# II. GROUPOIDS

## 11. Multiplication in sets

### 11.1. Basic concepts

By a *multiplication* or a *binary operation in the set* $G$ we mean a relationship between the elements of $G$ by which there corresponds, to every two-membered sequence of the elements $a, b \in G$, exactly one element $c \in G$; in other words, a relationship by which every two-membered sequence of the elements $a, b$ of $G$ is mapped onto an element $c$ of the same set $G$. The element $c$ is called the *product of* $a$ *and* $b$ and is denoted by $a \,.\, b$ or $ab$; so we have $c = ab$, where $a, b$ is the *first, second factor of the product* $c$, respectively.

From these definitions it is obvious that the word "multiplication" only expresses a relationship between the elements of $G$ which need not have, in actual cases, anything in common with arithmetic multiplication; the same applies to the product and the symbols $a \,.\, b$, $ab$.

In what sense the concept of multiplication in $G$ generalizes that of a mapping of $G$ into itself can easily be understood by comparing the two definitions: Every mapping of $G$ into itself associates, with each element of $G$, again an element of $G$; every multiplication in $G$ associates, with every two-membered sequence of elements of $G$, again an element of $G$.

It is obvious that a multiplication in $G$ may also be defined as a mapping of the Cartesian square $G \times G$ (1.8) into $G$. Then the products are images of the individual elements of this Cartesian square. The theory of groupoids which, as we shall see, is based on the concept of multiplication in a set can, in this way, be included in the general theory of mappings of sets. The following considerations are, however, based on the above concept of multiplication, since the theory of groupoids, developed in this way, without a detailed study of the properties of Cartesian squares, is simpler and better suited to our purpose. The reader is, nevertheless, advised to follow our study even from the aspect of mappings of sets; it will help him to gain an independent view of the single situations.

If the multiplication in $G$ is given, then, in particular, the product of each element $a \in G$ and $a$ itself is uniquely determined; instead of $aa$ we sometimes write, briefly, $a^2$.

## 11.2. Commutative (Abelian) multiplication

A multiplication in $G$ may have particular properties. The above definition does not exclude, e.g., that a multiplication associates, with two inversely arranged pairs of elements of $G$, two different elements; thus it may happen that the product of some elements $a$ and $b$ is different from the product of the elements $b$ and $a$, i.e., $ab \neq ba$.

If, for two elements $a, b \in G$, there holds $ab = ba$, then $a, b$ are called *interchangeable*; if every two elements of $G$ are interchangeable, then the multiplication is called *commutative* or *Abelian*.

Multiplication in a set may, of course, have other remarkable properties. In the following paragraph we shall give examples of multiplications that will often be referred to later.

## 11.3. Examples of multiplication in a set

a) Let $G$ be the set of all integers and let the multiplication be defined as follows: The product of an element $a \in G$ and an element $b \in G$ is the number $a + b$. The multiplication is, in this case, addition in the usual sense. From the equality $a + b = b + a$, true for every two elements $a, b \in G$, there follows that it is an Abelian multiplication.

b) Let $n$ be an arbitrary positive integer and $G$ a set consisting of non-negative integers and containing all the numbers $0, \ldots, n - 1$. The multiplication in $G$ is defined as follows: The product $ab$ of an element $a \in G$ and an element $b \in G$ is the remainder of the division of $a + b$ by $n$. The product $ab$ is therefore always one of the numbers $0, \ldots, n - 1$. This multiplication is called *addition modulo n*; evidently, it is also Abelian.

c) Assuming $G$ to be the set of all the permutations of a finite set of order $n$ ($\geqq 1$), let the multiplication be defined as follows: The product $\boldsymbol{p} \, . \, \boldsymbol{q}$ of two arbitrary elements $\boldsymbol{p} \in G$ and $\boldsymbol{q} \in G$ is the composite permutation $\boldsymbol{qp}$. Hence the multiplication is the composition of permutations. We know, from 8.7.2, that it need not be Abelian.

d) Suppose $G$ is the set of all decompositions of a certain set and let the multiplication be defined as follows: The product of two arbitrary elements $A \in G$ and $B \in G$ is the decomposition $[A, B]$ or $(A, B)$, respectively. By 3.4 and 3.5, both the multiplications are Abelian.

