ON DOMAINS OF MONOGENICITY IN CLIFFORD ANALYSIS

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1. Introduction

One of the basic problems encountered when passing from holomorphic functions of a single complex variable to holomorphic functions of several complex variables is caused by the fact that not any domain in C^n (n>1) is a domain of holomorphy, as it is the case in the complex plane. As is well known a classic counterexample is provided by the Hartogs Extension Theorem stating that any function which is holomorphic in $\Omega\setminus K$, where $\Omega\subset C^n$ is open, K is compact and $\Omega\setminus K$ is connected, may be extended to a holomorphic function in Ω.

Let us recall a classical characterization of domains of holomorphy in C^n.

<u>Theorem</u>. If Ω is a domain in C^n then the following conditions are equivalent:

(i) Ω is a domain of holomorphy;

(ii) Ω is holomorphically convex, i.e. for each compact subset $K\subset\Omega$ its holomorphic hull $\hat{K}_\Omega=\{z\in\Omega: |f(z)|\leq\sup_{u\in K}|f(u)|$, for all $f\in\mathcal{O}(\Omega)\}$ is again compact;

(iii) there exists a function $f\in\mathcal{O}(\Omega)$ which cannot be continued holomorphically beyond Ω, i.e. Ω is a holomorphic existence domain.

The aim of this paper is to investigate if, such as in the complex plane, any domain in R^{m+1} (m>1) satisfies all of the three conditions mentioned above, with respect to the monogenic functions. It can be shown in a straightforward manner that any domain Ω in R^{m+1} is a domain of monogenicity (§2) and that it is moreover mono-

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genically convex (§3); but it is not known yet if for each domain $\Omega\subset R^{m+1}$ there exists a monogenic function in Ω which cannot be extended monogenically beyond Ω. Nevertheless in section 4 some sufficient conditions are given for a domain $\Omega\subset R^{m+1}$ to be a monogenic existence domain.

For the definitions and properties concerning the monogenic functions the reader is referred to [3].

2. Domains of monogenicity.

DEFINITIONS 2.1. Let Ω be a domain in R^{m+1}.

(i) Ω is called a weak domain of monogenicity if for each domain $\Omega'\supsetneq\Omega$ there exists a monogenic function in Ω which is not the restriction to Ω of a monogenic function in Ω';

(ii) Ω is called a domain of monogenicity if it is impossible to find two domains U_1 and U_2 satisfying the following two conditions :

(a) $\phi\neq U_2\subset\Omega\cap U_1\subsetneq U_1$;

(b) for each monogenic function f in Ω there exists a monogenic function $\hat{f}$ in U_1 such that $f=\hat{f}$ on U_2.

REMARK. Clearly any domain of monogenicity is also a weak domain of monogenicity. The fact that both notions coincide is a consequence of the following theorem.

THEOREM 2.2. Every domain in R^{m+1} is a domain of monogenicity.

Proof. Let $\Omega\subset R^{m+1}$ be a domain which is not a domain of monogenicity. Then there exist two domains U_1 and U_2 satisfying the conditions (a) and (b) of Definition 2.1.(ii). Call U_2' the component of $U_1\cap\Omega$ which contains U_2 and let the points $a\in U_2$ and $b\in U_1\setminus\Omega$ be joint by means of a polygonal line Γ in U_1. Take $c\in\Gamma\cap\partial\Omega$. The function

$$g(x)=\frac{\overline{x}-\overline{c}}{|x-c|^{m+1}}$$

is monogenic in Ω , so, by the hypothesis made, there exists a function $\hat{g}$, monogenic in U_1, for which $g=\hat{g}$ on U_2 and also on U_2' by analytic continuation. Now, as $\hat{g}$ is monogenic in U_1, we have $\lim_{\substack{x\to c\\ x\in\Gamma}}\hat{g}(x)=\hat{g}(c)$, while $\lim_{\substack{x\to c\\ x\in\Gamma\cap\Omega}}|g(x)|_0=+\infty$, clearly a contradiction. ∎

REMARK. The proof of the above theorem depends heavily upon the existence of pointwise singularities. As in the more general two Clifford-variable theory of the biregular functions (see e.g.[1]) pointwise singularities do not occur anymore, it is expected that, in analogy with complex analysis, the study of the so-called domains of biregularity will be far from trivial (see [2]).

