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RESTRICTIONS OF 3-FORMS IN DIMENSION 7 TO SUBSPACES OF CODIMENSION 1

JIŘÍ VANŽURA

ABSTRACT. On a 6-dimensional real vector space there are six types of 3-forms. We take all types of 3-forms on a 7-dimensional space and determine types of restrictions to all subspaces of codimension 1.

Let V be a finite dimensional vector space. A k -form $\omega \in \Lambda^k V^*$ is called *multi-symplectic* or *regular* if the homomorphism

$$V \rightarrow \Lambda^{k-1} V^*, \quad v \mapsto \iota_v \omega = \omega(v, \dots)$$

is a monomorphism. If a k -form ω is not regular, we shall call it singular. We denote by $\Lambda_r^k V^* \subset \Lambda^k V^*$ ($\Lambda_s^k V^* \subset \Lambda^k V^*$) the subset consisting of all regular (singular) forms. The general linear group $GL(V)$ operates in a natural way on $\Lambda^k V^*$, and it is easy to see that this action preserves $\Lambda_r^k V^*$ ($\Lambda_s^k V^*$). Consequently, $\Lambda_r^k V^*$ ($\Lambda_s^k V^*$) decomposes into orbits of this action. In this paper we take $k = 3$, i.e. we consider 3-forms. It is known, that the number of orbits of 3-forms is finite if and only if $\dim V \leq 8$.

Let us treat first a 6-dimensional real vector space W . We choose its basis f_1, \dots, f_6 , and we denote β_1, \dots, β_6 the corresponding dual basis. There are three orbits consisting of singular forms represented by the forms

- (S1) $\sigma_1 = 0$,
- (S2) $\sigma_2 = \beta_1 \wedge \beta_2 \wedge \beta_3$,
- (S3) $\sigma_3 = \beta_1 \wedge (\beta_2 \wedge \beta_3 + \beta_4 \wedge \beta_5)$.

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There are also three orbits consisting of regular forms. They are represented by the forms

$$(R1) \quad \rho_1 = \beta_1 \wedge \beta_2 \wedge \beta_3 + \beta_4 \wedge \beta_5 \wedge \beta_6,$$

$$(R2) \quad \rho_2 = \beta_1 \wedge \beta_2 \wedge \beta_3 + \beta_1 \wedge \beta_4 \wedge \beta_5 + \beta_2 \wedge \beta_4 \wedge \beta_6 - \beta_3 \wedge \beta_5 \wedge \beta_6,$$

$$(R3) \quad \rho_3 = \beta_1 \wedge \beta_4 \wedge \beta_5 + \beta_2 \wedge \beta_4 \wedge \beta_6 + \beta_3 \wedge \beta_5 \wedge \beta_6.$$

Now, let us pass to a 7-dimensional real vector space V . We choose a basis e_1, \dots, e_7 of V , and we denote by $\alpha_1, \dots, \alpha_7$ the corresponding dual basis. Here the subset $\Lambda_r^3 V^*$ decomposes into eight orbits. They are represented by the following forms.

$$(1) \quad \omega_1 = \alpha_1 \wedge \alpha_2 \wedge \alpha_7 + \alpha_1 \wedge \alpha_3 \wedge \alpha_4 + \alpha_2 \wedge \alpha_5 \wedge \alpha_6,$$

$$(2) \quad \omega_2 = \alpha_1 \wedge \alpha_2 \wedge \alpha_5 + \alpha_1 \wedge \alpha_2 \wedge \alpha_7 + \alpha_1 \wedge \alpha_4 \wedge \alpha_7 \\ - \alpha_2 \wedge \alpha_3 \wedge \alpha_7 + \alpha_3 \wedge \alpha_4 \wedge \alpha_6 + \alpha_3 \wedge \alpha_4 \wedge \alpha_7,$$

$$(3) \quad \omega_3 = \alpha_1 \wedge (\alpha_2 \wedge \alpha_7 - \alpha_3 \wedge \alpha_6 + \alpha_4 \wedge \alpha_5),$$

$$(4) \quad \omega_4 = \alpha_1 \wedge (\alpha_2 \wedge \alpha_7 - \alpha_3 \wedge \alpha_6 + \alpha_4 \wedge \alpha_5) + \alpha_2 \wedge \alpha_4 \wedge \alpha_6,$$