## 11.4. Multiplication tables

1. *Description of a multiplication table.* If the set $G$ is finite and consists, for example, of the elements $a, b, \ldots, m$, then any multiplication in $G$ may be described by means of a *multiplication table* constructed as follows:

The first row and the first column, usually separated from the others by a horizontal and a vertical line, contains all the letters $a, b, \ldots, m$, generally in the same order: in the first row from left to right, in the first column from top to bottom. On the right-hand side of each letter $x$ in the first column, under the single letters $a$, $b, \ldots, m$ in the first row, there are the letters denoting the single products $xa, xb, \ldots$, $xm$.

The first row and the first column are the headings of the table. Every multiplication table contains, moreover, exactly as many rows and columns as the number of elements of $G$. If the letters $a, b, \ldots, m$ in both headings are written in the same order, then an Abelian multiplication is apparent from the symmetry of the table with regard to the main diagonal; that is to say, in any $j$-th row and any $k$-th column beyond both headings there is the same element as in the $k$-th row and the $j$-th column.

2. *Examples of multiplication tables.* Let us introduce, e.g., tables for the multiplication in $G$ of all the permutations of a set $H$ consisting of $n = 1, 2, 3$ elements, the multiplication being the composition of permutations described in 11.3c. Since the number of all the permutations in $H$ and, therefore, of all the elements of $G$ is $n! = 1, 2, 6$, these multiplication tables contain, besides the two headings, $n! = 1, 2, 6$ rows and the same number of columns.

For $n = 1$. The set $G$ consists of the identical permutation $\boldsymbol{e}$. If the unique element of $H$ is denoted by the letter $a$, then the symbol of the permutation is $\begin{pmatrix} a \\ a \end{pmatrix}$ and the multiplication table is:

$$\begin{array}{c|c} & \boldsymbol{e} \\ \hline \boldsymbol{e} & \boldsymbol{e} \end{array}$$

For $n = 2$. The set $G$ consists of two permutations. If the elements of the set $H$ are $a$ and $b$, then the symbols of the permutations are $\begin{pmatrix} a\,b \\ a\,b \end{pmatrix}$, $\begin{pmatrix} a\,b \\ b\,a \end{pmatrix}$. The former permutation is the identical permutation $\boldsymbol{e}$, the latter is denoted, e.g., by $\boldsymbol{a}$. The composite permutations are: $\boldsymbol{ee} = \boldsymbol{e}$, $\boldsymbol{ae} = \boldsymbol{a}$, $\boldsymbol{ea} = \boldsymbol{a}$, $\boldsymbol{aa} = \boldsymbol{e}$, whence we have the following multiplication table:

$$\begin{array}{c|cc} & \boldsymbol{e} & \boldsymbol{a} \\ \hline \boldsymbol{e} & \boldsymbol{e} & \boldsymbol{a} \\ \boldsymbol{a} & \boldsymbol{a} & \boldsymbol{e} \end{array}$$

For $n = 3$. The set $G$ consists of six permutations. If the elements of $H$ are $a$, $b$, $c$, then the symbols of the permutations are:

$$\begin{pmatrix} a & b & c \\ a & b & c \end{pmatrix}, \begin{pmatrix} a & b & c \\ b & c & a \end{pmatrix}, \begin{pmatrix} a & b & c \\ c & a & b \end{pmatrix}, \begin{pmatrix} a & b & c \\ a & c & b \end{pmatrix}, \begin{pmatrix} a & b & c \\ c & b & a \end{pmatrix}, \begin{pmatrix} a & b & c \\ b & a & c \end{pmatrix}.$$

The first symbol expresses the identical permutation $\boldsymbol{e}$, the others are denoted by $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}, \boldsymbol{f}$, respectively. The composite permutations are:

$$\begin{array}{llllll}
ee = e, & ae = a, & be = b, & ce = c, & de = d, & fe = f, \\
ea = a, & aa = b, & ba = e, & ca = d, & da = f, & fa = c, \\
eb = b, & ab = e, & bb = a, & cb = f, & db = c, & fb = d, \\
ec = c, & ac = f, & bc = d, & cc = e, & dc = b, & fc = a, \\
ed = d, & ad = c, & bd = f, & cd = a, & dd = e, & fd = b, \\
ef = f, & af = d, & bf = c, & cf = b, & df = a, & ff = e,
\end{array}$$

and we have the following multiplication table:

| | $e$ | $a$ | $b$ | $c$ | $d$ | $f$ |
|---|---|---|---|---|---|---|
| $e$ | $e$ | $a$ | $b$ | $c$ | $d$ | $f$ |
| $a$ | $a$ | $b$ | $e$ | $d$ | $f$ | $c$ |
| $b$ | $b$ | $e$ | $a$ | $f$ | $c$ | $d$ |
| $c$ | $c$ | $f$ | $d$ | $e$ | $b$ | $a$ |
| $d$ | $d$ | $c$ | $f$ | $a$ | $e$ | $b$ |
| $f$ | $f$ | $d$ | $c$ | $b$ | $a$ | $e$ |

