Mathematica Bohemica

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Mathematica Bohemica, Vol. 149 (2024), No. 1, 49–55

Persistent URL: http://dml.cz/dmlcz/152292

THE UNIT GROUP OF SOME FIELDS OF THE FORM $\mathbb{Q}(\sqrt{2}, \sqrt{p}, \sqrt{q}, \sqrt{-l})$

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Received June 3, 2022. Published online February 9, 2023.
Communicated by Clemens Fuchs

Abstract. Let p and q be two different prime integers such that $p \equiv q \equiv 3 \pmod 8$ with $(p/q) = 1$, and l a positive odd square-free integer relatively prime to p and q. In this paper we investigate the unit groups of number fields $\mathbb{L} = \mathbb{Q}(\sqrt{2}, \sqrt{p}, \sqrt{q}, \sqrt{-l})$.

Keywords: unit group; multiquadratic number fields; unit index

MSC 2020: 11R27, 11R04, 11R29

1. Introduction

Let k be a number field of degree n and let E_k denote the unit group of k that is the group of the invertible elements of the ring $\mathcal{O}_k$ of algebraic integers of the number field k. By Dirichlet's well known unit theorem, if $n = r_1 + 2r_2$, where r_1 is the number of real embeddings and r_2 the number of conjugate pairs of complex embeddings of k, then there exist $r = r_1 + r_2 - 1$ units of $\mathcal{O}_k$ that generate E_k (modulo the roots of unity), and these r units are called a *fundamental system of units of* k. Therefore

$$E_k \simeq \mu(k) \times \mathbb{Z}^{r_1+r_2-1},$$

where $\mu(k)$ is the group of roots of unity contained in k.

A major problem in algebraic number theory (and thus in the theory of units of number fields which is related to all areas of algebraic number theory) is the computation of a fundamental system of units. For quadratic fields, the problem is easily solved. For quartic bicyclic fields, Kubota (see [10]) gave a method for finding a fundamental system of units. Wada in [11] generalized Kubota's method, creating an algorithm for computing fundamental units in any given multiquadratic field. However, in general, it is not easy to compute the unit group of a number field especially

DOI: 10.21136/MB.2023.0077-22

for number fields of degree greater than 4. Very recently, Azizi, Chems-Eddin and Zekhnini used some very technical computations to determine the unit group of some number fields k of degree 16 (cf. [4]–[7], [9]). This paper is actually a continuation of these works. We determine 7 generators of the torsion-free subgroup of E_k for an infinite family of number fields k of degree 16 of the form $\mathbb{Q}(\sqrt{2}, \sqrt{p}, \sqrt{q}, \sqrt{-l})$, where $p \equiv q \equiv 3 \pmod 8$ are two different prime integers and l a positive odd square-free integer. We note that the computation of the unit group of these fields may be very important to deal with the problem of the 2-class field tower of biquadratic number fields (see, for example, [2]).

Let ε_m denote the fundamental unit of the quadratic field $\mathbb{Q}(\sqrt{m})$ and $(\cdot/\cdot)$ the Legendre symbol. Then the main theorem of this paper is the following.

Theorem 1.1. *Let $p \equiv q \equiv 3 \pmod 8$ be two different prime integers, l a positive odd square-free integer relatively prime to p and q, and $\mathbb{L} = \mathbb{Q}(\sqrt{2}, \sqrt{p}, \sqrt{q}, \sqrt{-l})$. Without loss of generality we may assume that $(p/q) = 1$. So we have:*

(1) *If $l = 1$, then a fundamental system of units of $\mathbb{L}$ is given by*

$$\Big\{\varepsilon_2, \sqrt{\varepsilon_p}, \sqrt{\varepsilon_q}, \sqrt{\varepsilon_{pq}}, \sqrt{\sqrt{\varepsilon_p}\sqrt{\varepsilon_q}\sqrt{\varepsilon_{2pq}}}, \sqrt{\sqrt{\varepsilon_{2p}}\sqrt{\varepsilon_{2q}}\sqrt{\varepsilon_{2pq}}}, \sqrt{\zeta_8\varepsilon_2\sqrt{\varepsilon_p}\sqrt{\varepsilon_{2p}}}\Big\},$$

where ζ_8 is a primitive 8th root of unity.

