Acta Universitatis Carolinae. Mathematica et Physica

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Acta Universitatis Carolinae. Mathematica et Physica, Vol. 29 (1988), No. 2, 27--36

Persistent URL: http://dml.cz/dmlcz/701941

Counterexamples in Minimally Generated Boolean Algebras

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Received 1 March, 1988

The class minimally generated algebras is introduced in [Ko 3]; the results of that paper might suggest that it is quite well-behaved. This hope is partially destroyed by counterexample to the questions (Q1) through (Q4) below.

Let us recall material from [Ko 3] as far as it is relevant to the questions and their answers; for general information on Boolean algebras resp. set theory see e.g. [Ko 2] resp. [Je]. For Boolean algebras A and B, $A \leqq B$ denotes that A is a subalgebra of B. $C \leqq_m D$ (D is *minimal* over C or a minimal extension of C) means that $C \leqq D$ and there is no subalgebra of D lying properly between C and D. $A \leqq_{mg} B$ (B is *minimally generated over* A) if there exists an ordinal ϱ and a sequence $(B_\alpha)_{\alpha<\varrho}$ such that $B_0 = A, \bigcup_{\alpha<\varrho} B_\alpha = B, B_\lambda = \bigcup_{\alpha<\lambda} B_\alpha$ for limit ordinals $\lambda < \varrho$, and $B_\alpha \leqq_m B_{\alpha+1}$ if $\alpha + 1 < \varrho$. We say that B is *minimally generated* if it is minimally generated over its two-element subalgebra 2.

Proposition 1 (cf. 1.7, 1.9 in [Ko 3]) *a) The class of minimally generated algebras is closed under taking subalgebras, quotients, and products of finitely many factors.*

b) If $A \leqq_{mg} B$, then for every $x \in B$, the subalgebra $A(x)$ of B generated by $A \cup \{x\}$ satisfies $A \leqq_{mg} A(x)$.

The subsequent proposition contains the most important easy examples on minimal generation.

Proposition 2 (cf. 2.1, 2.3, 3.3, 2.4 in [Ko 3]) *a) Every Boolean algebra embeddable into an interval algebra is minimally generated.*

b) *Every superatomic algebra is minimally generated.*

c) *If A is superatomic and B is minimally generated, then the free product $A \oplus B$ of A and B is minimally generated.*

d) No Boolean algebra with an uncountable free subalgebra is minimally generated.

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The author gratefully acknowledges the hospitality of the Fakultät für Mathematik, Universität Konstanz, during the preparation of this paper.

Interval algebras and superatomic algebras are crucial examples of minimally generated ones because of the following proposition. For $A \leqq B$ and $x \in B$, let $J_{x,A}$ be the ideal

$$J_{x,A} = \{a \in A : a \,.\, x \in A\}$$

of A. Then $A(x)$ is minimal over A iff $A/J_{x,A}$ is the one-element or the two-element algebra. For $T \subseteqq A$, call T a *tree in* A if $0 \notin T$, T is a tree (as defined in set theory) under the (restriction to T of the) converse of the Boolean partial order $<_A$ of A, and for any $x \neq y$ in T either $x <_A y$ or $y <_A x$ or $x \,.\, y = 0$. It is easily seen that if a tree $T \subseteqq A$ generates A, then A embeds into an interval algebra.

Proposition 3 (cf. 3.2, 4.3 in [Ko 3]) *a) A simple extension $A(x)$ of A is minimally generated over A iff $A/J_{x,A}$ is superatomic.*

b) For every minimally generated algebra B, there exists $A \leqq B$ such that: A is generated by a tree, A is dense in B and B is minimally generated over A.

We are ready to state our questions and their motivation.

(Q1) If $Fr\ \omega_1$, the free Boolean algebra over ω_1 generators, does not embed into B, does it follow that B is minimally generated?

(Q2) Is the free product of any two minimally generated algebras minimally generated?

(Q3) Does every infinite minimally generated algebra have cofinality ω?

(Q4) Is every retractive Boolean algebra minimally generated?