3. Monogenic convexity

DEFINITION 3.1. Let Ω be a domain in R^{m+1}, let K be a compact subset of Ω and let F be a family of left-monogenic functions containing the functions $\xi_i=x_i e_0-x_0 e_i$, $i=1,\dots,m$. The F-convex hull of K is the set $\hat{K}_F$ given by

$$\hat{K}_F=\{x\in\Omega: |f(x)|_0\leq \sup_{u\in K}|f(u)|_0, \text{ for all } f\in F\} .$$

In the particular case where $F=M(\Omega)$, i.e. the whole family of all left monogenic functions in Ω, $\hat{K}_F$ is called the monogenic hull of K and denoted by $\hat{K}_\Omega$.

The following properties of the F-convex hull are inmediate.

PROPOSITION 3.2. Let Ω,K and F be as in Definition 3.1. Then
(i) $\hat{K}_F$ is relatively closed in Ω;
(ii) $K\subset\hat{K}_F$ and $\sup_{x\in K}|f(x)|_0=\sup_{x\in\hat{K}_F}|f(x)|_0$ for all $f\in F$;
(iii) if $F_1\subset F_2$ then $\hat{K}_{F_2}\subset\hat{K}_{F_1}$.

DEFINITION 3.3. A domain $\Omega\subset R^{m+1}$ is called F-convex if for each $K\subset\Omega$ compact, the F-convex hull $\hat{K}_F$ is again a compact set. In the particular case where $F=M(\Omega)$ the domain Ω is called monogenically convex.

THEOREM 3.4. Every domain in R^{m+1} is monogenically convex.
Proof. Let $K\subset\Omega$ be compact. In view of Proposition 3.2 it suffices to prove that $\bar{\hat{K}}_\Omega\subset\Omega$. For $u\in R^{m+1}\setminus\Omega$ the function

$$g(x)=\frac{\overline{x}-\overline{u}}{|x-u|^{m+1}}$$ is left-monogenic in Ω and so

$$\sup_{x\in K}|g(x)|_0=\sup_{x\in\hat{K}_\Omega}|g(x)|_0 \text{ or } \sup_{x\in K}\frac{1}{|x-u|^m}=\sup_{x\in\hat{K}_\Omega}\frac{1}{|x-u|^m}$$

for any $u \in R^{m+1} \setminus \Omega$.

Hence $0 < \inf_{\substack{x\in K \\ u\in R^{m+1}\setminus\Omega}} |x-u| = \inf_{\substack{x\in \hat{K}_\Omega \\ u\in R^{m+1}\setminus\Omega}} |x-u|$

$d(K, R^{m+1}\setminus\Omega) = d(\hat{K}_\Omega, R^{m+1}\setminus\Omega) > 0$. ■

4. Monogenic existence domains

DEFINITION 4.1. Let Ω be a domain in R^{m+1}.

(i) Ω is called a weak monogenic existence domain for the monogenic function f in Ω if for each monogenic function F in Ω', where Ω' is a domain strictly containing Ω, $F|\Omega \neq f$.

(ii) Ω is called a monogenic existence domain for the monogenic function f in Ω if for each pair of domains U_1 and U_2 for which $\phi \neq U_2 \subset \Omega \cap U_1 \subsetneq U_1$, and for each monogenic function F in U_1, $F|U_2 \neq f|U_2$.

(iii) Ω is called a (weak) monogenic existence domain if there exists a monogenic function f in Ω for which Ω is a (weak) monogenic domain.

REMARKS.

(i) The first definition 4.1.(i) states that the function f cannot be extended monogenically beyond the boundary of Ω. Definition 4.1.(ii) has a local character and implies that the result of a monogenic extension might be a multi-valued function.

(ii) It is clear that a (weak) monogenic existence domain is also a (weak) domain of monogenicity. Moreover it is obvious that a monogenic existence domain is also a weak monogenic existence domain. Under an additional condition on Ω both notions can be made to coincide. To be more precise :

PROPOSITION 4.2. If the domain Ω is locally connected then for a monogenic function f in Ω the following statements are equivalent:

(i) Ω is a monogenic existence domain for f;

(ii) Ω is a weak monogenic existence domain for f.

As already mentioned in the introduction it is not known yet if any domain in R^{m+1} is a (weak) monogenic existence domain. Nevertheless it will be shown explicitly that a special class of monogenically convex domains are indeed monogenic existence domains.