(2) *If $l \neq 1$, then a fundamental system of units of $\mathbb{L}$ is given by*

$$\Big\{\varepsilon_2, \sqrt{\varepsilon_p}, \sqrt{\varepsilon_{2p}}, \sqrt{\varepsilon_q}, \sqrt{\varepsilon_{pq}}, \sqrt{\sqrt{\varepsilon_p}\sqrt{\varepsilon_q}\sqrt{\varepsilon_{2pq}}}, \sqrt{\sqrt{\varepsilon_{2p}}\sqrt{\varepsilon_{2q}}\sqrt{\varepsilon_{2pq}}}\Big\}.$$

The proof of this theorem needs long and technical computations. Therefore, we will expose it in the third section of the paper.

2. Preliminary results

In this section we recall some results that will be useful in what follows.

Lemma 2.1. *Let K_0 be a real number field, $K = K_0(i)$ a quadratic extension of K_0, $n \geqslant 2$ an integer and ξ_n a primitive 2^nth root of unity, then $\xi_n = \frac{1}{2}(\mu_n + \mathrm{i}\lambda_n)$, where $\mu_n = \sqrt{2 + \mu_{n-1}}$, $\lambda_n = \sqrt{2 - \mu_{n-1}}$, $\mu_2 = 0$, $\lambda_2 = 2$ and $\mu_3 = \lambda_3 = \sqrt{2}$. Let n_0 be the greatest integer such that ξ_{n_0} is contained in K, $\{\varepsilon_1, \ldots, \varepsilon_r\}$ a fundamental system of units of K_0 and ε a unit of K_0 such that $(2 + \mu_{n_0})\varepsilon$ is a square in K_0 (if it exists). Then a fundamental system of units of K is one of the following systems:*
(1) *$\{\varepsilon_1, \ldots, \varepsilon_{r-1}, \sqrt{\xi_{n_0}\varepsilon}\}$ if ε exists, in this case $\varepsilon = \varepsilon_1^{j_1} \ldots \varepsilon_{r-1}^{j_{r-1}}\varepsilon_r$, where $j_i \in \{0, 1\}$.*
(2) *$\{\varepsilon_1, \ldots, \varepsilon_r\}$ otherwise.*

P r o o f. See [1], Proposition 2. □

Lemma 2.2. *Let $K_0/\mathbb{Q}$ be an abelian extension such that K_0 is real and β a positive square-free algebraic integer of K_0. Assume that $K = K_0(\sqrt{-\beta})$ is a quadratic extension of K_0, which is abelian over $\mathbb{Q}$. Assume furthermore that $\mathrm{i} = \sqrt{-1} \notin K$. Let $\{\varepsilon_1, \ldots, \varepsilon_r\}$ be a fundamental system of units of K_0. Without loss of generality we may suppose that the units ε_i are positive. Let ε be a unit of K_0 such that $\beta\varepsilon$ is a square in K_0 (if it exists). Then a fundamental system of units of K is one of the following systems:*

(1) $\{\varepsilon_1, \ldots, \varepsilon_{r-1}, \sqrt{-\varepsilon}\}$ *if ε exists, in this case $\varepsilon = \varepsilon_1^{j_1} \ldots \varepsilon_{r-1}^{j_{r-1}} \varepsilon_r$, where $j_i \in \{0, 1\}$.*

(2) $\{\varepsilon_1, \ldots, \varepsilon_r\}$ *otherwise.*

Proof. See [1], Proposition 3. □

Lemma 2.3. *Let $p \equiv q \equiv 3 \pmod 8$ be two primes such that $(p/q) = 1$.*

(1) *Let x and y be two integers such that $\varepsilon_{2pq} = x + y\sqrt{2pq}$. Then*
   - (a) *$x - 1$ is a square in $\mathbb{N}$,*
   - (b) *$\sqrt{2\varepsilon_{2pq}} = y_1 + y_2\sqrt{2pq}$ and $2 = -y_1^2 + 2pqy_2^2$ for some integers y_1 and y_2 satisfying $y = y_1 y_2$.*

(2) *There are two integers a and b such that $\varepsilon_{pq} = a + b\sqrt{pq}$ and we have*
   - (a) *$2p(a + 1)$ is a square in $\mathbb{N}$,*
   - (b) *b is even, $\sqrt{\varepsilon_{pq}} = b_1\sqrt{p} + b_2\sqrt{q}$ and $1 = pb_1^2 - qb_2^2$ for some integers b_1 and b_2 such that $b = 2b_1 b_2$.*