A positive answer to (Q1) would give the very satisfactury characterization "B is minimally generated iff $Fr\ \omega_1$ does not embed into B", by Proposition 2d), and this would nicely parallel the well-known fact that B is superatomic iff $Fr\ \omega$ does not embed into B. It would also imply a positive answer to (Q2) because it is a result by Šapirovskii that for every infinite cardinal $\varkappa$, $\varkappa$, $Fr\ \varkappa$ embeds into a free product $A \oplus B$ iff it embeds either into A or into B – see e.g. Theorems 10.16 and 11.15 in [Ko 2].

In (Q3), the *cofinality* $cf\,B$ of an infinite algebra B is the least infinite cardinal $\varkappa$ such that $B = \bigcup_{\alpha<\kappa} B_\alpha$ for some strictly increasing chain $(B_\alpha)_{\alpha<\varkappa}$ of subalgebras. It has been shown in [Ko 1] that $\omega \leqq cf\,B \leqq 2^\omega$ and that for many algebras B, $cf\,B \leqq \omega_1$. In particular, $cf\,B = \omega$ if B is superatomic or embeds into an interval algebra; moreover $cf\,B \leqq \omega_1$ if $Fr\ \omega_1$ embeds into B. No algebra satisfying $cf\,B > \omega_1$ has been constructed up to now. Note that a positive answer to both (Q1) and (Q3) would imply that $cf\,B \leqq \omega_1$ for every infinite B, since in this case either $Fr\ \omega_1$ embeds into B and $cf\,B \leqq \omega_1$ or B is minimally generated and $cf\,B = \omega$.

In (Q4), a Boolean algebra B is called *retractive* if for any epimorphism $p\colon B \to Q$ onto some algebra Q, there exists a monomorphism $e\colon Q \to B$ such that $p \circ e = id_Q$. Subalgebras of interval algebras are retractive, as shown in [Rub], and by Propositions 2 and 3, they are important examples of minimally generated algebras; no other natural examples of retractive algebras seem to be known. Rubin has also constructed in [Rub] retractive algebras not embeddable into interval algebras, but only under additional set-theoretic assumptions. – It is easy to see that not every minimally generated algebra is retractive – e.g. the subalgebra of the power set algebra of ω generated by the singletons and an uncountable almost disjoint family is superatomic but not retractive.

All of the questions (Q1) through (Q4) will be answered in the negative. We shall use twice the following lemma.

Tree lemma *Let B be a Boolean algebra which admits a strictly positive finitely additive measure. Then every tree in B is countable.*

Proof. Let $\mu: B \to [0, 1]$ be the measure $T \subseteqq B$ a tree. B satisfies the countable chain condition, hence every branch and every level of T is countable. Assume T is uncountable; then its height must be ω_1. For $\alpha < \omega_1$, denote by T_α the α'th level of T and let

$$x_\alpha = \max\{\mu(t): t \in T_\alpha\} ;$$

x_α exists since T_α has at most n elements with measure $\geqq 1/n$, for $n \in \omega$. Then $(x_\alpha)_{\alpha<\omega_1}$ is a strictly decreasing sequence of reals, a contradiction. For if $\alpha < \beta < \omega_1$, fix $s \in T_\beta$ such that $\mu(s) = x_\beta$ and $t \in T_\alpha$ such that $t < s$ in T. Then $s <_\beta t$ and

$$x_\beta = \mu(s) < \mu(t) \leqq x_\alpha .$$

Our first example provides a negative answer to (Q2), hence (Q1).

Example 1. *The algebra*

$$B = Intalg[0, 1) \oplus Intalg[0, 1)_{\mathbf{Q}}$$

is not minimally generated; here $[0, 1) = \{x \in \mathbb{R}: 0 \leqq x < 1\}$, $[0, 1)_{\mathbf{Q}} = [0, 1) \cap \cap \mathbb{Q}$ and Intalg L is the interval algebra of a linear order L.

Proof. The elements of B are, without loss of generality, unions of finitely many disjoint rectangles in $[0, 1) \times [0, 1)_{\mathbf{Q}}$ of the form $u = [a, b) \times [c, d)$, where $a < b$ in $[0, 1]$ and $c < d$ in $[0, 1]_{\mathbf{Q}}$. Putting $\mu(u) = (b - a)(d - c)$, we see that B admits a strictly positive finitely additive measure.

