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Acta Universitatis Carolinae. Mathematica et Physica, Vol. 43 (2002), No. 2, 45--49

Persistent URL: http://dml.cz/dmlcz/702083

A Note on Forcing with Ideals and Hechler Forcing

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Received 14. March 2002

We present a simple proof of a theorem due to M. Gitik and S. Shelah stating that the Hechler forcing is not equivalent to a forcing with a uniform, κ-complete ideal on some uncountable cardinal κ. We also make some general remarks and comments[1].

1. Introduction

For an infinite cardinal λ let $\mathscr{C}_\lambda$ (resp. $\mathscr{R}_\lambda$) be the usual Boolean algebra (on the space $\{0,1\}^\lambda$) for adding λ Cohen (resp. random[2]) reals. Let $\mathscr{C} = \mathscr{C}_\omega$ and $\mathscr{R} = \mathscr{R}_\omega$. By *Hechler forcing* we mean the set $H = \{\langle f, n\rangle : f \in \omega^\omega, n < \omega\}$ with the ordering defined by: $\langle f, n\rangle \leq \langle g,m\rangle$ iff $n \geq m$, $f|m = g|m$ and $f(k) \geq g(k)$ for all $k \geq m$. We call $\mathscr{H} = \mathrm{RO}(H)$ the *Hechler algebra.* Recall that an ideal I on κ is *uniform* if $[\kappa]^{<\kappa} \subseteq I$. Also, I is *κ-complete* if $\bigcup A \in I$ whenever $A \subseteq I$ and $|A| < \kappa$.

Definition 1.1 *Let κ be an infinite cardinal. We say that a Boolean algebra $\mathscr{B}$ is* κ-representable *if $\mathscr{B}$ is isomorphic to the factor algebra* $\mathrm{P}(\kappa)/I$ *for some uniform, κ-complete ideal I on κ. We say that $\mathscr{B}$ is* representable *if $\mathscr{B}$ is κ-representable for some* **uncountable** *cardinal κ.*

In the next section we explain why the ω-representability is treated here as a separate notion. Using the above definition we can now formulate the following interesting result due to M. Gitik and S. Shelak ([GS1]).

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[1] 1991 Mathematics Subject Classification: 03E15, 06E10. Key words: Hechler forcing, forcing with ideals.

[2] Random reals are also called Solovay reals.

Theorem 1.2 *If $\mathscr{C}_\lambda$ (or $\mathscr{R}_\lambda$) is κ-representable and $\kappa > \omega$ then $\lambda \geq \kappa^+$. In particular $\mathscr{C}$ and $\mathscr{R}$ are not representable.*

In the same paper and also in [GS2] it is shown that Hechler algebra is not representable too. Proofs included in these papers are rather complicated. The aim of this note is to present a simpler proof of this result, based on some observation (due to J. Pawlikowski) concerning Hechler forcing.

2. ω-representability

In the first version of this note we had several examples of ω-representable algebras. But it was pointed to us by Balcar that the following is true.

Proposition 2.1 *Every complete Boolean algebra of cardinality $\leq 2^\omega$ is ω-representable.*

Proof. Assume that $\mathscr{B}$ is complete and write $\mathscr{B} = \{b_\alpha : \alpha < 2^\omega\}$. Let $\mathscr{I} = \{d_\alpha : \alpha < 2^\omega\}$ be an independent family in $\mathrm{P}(\omega)/\mathrm{fin}$. The map $d_\alpha \mapsto b_\alpha$ can be exatended to a surjective homomorphism defined on the subalgebra of $\mathrm{P}(\omega)/\mathrm{fin}$ generated by $\mathscr{I}$. Now $\mathscr{B}$ as a complete Boolean algebra is *injective* by a theorem due to Sikorski ([S]). Therefore the above homomorphism can be extended to a homomorphism from $\mathrm{P}(\omega)/\mathrm{fin}$ onto $\mathscr{B}$. The kernel of this final homomorphism gives us the ideal $I \supseteq \mathrm{fin}$ such that $\mathscr{B} \cong \mathrm{P}(\omega)/I$. □

It is perhaps interesting to note that the completion of a Souslin tree (if it exists) is ω-representable. Such an algebra is a special case of Souslin algebra (see [J] for more on Souslin algebras). We will show in the next section that Souslin agebras are not representable.