(3) *Let c and d be two integers such that $\varepsilon_{2q_i} = c + d\sqrt{2q_i}$. Then we have*
   - (a) *$c - 1$ is a square in $\mathbb{N}$,*
   - (b) *$\sqrt{2\varepsilon_{2q_i}} = d_1 + d_2\sqrt{2q_i}$ and $2 = -d_1^2 + 2q_i d_2^2$ for some integers d_1 and d_2 such that $d = d_1 d_2$.*

(4) *Let α and β be two integers such that $\varepsilon_{q_i} = \alpha + \beta\sqrt{q_i}$. Then we have*
   - (a) *$\alpha - 1$ is a square in $\mathbb{N}$,*
   - (b) *$\sqrt{2\varepsilon_{q_i}} = \beta_1 + \beta_2\sqrt{q_i}$ and $2 = -\beta_1^2 + q_i\beta_2^2$ for some integers β_1 and β_2 such that $\beta = \beta_1\beta_2$.*

Proof. See [8], Lemma 2.4. □

3. Proof of Theorem 1.1

Now we can prove Theorem 1.1. Let us prove the first statement.

(1) Without loss of generality we can suppose that $(p/q) = 1$. First we will need a fundamental system of units of $\mathbb{L}^+ = \mathbb{Q}(\sqrt{2}, \sqrt{p}, \sqrt{q})$ and then using Lemma 2.1 we deduce a fundamental system of units of $\mathbb{L}$.

Consider the following diagram of subfields of $\mathbb{L}^+/\mathbb{Q}(\sqrt{2})$.

$$\begin{array}{ccccc} & & \mathbb{L}^+ = \mathbb{Q}(\sqrt{2}, \sqrt{p}, \sqrt{q}) & & \\ & \nearrow & \uparrow & \nwarrow & \\ L_1 = \mathbb{Q}(\sqrt{2}, \sqrt{p}) & & L_2 = \mathbb{Q}(\sqrt{2}, \sqrt{q}) & & L_3 = \mathbb{Q}(\sqrt{2}, \sqrt{pq}) \\ & \nwarrow & \uparrow & \nearrow & \\ & & \mathbb{Q}(\sqrt{2}) & & \end{array}$$

Put $\mathrm{Gal}(\mathbb{L}^+/\mathbb{Q}) = \langle \sigma_1, \sigma_2, \sigma_3 \rangle$, where

$$\begin{array}{lll} \sigma_1(\sqrt{2}) = -\sqrt{2}, & \sigma_1(\sqrt{p}) = \sqrt{p}, & \sigma_1(\sqrt{q}) = \sqrt{q}, \\ \sigma_2(\sqrt{2}) = \sqrt{2}, & \sigma_2(\sqrt{p}) = -\sqrt{p}, & \sigma_2(\sqrt{q}) = \sqrt{q}, \\ \sigma_3(\sqrt{2}) = \sqrt{2}, & \sigma_3(\sqrt{p}) = \sqrt{p}, & \sigma_3(\sqrt{q}) = -\sqrt{q}. \end{array}$$

By [8], Proposition 2.7, we have

$$E_{\mathbb{L}^+} = \Big\langle -1, \varepsilon_2, \sqrt{\varepsilon_p}, \sqrt{\varepsilon_{2p}}, \sqrt{\varepsilon_q}, \sqrt{\varepsilon_{pq}}, \sqrt{\sqrt{\varepsilon_p}\sqrt{\varepsilon_q}\sqrt{\varepsilon_{2pq}}}, \sqrt{\sqrt{\varepsilon_{2p}}\sqrt{\varepsilon_{2q}}\sqrt{\varepsilon_{2pq}}} \Big\rangle.$$

Put

$$\xi^2 = (2+\sqrt{2})\varepsilon_2^a \sqrt{\varepsilon_p}^b \sqrt{\varepsilon_{2p}}^c \sqrt{\varepsilon_q}^d \sqrt{\varepsilon_{pq}}^e \sqrt[4]{\varepsilon_p\varepsilon_q\varepsilon_{2pq}}^f \sqrt[4]{\varepsilon_{2p}\varepsilon_{2q}\varepsilon_{2pq}}^g$$