Assume for contradiction that B is minimally generated; then by Proposition 3 and the tree lemma, B is minimally generated over some dense countable subalgebra A. The following definition and facts are what makes our proof work. If $u \in B$ and $a < b$ in $[0, 1]$, call u *full* in $[a, b)$ if for each $t \in [0, 1)_{\mathbf{Q}}$, $[a, b) \times \{t\}$ is either included in or disjoint from u.

Fact 1. The elements of B which are full in $[a, b)$ constitute a subalgebra of B.

Fact 2. For arbitrary $u \in B$ and $a < b$, there is $[a', b') \subseteqq [a, b)$ such that u is full in $[a', b')$.

Fact 3. If u is full in $[a, b)$ and $[a', b') \subseteqq [a, b)$ where $a' < b'$, then u is full in $[a', b')$.

Fact 4. If u intersects $[a, b) \times [0, 1)_{\mathbf{Q}}$, then there are $v \in A$ and $[a', b') \subseteqq [a, b)$ such that $v \leqq u$, the elements u, v, and $u \,.\, -v$ are full $[a', b')$, and both v and $u \,.\, -v$ intersect $[a', b') \times 0, 1)_{\mathbf{Q}}$ – this holds since A is dense in B.

For the rest of the proof, let us say that $c \in [0, 1]$ is a *relevant point* of $u \in B$ if there exists a non-empty interval I in $[0, 1)_{\mathbf{Q}}$, such that $\{c\} \times I$ is included in the boundary of u (computed in $[0, 1) \times [0, 1)_{\mathbf{Q}}$). E.g. a rectangle $[a, b) \times [c, d)$ has a and b as its relevant points, and each $u \in B$ has only finitely many relevant points. Let

$$R = \{c \in [0, 1] \colon c \text{ a relevant point of some } u \in A\} ,$$

a countable subset of $[0, 1]$.

Fact 5. Assume $u \in A$ is full in $[a, b)$ and intersects $[a, b) \times [0, 1)_{\mathbf{Q}}$, $a < c < b$ and $c \notin R$. Let $x = [0, c) \times [0, 1)_{\mathbf{Q}}$. Then $u \notin J_{x,A}$. – For otherwise, $u \,.\, x \in A$. But c is a relevant point of $u \,.\, x$ and hence $c \in R$, a contradiction.

Using the above facts, we construct, for $n \in \omega$, an interval I_n in $[0, 1]$ and an element u_n of A such that

(1) $I_n = [a_n, b_n)$ and $a_n < a_{n+1} < b_{n+1} < b_n$,

(2) I_n has length at most $1/2^n$,

(3) the unique element c of $\bigcap_{n \in \omega} [a_n, b_n]$ is not in R,

(4) u_n is full in I_n,

(5) for any $e \subseteqq n$, the elementary product $\prod_{i \in e} u_i \,.\, \prod_{i \in n \setminus e} - u_i$ in A intersects $I_n \times$ $\times\ [0, 1)_{\mathbf{Q}}$ (it is also full in I_n, by Facts 3 and 1).

Here (3) can be satisfied by Fact 3 and since R is countable, and (5) by Fact 4. Applying Fact 5 to $c \in \bigcap_{n \in \omega} [a_n, b_n]$ and $x = [0, c) \times [0, 1)_{\mathbf{Q}}$, we see that none of the elementary products displayed in (5) is in $J_{x,A}$. But then $A/J_{x,A}$ is not superatomic, $A(x)$ is not minimally generated over A by Proposition 3, and B is not minimally generated over A by Proposition 1.

We will now answer (Q4) and (Q3) in the negative, assuming the continuum hypothesis CH, resp. Jensen's principle $\diamondsuit$. In the proofs, we formulate some lemmas which are proved later on.

Example 2. (*CH*) *There a retractive Boolean algebra which is not minimally generated.*

Proof. Our algebra will be the union

$$C = \bigcup_{\alpha < \omega_1} C_\alpha$$

of a continuous chain $(C_\alpha)_{\alpha<\omega_1}$ of countable algebras such that C_0 is atomless and dense in C, i.e. $C_0 \leqq C \leqq (C_0)^{cm}$ (where A^{cm} denotes the completion of a Boolean algebra A) and $C_{\alpha+1} = C_\alpha(u_\alpha)$ for some $u_\alpha \in (C_0)^{cm}$. C will have the following properties:

(6) C is not minimally generated over any dense countable subalgebra,

(7) for any dense ideal K of C, C/K is countable.