$$(5) \quad \omega_5 = \alpha_1 \wedge \alpha_2 \wedge \alpha_3 - \alpha_1 \wedge \alpha_4 \wedge \alpha_5 + \alpha_1 \wedge \alpha_6 \wedge \alpha_7 \\ + \alpha_2 \wedge \alpha_4 \wedge \alpha_6 + \alpha_2 \wedge \alpha_5 \wedge \alpha_7 + \alpha_3 \wedge \alpha_4 \wedge \alpha_7 - \alpha_3 \wedge \alpha_5 \wedge \alpha_6,$$

$$(6) \quad \omega_6 = \alpha_1 \wedge \alpha_2 \wedge \alpha_7 - \alpha_1 \wedge \alpha_3 \wedge \alpha_6 + \alpha_1 \wedge \alpha_4 \wedge \alpha_5 \\ + \alpha_2 \wedge \alpha_3 \wedge \alpha_5 + \alpha_2 \wedge \alpha_4 \wedge \alpha_6,$$

$$(7) \quad \omega_7 = \alpha_1 \wedge \alpha_2 \wedge \alpha_5 + \alpha_1 \wedge \alpha_3 \wedge \alpha_6 + \alpha_1 \wedge \alpha_4 \wedge \alpha_7 \\ + \alpha_2 \wedge \alpha_3 \wedge \alpha_7 - \alpha_2 \wedge \alpha_4 \wedge \alpha_6 + \alpha_3 \wedge \alpha_4 \wedge \alpha_5,$$

$$(8) \quad \omega_8 = \alpha_1 \wedge \alpha_2 \wedge \alpha_3 + \alpha_1 \wedge \alpha_4 \wedge \alpha_5 - \alpha_1 \wedge \alpha_6 \wedge \alpha_7 \\ + \alpha_2 \wedge \alpha_4 \wedge \alpha_6 + \alpha_2 \wedge \alpha_5 \wedge \alpha_7 + \alpha_3 \wedge \alpha_4 \wedge \alpha_7 \\ - \alpha_3 \wedge \alpha_5 \wedge \alpha_6.$$

The subset $\Lambda_9^3 V^*$ decomposes into six orbits. They are represented by the following forms

$$(9) \quad \omega_9 = 0,$$

$$(10) \quad \omega_{10} = \alpha_1 \wedge \alpha_2 \wedge \alpha_3,$$

$$(11) \quad \omega_{11} = \alpha_1 \wedge (\alpha_2 \wedge \alpha_3 + \alpha_4 \wedge \alpha_5),$$

$$(12) \quad \omega_{12} = \alpha_1 \wedge \alpha_2 \wedge \alpha_3 + \alpha_4 \wedge \alpha_5 \wedge \alpha_6,$$

$$(13) \quad \omega_{13} = \alpha_1 \wedge \alpha_2 \wedge \alpha_3 + \alpha_1 \wedge \alpha_4 \wedge \alpha_5 + \alpha_2 \wedge \alpha_4 \wedge \alpha_6 - \alpha_3 \wedge \alpha_5 \wedge \alpha_6,$$

$$(14) \quad \omega_{14} = \alpha_1 \wedge \alpha_4 \wedge \alpha_5 + \alpha_2 \wedge \alpha_4 \wedge \alpha_6 + \alpha_3 \wedge \alpha_5 \wedge \alpha_6.$$

Let us recall that with a 3-form τ on a 6-dimensional space W we can associate an endomorphism $Q(\tau)$ in the following way. We choose a nonzero 6-form θ on W , and for $w \in W$ we define $Q(\tau)w$ by the formula

$$(\iota_w \tau) \wedge \tau = \iota_{Q(\tau)w} \theta.$$

We have

$$Q(\sigma_1) = 0, \quad Q(\sigma_2) = 0, \quad Q(\sigma_3)^2 = 0, \quad \dim \operatorname{im} Q(\sigma_3) = 1.$$

Replacing θ by $a\theta$ if necessary, we get moreover

$$Q(\rho_1)^2 = I, \quad \dim \operatorname{im}(Q(\rho_1) + I) = \dim \operatorname{im}(Q(\rho_1) - I) = 3, \\ Q(\rho_2)^2 = -I, \quad Q(\rho_3)^2 = 0, \dim \operatorname{im} Q(\rho_3) = 3.$$

More information about the endomorphism Q you can find in [BV1].

Further, let ω be a 3-form on a 7-dimensional space V . We choose again a 7-form θ on V . Then we can define a symmetric bilinear form q on V by the formula

$$(\iota_v \omega) \wedge (\iota_{v'} \omega) \wedge \omega = q(v, v') \theta.$$

It is obvious that the definition of the symmetric bilinear form q depends on the choice of the 7-form θ . In other words the form q is determined up to a nonzero scalar multiple. More information about 3-forms on a 7-dimensional space you can find in [BV2].