with $a, b, c, d, e, f, g \in \{0, 1\}$ (see also [3], Theorem 3.14). We use norm maps from $\mathbb{L}^+$ to its biquadratic subextensions. The computations of these norms are summarized in the following table (see Table 1). Note that the third line of Table 1 is constructed as follows (we similarly construct the rest of the table) By Lemma 2.3, we have $\sqrt{\varepsilon_p} = \frac{1}{\sqrt{2}}(\beta_1 + \beta_2\sqrt{p})$ and $2 = -\beta_1^2 + p\beta_2^2$. Thus

$$\begin{aligned}
\sqrt{\varepsilon_p}^{\sigma_1} &= \frac{1}{-\sqrt{2}}(\beta_1 + \beta_2\sqrt{p}) = -\sqrt{\varepsilon_p}, \\
\sqrt{\varepsilon_p}^{\sigma_2} &= \frac{1}{\sqrt{2}}(\beta_1 - \beta_2\sqrt{p}) = \frac{1}{\sqrt{2}}\frac{(\beta_1 - \beta_2\sqrt{p})(\beta_1 + \beta_2\sqrt{p})}{\beta_1 + \beta_2\sqrt{p}} \\
&= \frac{1}{\sqrt{2}}\frac{(\beta_1^2 - \beta_2^2 p)}{\sqrt{2}\sqrt{\varepsilon_p}} = \frac{1}{2}\frac{-2}{\sqrt{\varepsilon_p}} = \frac{-1}{\sqrt{\varepsilon_p}}, \\
\sqrt{\varepsilon_p}^{\sigma_3} &= \frac{1}{\sqrt{2}}(\beta_1 + \beta_2\sqrt{p}) = \sqrt{\varepsilon_p}, \\
\sqrt{\varepsilon_p}^{1+\sigma_1} &= \sqrt{\varepsilon_p}\sigma_1(\sqrt{\varepsilon_p}) = \sqrt{\varepsilon_p}(-\sqrt{\varepsilon_p}) = -\varepsilon_p, \\
\sqrt{\varepsilon_p}^{1+\sigma_2} &= \sqrt{\varepsilon_p}\sigma_2(\sqrt{\varepsilon_p}) = \sqrt{\varepsilon_p}\Big(\frac{-1}{\sqrt{\varepsilon_p}}\Big) = -1, \\
\sqrt{\varepsilon_p}^{1+\sigma_1\sigma_3} &= \sqrt{\varepsilon_p}\sigma_1(\sigma_3(\sqrt{\varepsilon_p})) = \sqrt{\varepsilon_p}\sigma_1(\sqrt{\varepsilon_p}) = \sqrt{\varepsilon_p}(-\sqrt{\varepsilon_p}) = -\varepsilon_p, \\
\sqrt{\varepsilon_p}^{1+\sigma_2\sigma_3} &= \sqrt{\varepsilon_p}\sigma_2(\sigma_3(\sqrt{\varepsilon_p})) = \sqrt{\varepsilon_p}\sigma_2(\sqrt{\varepsilon_p}) = \sqrt{\varepsilon_p}\Big(\frac{-1}{\sqrt{\varepsilon_p}}\Big) = -1.
\end{aligned}$$

ε	ε^{σ_1}	ε^{σ_2}	ε^{σ_3}	$\varepsilon^{1+\sigma_1}$	$\varepsilon^{1+\sigma_2}$	$\varepsilon^{1+\sigma_1\sigma_3}$	$\varepsilon^{1+\sigma_2\sigma_3}$
ε_2	$\frac{-1}{\sqrt{\varepsilon_2}}$	ε_2	ε_2	-1	ε_2^2	-1	ε_2^2
$\sqrt{\varepsilon_p}$	$-\sqrt{\varepsilon_p}$	$\frac{-1}{\sqrt{\varepsilon_p}}$	$\sqrt{\varepsilon_p}$	$-\varepsilon_p$	-1	$-\varepsilon_p$	-1
$\sqrt{\varepsilon_{2p}}$	$\frac{1}{\sqrt{\varepsilon_{2p}}}$	$\frac{-1}{\sqrt{\varepsilon_{2p}}}$	$\sqrt{\varepsilon_{2p}}$	1	-1	1	-1
$\sqrt{\varepsilon_q}$	$-\sqrt{\varepsilon_q}$	$\sqrt{\varepsilon_q}$	$\frac{-1}{\sqrt{\varepsilon_q}}$	$-\varepsilon_q$	ε_q	1	-1
$\sqrt{\varepsilon_{2q}}$	$\frac{1}{\sqrt{\varepsilon_{2q}}}$	$\sqrt{\varepsilon_{2q}}$	$\frac{-1}{\sqrt{\varepsilon_{2q}}}$	1	ε_{2q}	$-\varepsilon_{2q}$	-1
$\sqrt{\varepsilon_{pq}}$	$\sqrt{\varepsilon_{pq}}$	$\frac{-1}{\sqrt{\varepsilon_{pq}}}$	$\frac{1}{\sqrt{\varepsilon_{pq}}}$	ε_{pq}	-1	1	$-\varepsilon_{pq}$
$\sqrt{\varepsilon_{2pq}}$	$\frac{1}{\sqrt{\varepsilon_{2pq}}}$	$\frac{-1}{\sqrt{\varepsilon_{2pq}}}$	$\frac{-1}{\sqrt{\varepsilon_{2pq}}}$	1	-1	-ε_{2pq}	ε_{2pq}