All the above multiplication tables contain, in both headings, the symbols $\boldsymbol{e}, \boldsymbol{a}, \ldots, \boldsymbol{f}$ of the individual elements of $G$ in the same order and we observe that, for $n = 1, 2$, the tables are symmetric with regard to the main diagonal, whereas, for $n = 3$, the table is not symmetric. Consequently, the above multiplication in $G$ is Abelian for $n = 1, 2$, whereas, for $n = 3$, it is not.

We could give as many examples of multiplication in sets as we wished just by taking an arbitrary abstract nonempty set $G$ and uniquely associating, with every two-membered sequence of elements $a, b \in G$, an arbitrarily chosen element of $G$. If $G$ is finite, then the correspondence may be defined in a table where the symbols of the chosen elements are written in the single places under the horizontal and to the right of the vertical headings. Each choice of these elements determines a certain multiplication to which the resulting multiplication table applies.

## 11.5. Exercises

1. In the set of all Euclidian motions on an straight line, $\boldsymbol{f}[a]$, as well as in the set of all Euclidean motions on a straight line, $\boldsymbol{f}[a]$, $\boldsymbol{g}[a]$, (6.10.4), the multiplication may be defined by means of composing the motions in a similar way as in Example 11.3c). An analogous result applies to the set of all Euclidean motions in a plane, $\boldsymbol{f}[\alpha; a, b]$, and to the set of all Euclidean motions in a plane, $\boldsymbol{f}[\alpha; a, b]$, $\boldsymbol{g}[\alpha; a, b]$ (6.10.5).
2. In the set of $2n$ permutations of the vertices of a regular $n$-gon in a plane ($n \geqq 3$), described in Exercise 8.8.4, the multiplication may be defined by composing the permutations similarly as in Example 11.3c). Construct the appropriate multiplication tables for $n = 4, 5, 6$.

3. In **11.3** b, the set $G$ may consist only of the numbers $0, \ldots, n-1$. Construct the appropriate multiplication tables if $n = 1, 2, 3, 4, 5$.
4. If the positive integers $a, b$ are less than or equal to a positive integer $n \geqq 5$, then the number of the prime factors of the number $10a + b$ is $\leqq n$. Hence a multiplication in the set $G$, consisting of the numbers $1, 2, \ldots, n$, can be defined as follows: The product $a \,.\, b$ of an element $a \in G$ and an element $b \in G$ is the number of the prime factors of $10a + b$. The reader may verify that, for $n = 6$, the corresponding table is

| | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| 1 | 1 | 3 | 1 | 2 | 2 | 4 |
| 2 | 2 | 2 | 1 | 4 | 2 | 2 |
| 3 | 1 | 5 | 2 | 2 | 2 | 4 |
| 4 | 1 | 3 | 1 | 3 | 3 | 2 |
| 5 | 2 | 3 | 1 | 4 | 2 | 4 |
| 6 | 1 | 2 | 3 | 6 | 2 | 3 |

5. In the system of all the subsets of a nonempty set the multiplication can be defined by associating, with each ordered pair of subsets, their sum. May the multiplication be similarly defined by means of intersection?
6. Find some other examples of multiplication in sets.

## 12. Basic notions relative to groupoids

### 12.1. Definition

A nonempty set $G$ together with a multiplication $\boldsymbol{M}$ in $G$ is called a *groupoid*. $G$ is the *field* and $\boldsymbol{M}$ the *multiplication of* or *in the groupoid*. The groupoids will generally be denoted by German capitals corresponding to the Latin capitals used for their fields. Thus, for a groupoid whose field is denoted by $G$, we use the notation $\mathfrak{G}$; if a groupoid is denoted by $\mathfrak{G}$, then $G$ generally stands for its field.

### 12.2. Further notions. The groupoids $\mathfrak{Z}$, $\mathfrak{Z}_n$, $\mathfrak{S}_n$

To groupoids we may apply the notions and symbols defined for their fields. So we speak, for example, about elements of a groupoid instead of elements of the field of a groupoid and write $a \in \mathfrak{G}$ instead of $a \in G$; we speak about subsets of a groupoid and write, e.g., $A \subset \mathfrak{G}$ or $\mathfrak{G} \supset A$, we speak about decompositions in a groupoid and on a groupoid, about the order of a groupoid, a mapping of a group-