By (7), C will be retractive, as shown in the proof of Theorem 4.3c) in [Rub]. And by (6), Proposition (3) and the tree lemma, C is not minimally generated since it admits a strictly positive finitely additive measure. (To see this, fix an atomless complete algebra B with such a measure; clearly C_0 embeds into B and C embeds into B over C_0 since C_0 is dense in C and B is complete.)

The properties (6) and (7) of C are ensured by some bookkeeping device and the following lemma.

Lemma. Assume $MA(2^\omega)$ and let A be a countable atomless algebra, $A \leqq B \leqq$ $\leqq A^{cm}$ and $|B| < 2^\omega$. Let $\mathscr{J}$ be a family of dense ideals of B such that $|\mathscr{J}| < 2^\omega$. Then there exists $u \in A^{cm}$ such that $u \notin B$, $A(u)$ is not minimally generated over A, and for every $J \in \mathscr{J}$ there is $i \in J \cap A$ such that $u \,.\, -i \in A$.

For the bookkeeping, we assume that C_0 has underlying set ω and the algebra C to be constructed, evidently of size ω_1, has ω_1 as its underlying set. By (CH), there are enumerations

$$\{I \subseteqq C_0 : I \text{ a dense ideal of } C_0\} = \{I_\nu : \nu < \omega_1\}\,,$$
$$\{s \subseteqq \omega_1 : s \text{ countable}\} = \{s_\nu : \nu < \omega_1\}$$

such that each countable subset of ω_1 is enumerated ω_1 times.

The algebras C_α are constructed by induction as follows. Given C_α, put $A_\alpha = s_\alpha$ if s_α happens to be (the underlying set of) a dense subalgebra C_α (recall $C_\alpha \subseteqq \omega_1$), and $A_\alpha = C_0$ otherwise; in any case, A_α is a dense countable subalgebra of C_α. Also put

$$\mathscr{J}_\alpha = \{Ig_{C_\alpha}(I_\nu) : \nu < \alpha\}$$

(where for X a subset of a Boolean algebra C, $Ig_C(X)$ is the ideal of C generated by X), a countable family of dense ideals of C_α. Then define

$$C_{\alpha+1} = C_\alpha(u_\alpha)$$

where $u_\alpha \in (C_\alpha)^{cm} = (C_0)^{cm}$ is chosen by the lemma to take care of $A = A_\alpha$, $B = C_\alpha$ and $\mathscr{J} = \mathscr{J}_\alpha$. – This finishes the construction of C.

To prove (6), assume A is a countable dense subalgebra of C, say $A \leqq C_\nu$ for some $\nu < \omega_1$. Pick $\alpha > \nu$ such that $s_\alpha = A$. Thus in the construction of $C_{\alpha+1}$, $s_\alpha = A \leqq C_\nu \leqq C_\alpha$ and $A_\alpha = A$. Now u_α has been chosen by the the lemma such that $A(u_\alpha)$ is not minimally generated over A. By Proposition 1, C is not minimally generated over A.

To prove (7), let K be a dense ideal of C and $\pi\colon C \to C/K$ canonical. $I = K \cap C_0$ is a dense ideal of C_0, say $I = I_\nu$. We show that $\pi[C_\alpha] = \pi[C_{\alpha+1}]$ for every $\alpha > \nu$, hence $C/K = \pi[C_{\nu+1}]$ is countable. So let $\alpha > \nu$. Put $J = Ig_{C_\alpha}(I)$; so in the construction of $C_{\alpha+1}$,

$$J = Ig_{C_\alpha}(I_\nu) \in \mathscr{J}_\alpha ,$$

and $J \subseteq K$. By the lemma, u_α has been chosen such that there is $i \in J \cap A_\alpha$ satisfying $u_\alpha \cdot -i \in A_\alpha \subseteq C_\alpha$. Since $i \in J \subseteq K$, it follows that $\pi(-i) = 1$ and $\pi(u_\alpha) \in \pi[C_\alpha]$, i.e. by $C_{\alpha+1} = C_\alpha(u_\alpha)$ that $\pi[C_{\alpha+1}] = \pi[C_\alpha]$.