Table 1. Norms in $\mathbb{L}^+/\mathbb{Q}(\sqrt{2})$.

Let us eliminate some forms of ξ^2 such that ξ cannot be in $\mathbb{L}$. Considering $L_4 = \mathbb{Q}(\sqrt{p}, \sqrt{q})$, we apply the norm $N_{\mathbb{L}/L_4} = 1 + \sigma_1$,

$$\begin{aligned} N_{\mathbb{L}/L_4}(\xi^2) &= 2(-1)^a(-1)^b\varepsilon_p^b 1(-1)^d(\varepsilon_q)^d\varepsilon_{pq}^e(-1)^{uf}\sqrt{\varepsilon_p}^f\sqrt{\varepsilon_q}^f(-1)^{gv} \\ &= (-1)^{a+b+d+uf+gv}2\varepsilon_p^b\varepsilon_q^d\varepsilon_{pq}^e\sqrt{\varepsilon_p}^f\sqrt{\varepsilon_q}^f. \end{aligned}$$

Therefore, $a+b+d+uf+gv \equiv 0 \pmod 2$. One can easily deduce that $f = 0$. Thus $a+b+d+gv \equiv 0 \pmod 2$ and

$$\xi^2 = (2+\sqrt{2})\varepsilon_2^a\sqrt{\varepsilon_p}^b\sqrt{\varepsilon_{2p}}^c\sqrt{\varepsilon_q}^d\sqrt{\varepsilon_{pq}}^e\sqrt[4]{\varepsilon_{2p}\varepsilon_{2q}\varepsilon_{2pq}}^g.$$

Now we apply the norm $N_{\mathbb{L}/L_3} = 1 + \sigma_2\sigma_3$, where $L_3 = \mathbb{Q}(\sqrt{2}, \sqrt{pq})$. We have

$$\begin{aligned} N_{\mathbb{L}/L_3}(\xi^2) &= (2+\sqrt{2})^2\varepsilon_2^{2a}(-1)^b(-1)^c\ (-1)^d(-1)^e\varepsilon_{pq}^e(-1)^{tg}\sqrt{\varepsilon_{2pq}}^g \\ &= (2+\sqrt{2})^2\varepsilon_2^{2a}(-1)^{b+c+d+e+tg}\varepsilon_{pq}^e\sqrt{\varepsilon_{2pq}}^g. \end{aligned}$$

Using Lemma 2.3, it is easy to deduce that $e = g = 0$. Thus $b+c+d \equiv 0 \pmod 2$ and $a+b+d \equiv 0 \pmod 2$. It follows that $a = c$ and

$$\xi^2 = (2+\sqrt{2})\varepsilon_2^a\sqrt{\varepsilon_p}^b\sqrt{\varepsilon_{2p}}^a\sqrt{\varepsilon_q}^d.$$

Let us apply $N_{\mathbb{L}/L_5} = 1 + \sigma_1\sigma_3$ with $L_5 = \mathbb{Q}(\sqrt{p}, \sqrt{2q})$. We have

$$N_{\mathbb{L}/L_3}(\xi^2) = 2(-1)^a(-1)^b\varepsilon_p^b 11 = (-1)^{a+b}2\varepsilon_p^b.$$

So $a+b\equiv 0 \pmod 2$. Since 2 is not a square in L_5, then using Lemma 2.3, one easily deduces that $b=1$ and so $a=1$. Since $a+b+d\equiv 0 \pmod 2$, then $d=0$. Therefore,

$$\xi^2=(2+\sqrt{2})\varepsilon_2\sqrt{\varepsilon_p}\sqrt{\varepsilon_{2p}}.$$

Since Hasse's unit index $Q_{\mathbb{L}}$ equals 2 (cf. the proof of the main theorem of [8]), then by Lemma 2.1, $(2+\sqrt{2})\varepsilon_2\sqrt{\varepsilon_p}\sqrt{\varepsilon_{2p}}$ is a square and therefore the first statement holds.

