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Descriptive Properties of Spaces of Signed Measures

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Topological spaces of signed Radon measures on a Tychonoff space X inherit some descriptive properties of the space X. We show in particular that the space of signed Radon measures on an absolute Borel (Čech-analytic) space is again absolute Borel (Čech-analytic). Main tool in proving such results is study of measurability of evaluation functions.

1. Introduction

We study possibilities to derive descriptive properties of topological spaces of signed measures on a topological space X from the respective properties of X. In [1] such problems are investigated for spaces of non-negative measures. We show, in answer to questions formulated in the final section of [1], that in case of signed measures the situation is quite similar but not completely the same.

Along this paper X will denote a topological space. All the measures considered are real valued and finite Borel measures. The main tool will be, similarly as in [1], measurability of evaluation mappings. Recall that for a measure μ we denote by μ^+ $(\mu^-, |\mu|)$ its positive part (negative part, absolute variation, respectively). I.e., μ^+ and μ^- are mutually singular non-negative measures satisfying $\mu = \mu^+ - \mu^-$, and $|\mu| = \mu^+ + \mu^-$.

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We shall consider a space $\mathfrak{M}$ of measures on X. The most interesting case of $\mathfrak{M}$ is the space $\mathfrak{M}_t(X)$ of all Radon measures on X which we study in fourth section. However, we formulate abstract results for a general $\mathfrak{M}$. Unlike in case of non-negative measures there is not a unique natural topology on $\mathfrak{M}$. Before formulating which topologies we will study we fix the following notation for evaluation mappings.

If $A \subset X$ is μ-measurable (i.e., both μ^+-measurable and μ^--measurable) for all $\mu \in \mathfrak{M}$, we put

$$\begin{aligned} \Psi_A(\mu) &= \mu(A), \\ \Psi_A^+(\mu) &= \mu^+(A), \\ \Psi_A^-(\mu) &= \mu^-(A) \end{aligned}$$

for all $\mu \in \mathfrak{M}$.

Along the paper we shall assume that $\mathfrak{M}$ is endowed with a topology satisfying the following condition:

(*) Ψ_A^+ and Ψ_A^- are lower semicontinuous on $\mathfrak{M}$ for every open $A \subset X$.

Condition (*) holds for the topology of pointwise convergence on a subset $\mathfrak{F} \subset C(X)$ in the following cases:

(1) X is a subset of Tychonoff space Y, $\mathfrak{M}$ is the set of Radon measures and $\mathfrak{F} = \{f \mid X : f \in C_b(Y)\}$, where $C_b(Y)$ denotes the bounded continuous functions on Y. In particular, in case $Y = X$ we consider $\mathfrak{M}_t(X)$ naturally embedded into $(C_b(X)^*, w^*)$.

(2) X is locally compact, $\mathfrak{M}$ is the set of Radon measures and $\mathfrak{F} = C_0(X)$. In that case $\mathfrak{M}$ can be identified with the dual of $C_0(X)$ and the topology considered is the weak* topology.

(3) X is normal, $\mathfrak{M}$ the set of regular measures and $\mathfrak{F} = C_b(X)$.

In all these cases lower semicontinuity follows easily from the formulae

$$\Psi_A^+(\mu) = \sup\left\{\int f \,\mathrm{d}\mu : f \in \mathfrak{F}, f(X) \subset [0, 1], \operatorname{supp}(f) \subset A\right\},$$

$$\Psi_A^-(\mu) = \sup\left\{\int f \,\mathrm{d}\mu : f \in \mathfrak{F}, f(X) \subset [-0, 1], \operatorname{supp}(f) \subset A\right\},$$

for every $\mu \in \mathfrak{M}$ and for every open $A \subset X$ which expresses Ψ_A^+ and Ψ_A^- as a supremum of continuous functionals on $\mathfrak{M}$.

2. Some abstract auxiliary results

In this section we name some auxiliary results which are analogous to the results in [1, Section 2].

We will deal with the sets of the form

$$\mathfrak{M}^+(B, c) = \{\mu \in \mathfrak{M} \mid \mu^+(B) > c\} \quad \text{and} \quad \mathfrak{M}^-(B, c) = \{\mu \in \mathfrak{M} \mid \mu^-(B) > c\}$$

where B is μ-measurable for every $\mu \in \mathfrak{M}$, and $c \geq 0$. We formulate the results for the sets $\mathfrak{M}^+(B, c)$ but it is clear that they hold for the sets $\mathfrak{M}^-(B, c)$, too.

We begin by the following easy lemma ($\mathbb{Q}^+$ denotes the set of all positive rational numbers).

Lemma 2.1. *Let X be a Hausdorff topological space, $\mathfrak{M}$ be a set of measures on X, the sets A_n, $n \in \mathbb{N}$, be μ-measurable for every $\mu \in \mathfrak{M}$, and c be a non-negative constant. Then*

(a) $\mathfrak{M}^+(A, c) = \bigcup_{n \in \mathbb{N}} \mathfrak{M}^+(A_n, c)$ *if* $A_n \nearrow A$,

(b) $\mathfrak{M}^+(A, c) = \bigcup_{p \in \mathbb{N}} \bigcap_{n \in \mathbb{N}} \mathfrak{M}^+(A_n, c + \frac{1}{p})$ *if* $A_n \searrow A$,

(c) $\mathfrak{M}^+(A_2 \setminus A_1, c) = \bigcup_{p \in \mathbb{Q}^+} (\mathfrak{M}^+(A_2, c + p) \setminus \mathfrak{M}^+(A_1, p))$ *if* $A_1 \subset A_2$.