3. Quasi-measurable cardinals and names

Recall the definition introduced by Fremlin ([F1]). An uncountable cardinal κ is called *quasi-measurable* if there exists a uniform, κ-complete ideal I on κ such that the Boolean algebra $\mathrm{P}(\kappa)/I$ satisfies the c.c.c. (countable chain condition). Quasi-measurable cardinals appear naturally when we are dealing with κ-representability of c.c.c. algebras. In this section we assume that κ is quasi-measurable and $\mathscr{B} = \mathrm{P}(\kappa)/I$, where I witness that κ is quasi-measurable. Note that $\mathscr{B}$ is complete.

It is known (see [F1]) that κ must be a large cardinal. In fact, if $\mathscr{B}$ has an atom then κ is *measurable*. If $\mathscr{B}$ is atomless then $\kappa \leq 2^\omega$ but κ is still very large, for example it is *greatly Mahlo*.

We look at $\mathscr{B}$-names of reals and Borel sets. The main idea is that such names translate into κ-sequences of objects from the ground model. Reader may recognise connections with the method of *generic ultrapower* (see [So] and [JP]).

Lemma 3.1 *Assume that r is a $\mathcal{B}$-name for a real, i.e.,* $[\![r \text{ is a real}]\!] = \mathbf{1}$. *There exists a κ-sequence $\langle r_\alpha : \alpha < \kappa \rangle$ of reals such that for every Borel set C we have* $[\![r \in C^*]\!] = [\{\alpha < \kappa : r_\alpha \in C\}]$. *Moreover, any κ-sequence of reals defines a $\mathcal{B}$-name of this form.*

Note. We treat Borel sets (from the ground model) as *codes*(see [J]) and C^* denotes (a name for) C encoded in $V^{\mathcal{B}}$. We write $[A]$ for the equivalence class (modulo I) of A.

Proof. We shall use the Baire space ω^ω as our "model" of reals. Similar proof works for the Cantor space $\{0,1\}^\omega$. Assume that $[\![r \in \omega^\omega]\!] = \mathbf{1}$. Choose sets $A_{n,m} \subseteq \kappa$ such that $[\![r(n) = m]\!] = [A_{n,m}]$ for every $n, m < \omega$. We can assume that $A_{n,m} \cap A_{n,k} = \emptyset$ for every n and distinct m, k, and that $\bigcup_{m<\omega} A_{n,m} = \kappa$ for every n. Now define $r_\alpha \in \omega^\omega$ as follows: $r_\alpha(n) = m$ iff $\alpha \in A_{n,m}$. The claim about Borel sets can be easy proved by induction on complexity of Borel set C. Given a κ-sequence $\langle r_\alpha : \alpha < \kappa \rangle$ of reals define a $\mathcal{B}$-name r as follows: put $[\![r(n) = m]\!] = [\{\alpha < \kappa : r_\alpha(n) = m\}]$. □

Corollary 3.2 *No Souslin algebra is representable.*

Proof. It suffices to show that if $\mathcal{B}$ is atomless then it is not ω-distributive. So assume that $\mathcal{B}$ has no atom. Then $\kappa \leq 2^\omega$. Choose any sequence $\langle r_\alpha : \alpha < \kappa \rangle$ of *distinct* reals and let r be the associated $\mathcal{B}$-name. Then, for any fixed *old* real number x we have $[\![r = x]\!] = [\{\alpha < \kappa : r_\alpha = x\}] = \mathbf{0}$. Thus r is a *new* real number. □

A slightly more involved construction is used to define a $\mathcal{B}$-name of a (possibly new) Borel set. Let $\langle D_\alpha : \alpha < \kappa \rangle$ be a sequence of Borel sets. Define a $\mathcal{B}$-name D as follows: let r be a name of a real, put $[\![r \in D]\!] = [\{\alpha < \kappa : r_\alpha \in D_\alpha\}]$ where $\langle r_\alpha : \alpha < \kappa \rangle$ is the sequence associated with the name r. We leave as an exercise the proof of the following lemma.

Lemma 3.3 *If C is a Borel set then* $[\![C^* \subseteq D]\!] = [\{\alpha < \kappa : C \subseteq D_\alpha\}]$.

Let $\mathcal{K}$ denote the ideal of all sets of first category (meager).