(2) For the proof of the second statement we similarly put

$$\xi^2=l\varepsilon_2^a\sqrt{\varepsilon_p}^b\sqrt{\varepsilon_{2p}}^c\sqrt{\varepsilon_q}^d\sqrt{\varepsilon_{pq}}^e\sqrt[4]{\varepsilon_p\varepsilon_q\varepsilon_{2pq}}^f\sqrt[4]{\varepsilon_{2p}\varepsilon_{2q}\varepsilon_{2pq}}^g$$

with $a,b,c,d,e,f\in\{0,1\}$. We proceed as above to eliminate all forms of ξ^2 and we deduce the result by using Lemma 2.2.

Let us eliminate some forms of ξ^2 such that ξ cannot be in $\mathbb{L}$. Considering $L_4=\mathbb{Q}(\sqrt{p},\sqrt{q})$, we apply the norm $N_{\mathbb{L}/L_4}=1+\sigma_1$,

$$\begin{aligned}N_{\mathbb{L}/L_4}(\xi^2)&=l^2(-1)^a(-1)^b\varepsilon_p^b1(-1)^d(\varepsilon_q)^d\varepsilon_{pq}^e(-1)^{uf}\sqrt{\varepsilon_p}^f\sqrt{\varepsilon_q}^f(-1)^{gv}\\&=l^2(-1)^{a+b+d+uf+gv}\varepsilon_p^b\varepsilon_q^d\varepsilon_{pq}^e\sqrt{\varepsilon_p}^f\sqrt{\varepsilon_q}^f.\end{aligned}$$

Therefore, $a+b+d+uf+gv\equiv 0 \pmod 2$. One can easily deduce that $f=0$. Thus $a+b+d+gv\equiv 0 \pmod 2$ and

$$\xi^2=l\varepsilon_2^a\sqrt{\varepsilon_p}^b\sqrt{\varepsilon_{2p}}^c\sqrt{\varepsilon_q}^d\sqrt{\varepsilon_{pq}}^e\sqrt[4]{\varepsilon_{2p}\varepsilon_{2q}\varepsilon_{2pq}}^g.$$

Now we apply the norm $N_{\mathbb{L}/L_3}=1+\sigma_2\sigma_3$, where $L_3=\mathbb{Q}(\sqrt{2},\sqrt{pq})$. We have

$$\begin{aligned}N_{\mathbb{L}/L_3}(\xi^2)&=l^2\varepsilon_2^{2a}(-1)^b(-1)^c\ (-1)^d(-1)^e\varepsilon_{pq}^e(-1)^{tg}\sqrt{\varepsilon_{2pq}}^g\\&=l^2\varepsilon_2^{2a}(-1)^{b+c+d+e+tg}\varepsilon_{pq}^e\sqrt{\varepsilon_{2pq}}^g.\end{aligned}$$

Using Lemma 2.3, it is easy to deduce that $e=g=0$. Thus $b+c+d\equiv 0 \pmod 2$ and $a+b+d\equiv 0 \pmod 2$. It follows that $a=c$ and

$$\xi^2=l\varepsilon_2^a\sqrt{\varepsilon_p}^b\sqrt{\varepsilon_{2p}}^a\sqrt{\varepsilon_q}^d.$$

Let us apply $N_{\mathbb{L}/L_5}=1+\sigma_1\sigma_3$ with $L_5=\mathbb{Q}(\sqrt{p},\sqrt{2q})$. We have

$$N_{\mathbb{L}/L_3}(\xi^2)=l(-1)^a(-1)^b\varepsilon_p^b11=l(-1)^{a+b}\varepsilon_p^b.$$

Therefore, $a+b\equiv 0 \pmod 2$ and by Lemma 2.3, it is clear that $b=0$. Thus, $a=0$. Since $a+b+d\equiv 0 \pmod 2$, this implies that $d=0$. Hence Lemma 2.2 gives the second statement of Theorem 1.1. □

A c k n o w l e d g m e n t. The author would like to thank the referee for several advices and helpful suggestions that helped improve this article, and for his calling attention to the missing details.

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