. Int. Research J.* *4* (2015), 121–125.
- [18] *M. M. Pawar, V. P. Bhamre*: Covering energy of some classes of posets. *J. Indian Math. Soc., New Ser.* *87* (2020), 193–205. [zbl](#) [MR](#) [doi](#)
- [19] *M. M. Pawar, S. T. Bhangale*: Minimum independent dominating energy of graphs. *Asian-Eur. J. Math.* *14* (2021), Article 2150127, 14 pages. [zbl](#) [MR](#) [doi](#)
- [20] *I. Rival*: Lattices with doubly irreducible elements. *Can. Math. Bull.* *17* (1974), 91–95. [zbl](#) [MR](#) [doi](#)
- [21] *N. K. Thakare, M. M. Pawar, B. N. Waphare*: A structure theorem for dismantlable lattices and enumeration. *Period. Math. Hung.* *45* (2002), 147–160. [zbl](#) [MR](#) [doi](#)

Authors’ addresses: Vandana P. Bhamre, Department of Mathematics, Pratap College, Amalner, Maharashtra 425401 and Kaviyitri Bahinabai Choudhari North Maharashtra University, Jalgaon, India, e-mail: patilvandanapatil1915@yahoo.com; Madhukar M. Pawar (corresponding author), Department of Mathematics, S.S.V.P.S’s. Late Kr.Dr.P.R. Ghogrey Science College, Dhule, Maharashtra 424005, and Kaviyitri Bahinabai Choudhari North Maharashtra University, Jalgaon, India e-mail: m2pawar@yahoo.com, mm2pawar@gmail.com.