DEFINITION 4.3. Call ${}_2M(\Omega)$ the family of functions such that

(i) f is left-monogenic in Ω;

(ii) $|f.\bar{f}|=|f|^2$, with $|f|=2^{-n/2}|f|_0$;

(iii) all the functions f^{2^n}, $n\in N$, satisfy (i) and (ii).

Notice that in the complex case this family consists of all holomorphic functions in Ω.
That ${}_2M(\Omega)$ is not empty may be illustrated by the following examples : any monogenic function in Ω of the form $f(x)e_0+g(x)e_i$ $(0<i\leq m)$, or $f(x)e_0+g(x)e_ie_j$ $(0<i<j\leq m)$, f and g being real-valued, belongs to ${}_2M(\Omega)$. In particular the hypercomplex variables ξ_i $(1\leq i\leq m)$ all belong to ${}_2M(\Omega)$.

Now we construct in a ${}_2M(\Omega)$-convex domain a very peculiar monogenic function.

THEOREM 4.4. Let $\Omega\neq R^{m+1}$ be ${}_2M(\Omega)$-convex. Then there exists a monogenic function F in Ω satisfying the following condition : for each point $x\in\Omega$ with rational coordinates there exists a sequence

$(x^{(\nu)})_{\nu=1}^{\infty}$ in $\mathring{B}(x,d(x,\partial\Omega))$ on which F is unbounded.

Proof. Take all points in Ω with rational co-ordinates $(\eta^{(\nu)})_{\nu=1}^{\infty}$ and arrange them as follows :

$$(w^{(\nu)})_{\nu=1}^{\infty}=(\eta^{(1)},\eta^{(1)},\eta^{(2)},\eta^{(1)},\eta^{(2)},\eta^{(3)},\eta^{(1)},\dots).$$

Put $U_\nu=\mathring{B}(w^{(\nu)},d(w^{(\nu)},\partial\Omega))$; then by definition we have $\bar{U}_1\cap\partial\Omega\neq\phi$, and so we may construct a sequence $(w_1^{(\mu)})_{\mu=1}^{\infty}$ in U_1 converging to a point of $\partial\Omega$.
Consider the compact exhaustion $(K_\lambda)_{\lambda=1}^{\infty}$ of Ω. As Ω is ${}_2M(\Omega)$-convex, $\hat{K}_{1,{}_2M}$ is compact; hence there exists a function $\tilde{f}_1\in{}_2M(\Omega)$ and a

point $x^{(1)}=w_1^{(\mu_1)}$ such that $x_1\notin\hat{K}_{1,{}_2M}$ and $|\tilde{f}_1(x^{(1)})|_0>\sup_{u\in K_1}|\tilde{f}_1(u)|_0$.

After dividing by a suitable real number we can reduce $\tilde{f}_1$ to $f_1\in{}_2M(\Omega)$ for which

$$2^{-n/2}\sup_{u\in K_1}|f_1(u)|_0<1<2^{-n/2}|f_1(x^{(1)})|_0.$$

Also $x^{(1)}$ is contained in a certain K_λ which is called now $K_{\lambda(1)}$.

Proceeding in the same way, we may define sequences $(x^{(\nu)})_{\nu=1}^{\infty}$, $(K_{\lambda(\nu)})_{\nu=1}^{\infty}$ and $(f_\nu)_{\nu=1}^{\infty}$ such that $x^{(\nu)} \in U_\nu \cap (K_{\lambda(\nu+1)} \setminus \hat{K}_{\lambda(\nu),2M})$ and $2^{-n/2} \sup_{u \in K_{\lambda(\nu)}} |f_\nu(u)|_0 < 1 < 2^{-n/2} |f_\nu(x^{(\nu)})|_0$.

In view of the structure of the ${}_2M(\Omega)$-functions one now determines inductively a sequence of natural numbers $(a_\nu)_{\nu=1}^{\infty}$ such that

$$\frac{|f_{\nu+1}^{a_{\nu+1}}(x^{(\nu+1)})|_0}{(\nu+1)^2} - \sum_{\alpha=1}^{\nu} \frac{|f_\alpha^{a_\alpha}(x^{(\nu+1)})|_0}{\alpha^2} > \nu+1.$$