Proof of the Lemma. We work in the Stone space $X = Ult A$ of A, a second countable compact (and hence completely metrizable) zero-dimensional space. We identify A with $Clop\, X$ and A^{cm} with $RO(X)$, the regular open algebra of X.

For $J \in \mathscr{J}$, $J \cap A$ is a dense ideal of A; hence the open subset U_J of X dual to $J \cap A$ is dense in X and $N_J = X \setminus U_J$ is nowhere dense. By Martin's axiom, there are countably many nowhere dense closed subsets M_n of X, $n \in \omega$, such that $\bigcup_{J \in J} N_J \subseteq \bigcup_{n \in \omega} M_n$ (see e.g. [Rud], Theorem 14). Now

$$G = X \setminus \bigcup_{n \in \omega} M_n$$

is a G_δ-subset of X which is uncountable by Baire's theorem. Thus there exists a perfect subset N of G, i.e. a closed set without isolated points (cf. e.g. Theorem 94(c) in [Je]). Passing if necessary to a subset of N, we may assume that N is nowhere dense. Finally, we may assume that

$$(8) \qquad N \neq bd\, b\,, \quad \text{for all} \quad b \in B$$

where $bd\, b$ denotes the boundary of the regular open set $b \in B$. This is possible since N, being homeomorphic to the Cantor space X, is homeomorphic to $N \times N$, i.e. N can be split into 2^ω subspaces N_α homeomorphic to N; since $|B| < 2^\omega$, we can satisfy (8) replacing N by some N_α.

N being closed and nowhere dense in X, there exists a regular open subset u of X such that

$$bd\, u = N\,;$$

we show that u works for the Lemma. Clearly $u \notin B$, by (8). $A(u)$ is not minimally generated over A by Proposition 3 and since

$$A/J_{u,A} \cong Clop(bd\, u) \cong Clop\, N$$

and N was perfect, i.e. $A/J_{u,A}$ is atomless. Finally, for $J \in \mathscr{J}$, we have that $N \subseteq U_J$, Now N is compact and U_J is open in X; so there is $i \in Clop\, X = A$ such that $N \subseteq i \subseteq U_J$. It follows from $N = bd\, u \subseteq i$ that $u \cap (X \setminus i)$ is clopen, i.e. an element of A.

Example 3. *Assume $\Diamond$ holds. Then there exists a minimally generated Boolean algebra B such that $|B| = \omega_1 = cf(B)$.*

Proof. For any Boolean algebra A, a sequence $(A_n)_{n\in\omega}$ of subalgebras of A demonstrating $cf\, A = \omega$ can be coded by the function $v: A \to \omega$ defined by

$$v(a) = \min \{n \in \omega: a \in A_n\} .$$

Let us call a function a *valuation* of A. I.e. v is a valuation of A iff $v: A \to \omega$ and

(9) $v(0) = 0$,

(10) $v(-x) = v(x)$, $v(x + y) \leqq \max (v(x), v(y))$ for $x, y \in A$,

(11) $v[A]$ is unbounded in ω.

Our strategy will, of course, be to construct B in ω_1 steps and killing, by $\Diamond$, all possible valuations of B. This depends essentially on the following definitions and two lemmas. Given a valuation v of A, we put

$$h\, v(x) = \sup \{v(y): y \leqq x\} \quad \text{for} \quad x \in A ,$$

thus $h\, v(x) \leqq \omega$. Moreover, we let

$$I(v) = \{x \in A: h\, v(x) < \omega\} ,$$

a proper ideal of A. We say that v has multiplicity n and write mult $(v) = n$, where $n < \omega$, if $A/I(v)$ is finite with exactly n atoms; $A/I(v)$ is infinite, put mult $(v) = \omega$. Thus mult $(v) \geqq n$ (resp. $= \omega$) means that there are at least n (resp. infinitely many) pairwise disjoint elements in the set $\{x \in A: h\, v(x) = \omega\}$. The following observation will be used several times: if $A \leqq C$ and v resp. w are valuations of A resp. C such that w extends v, then $h\, v(x) \leqq h\, w(x)$ for $x \in A$, and hence mult $(v) \leqq$ mult (w).