Lemma 3.4 $[\![D \in \mathcal{K}]\!] = [\{\alpha < \kappa : D_\alpha \in \mathcal{K}\}]$.

Proof. $[\![G \text{ is open dense}]\!] = [\{\alpha < \kappa : G_\alpha \text{ is open dense}\}]$. □

4. Main result

We prove that Hechler algebra is not representable. First we need some definitions. For $f, d \in \omega^\omega$ we write $f \prec d$ if there exists $m < \omega$ such that $f(n) < d(n)$ for all $n > m$. Also, for $F \subseteq \omega^\omega$ we write $F \prec d$ if $f \prec d$ for all $f \in F$. We say that a Boolean algebra $\mathcal{B}$ *adds a dominating real* if there is a $\mathscr{B}$-name, say d such that $[\![\omega^\omega \cap V \prec d]\!] = \mathbf{1}$. Obviously the Hechler algebra adds a dominating real. This (canonical) dominating real will be called the *Hechler real.* It is different from dominating reals added by Mathias or Laver forcing.

We say that $\mathscr{B}$ *adds a Cohen real* if there is a $\mathscr{B}$-name, say c such that $[\![c \notin N^*]\!] = \mathbf{1}$ for every Borel set N from $\mathscr{K}$. It is also well known that Hechler algebra adds a Cohen real. Namely, if h is the canonical Hechler real then let $c \in \{0,1\}^\omega$ be defined by: $c(n) = 1$ if $h(n)$ is odd.

Finally, let us say that $\mathscr{B}$ *kills* $\mathscr{K}$ if $[\![\bigcup \mathscr{K} \cap V \in \mathscr{K}]\!] = \mathbf{1}$ where by $\mathscr{K} \cap V$ we mean the Borel sets from $\mathscr{K}$ coded in V. We shall use the following result of J. Pawlikowski ([P]).

Proposition 4.1 *Hechler algebra does not kill* $\mathscr{K}$.

We can now state our main result.

Proposition 4.2 *Assume that* κ *is quasi-measurable and* I *is a witnessing ideal. If* $\mathrm{P}(\kappa)/I$ *adds a dominating real and a Cohen real then it kills* $\mathscr{K}$.

Proof. We shall use some cardinals from Cichoń's Diagram (see [F2]). Let d be a name such that $[\![\omega^\omega \cap V \prec d]\!] = \mathbf{1}$ and let $\langle d_\alpha : \alpha < \kappa\rangle$ be the associated κ-sequence. It is easy to verify that for every $f \in \omega^\omega$ we have $\{\alpha < \kappa : f \not\prec d_\alpha\} \in I$. It follows that $\mathbf{b} = \mathbf{d} = \kappa$. Let c be a name for a Cohen real and let $\langle c_\alpha : \alpha < \kappa\rangle$ be the associated sequence. Then, for every Borel set $N \in \mathscr{K}$ we have $\{\alpha < \kappa : c_\alpha \in N\} \in I$. Hence $\mathrm{Cov}(\mathscr{K}) \geq \kappa$ and $\mathrm{Non}(\mathscr{K}) \leq \kappa$. Now $\mathrm{Add}(\mathscr{K}) = \min\{\mathrm{Cov}(\mathscr{K}), \mathbf{b}\} = \kappa = \max\{\mathrm{Non}(\mathscr{K}), \mathbf{d}\} = \mathrm{Cof}(\mathscr{K})$. It follows that there is an increasing κ-sequence $\langle D_\alpha : \alpha < \kappa\rangle \subseteq \mathscr{K}$ consisting of Borel sets and cofinal in $\mathscr{K}$. Any such sequence defines a name of a set from $\mathscr{K}$ which contains all Borel sets from $\mathscr{K} \cap V$. □

Corollary 4.3 *Hechler algebra* $\mathscr{H}$ *is not representable. Moreover, any finite support product of Hechler algebras is not representable.*

Proof. The first statement follows from propositions 4.1 and 4.2. For the second, note that finite support product of $\mathscr{H}$ satisfies the c.c.c. and still does not kill $\mathscr{K}$ because any name of a Borel set is defined from some *countable* (sub)product. But countable product of Hechler algebras (with finite supports) is isomorphic to $\mathscr{H}$. □

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