The series $\sum_{\alpha=1}^{\infty} \frac{f_\alpha^{a_\alpha}(x)}{\alpha^2}$ is normally convergent in Ω, and so represents a monogenic function in Ω, say F. However it may be shown that $|F(x^{(\nu)})|_0 \geq \nu - C$, C being a positive constant.
Now take an arbitrary point x in Ω with rational co-ordinates; then x coincides with a certain $\eta^{(\nu_0)}$ and this yields by construction a subsequence $(w^{(\nu_\mu)})_{\mu=1}^{\infty}$ of $(w^{(\nu)})_{\nu=1}^{\infty}$ such that $w^{(\nu_\mu)} = \eta^{(\nu_0)} = x$ for all $\mu = 1,2,\ldots$, whence the corresponding subsequence $(x^{(\nu_\mu)})_{\mu=1}^{\infty}$ is obviously contained in

$$U_{\nu_\mu} \mathring{B}(w^{(\nu_\mu)}, d(w^{(\nu_\mu)}, \partial\Omega)) = \mathring{B}(x, d(x, \partial\Omega))$$

and has the property that $|F(x^{(\nu_\mu)})|_0 \geq \nu_\mu - C$. ■

Now we study the behaviour of F on the boundary of Ω.

PROPOSITION 4.5. With the same notations as in Theorem 4.4 we have for any $\eta \in \partial\Omega$,

$$\overline{\lim_{\substack{x \to \eta \\ x \in \Omega}}} |F(x)|_0 = +\infty.$$

Proof. Take a sequence $(u^{(\mu)})_{\mu=1}^{\infty}$ in Ω converging to η; then we may select a sequence $(\eta^{(\nu_\mu)})_{\mu=1}^{\infty}$ such that $|u^{(\mu)} - \eta^{(\nu_\mu)}| < \frac{1}{\mu}$.

Reviewing the construction of $(w^{(\nu)})_{\nu=1}^{\infty}$ we may find a subsequence $(w^{(\nu_\lambda)})_{\lambda=1}^{\infty}$ for which $w^{(\nu_\lambda)}=\eta^{(\nu_\mu)}$, and clearly

$$\lim_{\lambda\to\infty} w^{(\nu_\lambda)}=\eta.$$

On the other hand, analyzing the proof of Theorem 4.4, we have that $|x^{(\nu_\lambda)}-w^{(\nu_\lambda)}|<d(w^{(\nu_\lambda)},\partial\Omega)$, $x^{(\nu_\lambda)}$ being the point corresponding to $w^{(\nu_\lambda)}$, and so $\lim_{\lambda\to\infty} x^{(\nu_\lambda)}=\eta$.

As moreover $|F(x^{(\nu_\lambda)})|_0>\nu_\lambda-C$, we obtain that

$$\lim_{\lambda\to\infty} |F(x^{(\nu_\lambda)})|_0=+\infty. \blacksquare$$

In the same way as for Theorem 2.2 we may now prove

THEOREM 4.6. If Ω is a ${}_2M(\Omega)$-convex domain in R^{m+1}, then it is a monogenic existence domain.

EXAMPLES.

(i) Take an arbitrary domain ω in a co-ordinate plane of R^{m+1} and consider a tube domain of the form $\Omega=\omega x R^{m-1}$. As each compact subset K of Ω can be written as a subset of a certain $K' x R^{m-1}$, K' being compact in ω, we have that $\hat{K}_\Omega\subset\hat{K}'_\omega x R^{m-1}$, $\hat{K}'_\omega$ being the holomorphically convex hull of K'. As $\hat{K}'_\omega$ is compact and $\hat{K}_\Omega$ is bounded and relatively closed in Ω, it follows that $\hat{K}_\Omega$ is compact in Ω. So Ω is ${}_2M(\Omega)$-convex and hence a monogenic existence domain.

(ii) We know from [3], Proposition 15.7.4 that the series $F(x)=\sum_{\alpha=0}^{\infty} p(x)^{\alpha!}$, where $p(x)=\xi_1+\ldots+\xi_m$, converges normally in the tube domain $\Omega=\{x\in R^{m+1}:(x_1+\ldots+x_m)^2+mx_0^2<1\}$
Moreover it can be proved that this series becomes unbounded on $\partial\Omega$, whence Ω is a monogenic existence domain for the function F.

REMARK. The family ${}_2M(\Omega)$ may be replaced by any larger family of monogenic functions f which possess in Ω an infinite number of monogenic powers f^α, $\alpha\in I\subset N$, satisfying the supplementary condition

$|f^{\alpha}|=|f|^{\alpha}$ for all $\alpha\in I$. Convexity with respect to that new family will also be sufficient for a domain to be a monogenic existence domain.

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