Lemma 1. Let v be a valuation of a countable Boolean algebra A such that mult $(v) = n < \omega$. Then there is a minimal extension C of A such that mult $(w) \geqq n + 1$ holds for each valuation w of C extending v.

Lemma 2. Let v be a valuation of a countable Boolean algebra A such that mult $(v) = \omega$. Then there is a minimal extension C of A such that v does not extend to a valuation of C.

To begin the construction of the algebra B for Example 3, fix by $\Diamond$ a sequence $(s_\alpha)_{\alpha<\omega_1}$ such that $s_\alpha: \alpha \to \omega$ and for each $s: \omega_1 \to \omega$, the set $\{\alpha < \omega_1: s \upharpoonright \alpha = s_\alpha\}$ is stationary. We let B be the union of a continuous chain of countable algebras B_α, $\alpha < \omega_1$. Without loss of generality, we may assume that the underlying set of B is ω_1; hence each B_α will be a countable subset of ω_1. Put $B_0 = 2 = \{0, 1\} \subseteq \omega_1$ and $B_\lambda = \bigcup_{\alpha<\lambda} B_\alpha$ for limit λ.

Given B_α, let $B_{\alpha+1}$ be a proper minimal extension of B_α. Moreover, if B_α happens to have α as its underlying set and $s_\alpha: \alpha \to \omega$ happens to be a valuation of B_α, pick $B_{\alpha+1}$ by applying Lemma 1 to B_α and s_α if mult $(s_\alpha) < \omega$ and Lemma 2 if mult $(s_\alpha) = \omega$.

Assume for contradiction that $\operatorname{cf} B = \omega$, i.e. there exists a valuation v of B. By $\Diamond$ and continuity of the chain $(B_\alpha)_{\alpha < \omega_1}$, the set

$$X = \{\alpha < \omega_1 \colon B_\alpha \text{ has underlying set } \alpha \text{ and } v \upharpoonright \alpha = s_\alpha\}$$

is stationary, hence unbounded. Fix a sequence $\alpha(0) < \alpha(1) < \ldots < \alpha(\omega)$ in X where $\alpha(0)$ is large enough to guarantee that $v[B_{\alpha(0)}]$ is unbounded in ω. It follows from the construction of $B_{\alpha(0)+1}, B_{\alpha(1)+1}, \ldots$ that

$$\operatorname{mult}(v \upharpoonright B_{\alpha(0)}) < \operatorname{mult}(v \upharpoonright B_{\alpha(1)}) < \ldots < \operatorname{mult}(v \upharpoonright B_{\alpha(\omega)}),$$

hence $\operatorname{mult}(v \upharpoonright B_{\alpha(\omega)}) = \omega$. But then $B_{\alpha(\omega)+1}$ has been constructed by Lemma 2, i.e. $v \upharpoonright B_{\alpha(\omega)}$ does not extend to $B_{\alpha(\omega)+1}$, a contradiction.

Proof of Lemma 1. Let M be, in A, a set of representatives of the n atoms of $A/I(v)$; we may assume that the elements of M are pairwise disjoint. For the rest of the proof, fix an element a of M.

Since a is an atom modulo $I(v)$, the set

$$I = A \upharpoonright a \cap I(v)$$

is a prime ideal in $A \upharpoonright a$. Also I is the union of the increasing sequence $(I_n)_{n \in \omega}$ of ideals $I_n = \{x \in A \upharpoonright a \colon h\, v(x) \leqq n\}$ of $A \upharpoonright a$, and each I_n is a proper subset of I. By countability of A there are elements a_n, $n \in \omega$, of I satisfying

$$h\, v(a_n) < h\, v(a_{n+1}),$$

the a_n are pairwise disjoint

I is the ideal generated by the a_n.

Working in the completion of A, set

$$t = \sum_{n \in \omega} a_{2n}, \quad C = A(t);$$

note that $(-t) \cdot a = \sum_{n \in \omega} a_{2n+1}$. It follows that each a_n is in $J_{t,A}$ and $J_{t,A} = \{x \in A \colon x \cdot a \in I\}$, a prime ideal of A; hence C is minimal over A.