Finally, for any 3-form ζ on a vector space Z we define

$$\Delta^2(\zeta) = \{z \in Z; (\iota_z \zeta)^{\wedge 2} = 0\}, \quad \Delta^3(\zeta) = \{z \in Z; (\iota_z \zeta)^{\wedge 3} = 0\}.$$

In the sequel we take the 3-forms $\omega_1, \dots, \omega_{14}$ on the 7-dimensional space V , and consider their restrictions on all 6-dimensional subspaces $W \subset V$. I present the results without proofs. The proofs have computational character. For every restriction $\omega_i|_W$ I have computed the corresponding endomorphism $Q(\omega_i|_W)$, which (with the exceptions of the types (S1) and (S2)) enables to recognize type of the restriction $\omega_i|_W$.

TYPE 1

For this form we have

$$\Delta^2(\omega_1) = V_3^a \cup V_3^b, \quad \text{where } V_3^a = [e_3, e_4, e_7], \quad V_3^b = [e_5, e_6, e_7], \quad V_1 = V_3^a \cap V_3^b$$

$$\Delta^3(\omega_1) = V_6^a \cup V_6^b, \quad \text{where } V_6^a = [e_1, e_3, e_4, e_5, e_6, e_7], \quad V_6^b = [e_2, e_3, e_4, e_5, e_6, e_7].$$

1. Proposition.

- (S1) *There is no W such that $\omega_1|W$ is of type (S1).*
 (S2) *$\omega_1|W$ is of type (S2) if and only if $W = V_6^a$ or $W = V_6^b$.*
 (S3) *$\omega_1|W$ is of type (S3) if and only if $W \supset V_3^a$ or $W \supset V_3^b$ and $W \neq V_6^a, V_6^b$.*
 (R1) *$\omega_1|W$ is of type (R1) if and only if $W \not\supset V_1$.*
 (R2) *There is no W such that $\omega_1|W$ is of type (R2).*
 (R3) *$\omega_1|W$ is of type (R3) if and only if $W \supset V_1$, $W \not\supset V_3^a$, and $W \not\supset V_3^b$.*

TYPE 2

Let us write $v = c_1e_1 + \dots + c_7e_7$ and $v' = c'_1e_1 + \dots + c'_7e_7$. For this form we have

$$\Delta^2(\omega_2) = \{v \in V; c_1 = c_2 = c_3 = c_4 = 0, c_5c_6 + c_6c_7 + c_7c_5 = 0\},$$

$$\Delta^3(\omega_2) = \{v \in V; c_1c_4 - c_2c_3 = 0\}.$$

Obviously, $\Delta^2(\omega_2)$ determines a subspace $V_3 \subset V$, $V_3 = [e_5, e_6, e_7]$. Moreover, on V we have a symmetric bilinear form q (determined up to a nonzero multiple) defined by the formula

$$q(v, v') = c_1c'_4 - c_2c'_3 - c_3c'_2 + c_4c'_1.$$

We can immediately see that $\ker q = V_3$. Consequently, q determines a regular symmetric bilinear form on V/V_3 , and this one in turn determines a quadric \mathcal{Q} in the projective space $P(V/V_3)$ associated with the vector space V/V_3 . If $W \subset V$ is a subspace of codimension 1 such that $W \supset V_3$, then W determines a subspace of codimension 1 in V/V_3 , and this one in turn determines a hyperplane \mathcal{W} in the projective space $P(V/V_3)$. Finally, on V_3 we have a regular symmetric bilinear form q_3 (determined up to a nonzero multiple) defined by the formula

$$q_3(v, v') = c_5c'_6 + c_5c'_7 + c_6c'_5 + c_6c'_7 + c_7c'_5 + c_7c'_6.$$

Let us remark that for each 2-dimensional subspace $Z \subset V_3$ the restriction $q_3|Z$ is a regular bilinear form.

2. Proposition.

- (S1) *There is no W such that $\omega_2|W$ is of type (S1).*
 (S2) *There is no W such that $\omega_2|W$ is of type (S2).*
 (S3) *$\omega_2|W$ is of type (S3) if and only if $W \supset V_3$ and the hyperplane \mathcal{W} is tangent to the quadric \mathcal{Q} .*

- (R1) $\omega_2|W$ is of type (R1) if and only if $W \not\supset V_3$ and the restriction $q_3|(W \cap V_3)$ is indefinite.
- (R2) $\omega_2|W$ is of type (R2) if and only if $W \not\supset V_3$ and the restriction $q_3|(W \cap V_3)$ is definite.
- (R3) $\omega_2|W$ is of type (R3) if and only if $W \supset V_3$ and the hyperplane W is not tangent to the quadric Q .