Now let w be an arbitrary valuation of C extending v. The elements of

$$(M \setminus \{a\}) \cup \{t, (-t) \cdot a\}$$

are pairwise disjoint and for $m \in (M \setminus \{a\})$, we know that $h\, w(m) = h\, v(m) = \omega$. Also $h\, w(t) = \omega$, since for each $n \in \omega$, we have $a_{2n} \leqq t$ and thus $2n \leqq h\, v(a_{2n}) \leqq h\, w(a_{2n}) \leqq h\, w(t)$. Similarly, $h\, w((-t) \cdot a) = \omega$, which proves that $\operatorname{mult}(w) \geqq n + 1$.

Proof of Lemma 2. Let $\pi \colon A \to A/I(v)$ be canonical. In the infinite algebra $A/I(v)$, we fix a non-principal ultrafilter q and let p be its preimage under π.

Claim. For $a, b \in p$ and $n \in \omega$, there is $a' \in p$ such that $a' < a \,.\, b$ and, letting $d = a \,.\, -a'$, we have $v(d) > n$ and $h\, v(d) = \omega$.

To prove the claim, first choose $y \leqq a . b$ such that the element $c = a . b . -y$ is in p and $\pi(y) > 0$; this is possible since $a . b \in p$ and the image of $A \restriction (a . b)$ under π has more than two elements. Now $h\, v(c) = \omega$, so there is $a' \leqq c$ such that

$$v(a') > v(c)\,, \quad v(a') > v(a . -c)\,, \quad v(a') > n\,.$$

$v(a') > v(c)$ implies that $v(a') = v(c . -a')$, and either a' or $c . -a'$ is in p; so we may assume that $a' \in p$. Consider $d = a . -a'$. We have $y \leqq d$ and hence, by $\pi(y) > 0$, $h\, v(d) \geqq h\, v(y) = \omega$. Finally, d is the disjoint sum of $a . -c$ and $c . -a'$; since $v(c . -a') = v(a') > v(a . -c)$, it follows that $v(d) = v(a') > n$. This proves claim.

Using the claim and the fact that p is countably generated as a filter, it is now easy to construct a sequence $a_0 > a_1 > \ldots$ in p such that the a_n generate p and the elements $d_n = a_n . -a_{n+1}$ satisfy $h\, v(d_n) = \omega$ and $v(d_n) > n$. By $h\, v(d_n) = \omega$, choose for $n \in \omega$ elements b_n and c_n such that

$$b_n . c_n = 0\,, \quad b_n + c_n = d_n\,, \quad v(b_n) > v(d_n)\,.$$

Working in the completion of A, put

$$t = \sum_{n \in w} b_n\,, \quad C = A(t)\,.$$

C is a minimal extension of A since $b_n \leqq t$, $c_n \leqq -t$, and hence $J_{t,A} = A \setminus p$. Assume that w is a valuation of C extending A. Letting $n^* = w(t)$, pick $n \in \omega$ such that $v(d_n) \geqq n^*$ (which is possible by $v(d_k) > k$). Now $t . d_n = b_n$, but $w(t . d_n) \leqq \max(n^*, v(d_n)) = v(d_n)$ and $w(b_n) = v(b_n) > v(d_n)$, a contradiction.

Example 3 gives a particularly strong counterexample for a conjecture due to Efimov — unfortunately, only under $\Diamond$. Efimov's conjecture states that every infinite compact Hausdorff space has a closed subspace either homeomorphic to the one-point compactification or to the Stone-Čech compactification of the integers. Restricting to Boolean spaces and applying Stone duality, we obtain the conjecture that for any infinite Boolean algebra B: either B has the finite-cofinite algebra of ω as a homomorphic image (i.e. B has a countable homomorphic image) or B has the power set algebra of ω as a homomorphic image (i.e. B has an independent subset of size 2^{ω}). But algebra B of Example 3 has cofinality greater than ω, hence no countable homomorphic image, and it is minimally generated, whence it has no uncountable independent subset, by Proposition 2.

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