TYPE 3

For this form we have

$$\Delta^2(\omega_3) = \Delta^3(\omega_3) = V_6 = [e_2, e_3, e_4, e_5, e_6, e_7].$$

3. Proposition.

- (S1) $\omega_3|W$ is of type (S1) if and only if $W = V_6$.
- (S2) There is no W such that $\omega_3|W$ is of type (S2).
- (S3) $\omega_3|W$ is of type (S3) if and only if $W \neq V_6$.
- (R1) There is no W such that $\omega_3|W$ is of type (R1).
- (R2) There is no W such that $\omega_3|W$ is of type (R2).
- (R3) There is no W such that $\omega_3|W$ is of type (R2).

TYPE 4

For this form we have

$$\Delta^2(\omega_4) = V_3 = [e_3, e_5, e_7], \quad \Delta^3(\omega_4) = V_6 = [e_2, e_3, e_4, e_5, e_6, e_7].$$

4. Proposition.

- (S1) There is no W such that $\omega_4|W$ is of type (S1).
- (S2) $\omega_4|W$ is of type (S2) if and only if $W = V_6$.
- (S3) $\omega_4|W$ is of type (S3) if and only if $W \supset V_3$ and $W \neq V_6$.
- (R1) There is no W such that $\omega_4|W$ is of type (R1).
- (R2) There is no W such that $\omega_4|W$ is of type (R2).
- (R3) $\omega_4|W$ is of type (R3) if and only if $W \not\supset V_3$.

TYPE 5

Let us write again $v = c_1e_1 + \dots + c_7e_7$ and $v' = c'_1e_1 + \dots + c'_7e_7$. For this form we have

$$\Delta^2(\omega_5) = \{0\}, \quad \Delta^3(\omega_5) = \{v \in V; -c_1^2 - c_2^2 - c_3^2 + c_4^2 + c_5^2 + c_6^2 + c_7^2 = 0\}.$$

This time again, on V we have a symmetric bilinear form q (determined up to a nonzero multiple) defined by the formula

$$q(v, v') = -c_1c'_1 - c_2c'_2 - c_3c'_3 + c_4c'_4 + c_5c'_5 + c_6c'_6 + c_7c'_7.$$

This form has obviously signature $\{3, 4\}$. (We use this notation in order to underline that the bilinear form is determined up to a nonzero multiple. Depending on our choice it can have signature $(4, 3)$ or $(3, 4)$.)

5. Proposition.

- (S1) *There is no W such that $\omega_5|W$ is of type (S1).*
- (S2) *There is no W such that $\omega_5|W$ is of type (S2).*
- (S3) *There is no W such that $\omega_5|W$ is of type (S3).*
- (R1) *$\omega_5|W$ is of type (R1) if and only if the restriction $q|W$ is a regular form of signature $\{3, 3\}$.*
- (R2) *$\omega_5|W$ is of type (R2) if and only if the restriction $q|W$ is a regular form of signature $\{2, 4\}$.*
- (R3) *$\omega_5|W$ is of type (R3) if and only if the restriction $q|W$ is a singular form.*

TYPE 6

For this form we have

$$\Delta^2(\omega_6) = V_1 = [e_7], \quad \Delta^3(\omega_6) = V_5 = [e_3, e_4, e_5, e_6, e_7].$$

6. Proposition.

- (S1) *There is no W such that $\omega_6|W$ is of type (S1).*
- (S2) *There is no W such that $\omega_6|W$ is of type (S2).*
- (S3) *$\omega_6|W$ is of type (S3) if and only if $W \supset V_5$.*
- (R1) *There is no W such that $\omega_6|W$ is of type (R1).*
- (R2) *$\omega_6|W$ is of type (R2) if and only if $W \not\supset V_1$.*
- (R3) *$\omega_6|W$ is of type (R3) if and only if $W \supset V_1$ and $W \not\supset V_5$.*

TYPE 7

For this form we have

$$\Delta^2(\omega_7) = \{0\}, \quad \Delta^3(\omega_7) = V_3 = [e_5, e_6, e_7].$$

7. Proposition.

- (S1) *There is no W such that $\omega_7|W$ is of type (S1).*
- (S2) *There is no W such that $\omega_7|W$ is of type (S2).*
- (S3) *There is no W such that $\omega_7|W$ is of type (S3).*
- (R1) *There is no W such that $\omega_7|W$ is of type (R1).*
- (R2) *$\omega_7|W$ is of type (R2) if and only if $W \not\supset V_3$.*
- (R3) *$\omega_7|W$ is of type (R3) if and only if $W \supset V_3$.*

TYPE 8

For this form we have

$$\Delta^2(\omega_8) = \Delta^3(\omega_8) = \{0\}.$$

8. Proposition. *The restriction $\omega_8|W$ is always of type (R2).*

TYPE 9

9. Proposition. *The restriction $\omega_9|W$ is always of type (S1).*

TYPE 10

For this form we have $\ker \omega_{10} = V_4 = [e_4, e_5, e_6, e_7]$.

10. Proposition.

- (S1) $\omega_{10}|W$ is of type (S1) if and only if $W \supset V_4$.
- (S2) $\omega_{10}|W$ is of type (S2) if and only if $W \not\supset V_4$.
- (S3) There is no W such that $\omega_{10}|W$ is of type (S3).
- (R1) There is no W such that $\omega_{10}|W$ is of type (R1).
- (R2) There is no W such that $\omega_{10}|W$ is of type (R2).
- (R3) There is no W such that $\omega_{10}|W$ is of type (R3).

TYPE 11

For this form we have

$$\ker \omega_{11} = V_2 = [e_6, e_7] \quad \text{and} \quad \Delta^2(\omega_{11}) = V_6 = [e_2, e_3, e_4, e_5, e_6, e_7].$$

11. Proposition.

- (S1) $\omega_{11}|W$ is of type (S1) if and only if $W = V_6$.
- (S2) $\omega_{11}|W$ is of type (S2) if and only if $W \supset V_2$ and $W \neq V_6$.
- (S3) $\omega_{11}|W$ is of type (S3) if and only if $W \not\supset V_2$.
- (R1) There is no W such that $\omega_{11}|W$ is of type (R1).
- (R2) There is no W such that $\omega_{11}|W$ is of type (R2).
- (R3) There is no W such that $\omega_{11}|W$ is of type (R3).

TYPE 12

For this form we have

$$\ker \omega_{12} = V_1 = [e_7] \quad \text{and} \quad \Delta^2(\omega_{12}) = V_4^a \cup V_4^b,$$

where $V_4^a = [e_1, e_2, e_3, e_7]$ and $V_4^b = [e_4, e_5, e_6, e_7]$.

12. Proposition.

- (S1) There is no W such that $\omega_{12}|W$ is of type (S1).
- (S2) $\omega_{12}|W$ is of type (S2) if and only if $W \supset V_4^a$ or $W \supset V_4^b$.
- (S3) $\omega_{12}|W$ is of type (S3) if and only if $W \supset V_1$, $W \not\supset V_4^a$, and $W \not\supset V_4^b$.
- (R1) $\omega_{12}|W$ is of type (R1) if and only if $W \not\supset V_1$.
- (R2) There is no W such that $\omega_{12}|W$ is of type (R2).
- (R3) There is no W such that $\omega_{12}|W$ is of type (R3).

TYPE 13

For this form we have

$$\ker \omega_{13} = \Delta^2(\omega_{13}) = V_1 = [e_7].$$

13. Proposition.

- (S1) *There is no W such that $\omega_{13}|_W$ is of type (S1).*
- (S2) *There is no W such that $\omega_{13}|_W$ is of type (S2).*
- (S3) *$\omega_{13}|_W$ is of type (S3) if and only if $W \supset V_1$.*
- (R1) *There is no W such that $\omega_{13}|_W$ is of type (R1).*
- (R2) *$\omega_{13}|_W$ is of type (R2) if and only if $W \not\supset V_1$.*
- (R3) *There is no W such that $\omega_{13}|_W$ is of type (R3).*

TYPE 14

For this form we have

$$\ker \omega_{14} = V_1 = [e_7] \quad \text{and} \quad \Delta^2(\omega_{14}) = V_4 = [e_1, e_2, e_3, e_7].$$

14. Proposition.

- (S1) *There is no W such that $\omega_{14}|_W$ is of type (S1).*
- (S2) *$\omega_{14}|_W$ is of type (S2) if and only if $W \supset V_4$.*
- (S3) *$\omega_{14}|_W$ is of type (S3) if and only if $W \not\supset V_4$ and $W \supset V_1$.*
- (R1) *There is no W such that $\omega_{14}|_W$ is of type (R1).*
- (R2) *There is no W such that $\omega_{14}|_W$ is of type (R2).*
- (R3) *$\omega_{14}|_W$ is of type (R3) if and only if $W \not\supset V_1$.*

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