In the remaining results of this section we will use the following assumption (cf. the analogous assumption used in [1, Section 2]).

Assumption (S). *Let X be a Haussdorff topological space and $\mathfrak{M}$ a set of measures on X. Let $\mathscr{R}$ be a family of subsets of X which are μ-measurable for all $\mu \in \mathfrak{M}$, and $\mathscr{V}$ be a family of subsets of $\mathfrak{M}$ such that $\mathfrak{M}^+(R, c) \in \mathscr{V}$ for every $R \in \mathscr{R}$ and every $c \geq 0$.*

If $\mathscr{A}$ is a family of subsets of a given set, we denote by $\mathscr{A}^c$ the family of all complements of elements of $\mathscr{A}$, by $\mathscr{A}_\sigma$ $(\mathscr{A}_\delta)$ the familly of all countable unions (countable intersections, respectively) of elements of $\mathscr{A}$. The symbol $\sigma(\mathscr{A})$ stands for the smallest σ-algebra containing $\mathscr{A}$, Suslin$(\mathscr{A})$ is the family of sets which result from a Suslin operation applied to elements of $\mathscr{A}$, co-Suslin$(\mathscr{A}) =$ $(\text{Suslin}(\mathscr{A}))^c$. If $\mathscr{A}_1$ and $\mathscr{A}_2$ are two families of sets, we denote by $\mathscr{A}_1 \wedge \mathscr{A}_2$ the family of all intersections $A_1 \cap A_2$ with $A_1 \in \mathscr{A}_1$ and $A_2 \in \mathscr{A}_2$. The family of unions $A_1 \cup A_2$ with $A_1 \in \mathscr{A}_1$ and $A_2 \in \mathscr{A}_2$ is then denoted by $\mathscr{A}_1 \vee \mathscr{A}_2$.

The following two propositions are analogous of Proposition 1 and Proposition 2 of [1]. Their proofs are exactly the same as the proofs of the named results of [1] and so we omit them.

Proposition 2.2. (under Assumption (S)) *Let us suppose that the system $\mathscr{R}$ contains X and is closed either to finite unions or to finite intersections. Then $\mathfrak{M}^+(R, c) \in \sigma(\mathscr{V})$ for every $R \in \sigma(\mathscr{R})$ and $c \geq 0$.*

Proposition 2.3. (under Assumption (S)) *Let $\mathscr{R}$ be closed to finite unions and finite intersections. Then $\mathfrak{M}^+(R, c) \in$ Suslin$(\mathscr{V})$ for every $R \in$ Suslin$(\mathscr{R})$ and every $c \geq 0$.*

In addition to these results we give one more result of this type.

Proposition 2.4. (under Assumption (S)) *Let $\mathscr{R}$ be closed to finite unions and finite intersections. Then* $\mathfrak{M}^+(R, c) \in$ co-Suslin$(\mathscr{V}^c)$ *for every* $R \in$ co-Suslin$(\mathscr{R}^c)$ *and every* $c \geq 0$.

Proof. The proof is done using some ídeas of the proof of [1, Proposition 2]. Let $R \in$ co-Suslin$(\mathscr{R}^c)$. We will show that $\{\mu \in \mathfrak{M} : \mu^+(R) \leq c\} \in$ Suslin$(\mathscr{V}^c)$ for all $c \geq 0$.

We can express $X \setminus R$ by a monotone Suslin operation, i.e. there are elements $A_s \in \mathscr{R}$ indexed by finite sequences of positive integers such that $A_s \subset A_t$ if s is a beginning of t and such that

$$X \setminus R = \bigcup_{\nu \in \mathbb{N}^{\mathbb{N}}} \bigcap_{n=1}^{\infty} (X \setminus A_{\nu|n}).$$

Then clearly

$$R = \bigcap_{\nu \in \mathbb{N}^{\mathbb{N}}} \bigcup_{n=1}^{\infty} A_{\nu|n}.$$

Denote by $\mathbb{S}$ the set of all nonempty finite subsets of $\mathbb{N}$ and for any $(\sigma_1, \ldots, \sigma_n) \in \mathbb{S}^n$ put

$$B_{(\sigma_1, \ldots, \sigma_n)} = \bigcap \{A_{k_1, \ldots, k_n} : (k_1, \ldots, k_n) \in \sigma_1 \times \ldots \times \sigma_n\}.$$

Now we have (cf. [1, Proposition 2])

$$R = \bigcap_{\sigma \in \mathbb{S}^{\mathbb{N}}} \bigcup_{n=1}^{\infty} B_{\sigma|n}.$$

We claim that for any $c > 0$ we have

$$\{\mu \in \mathfrak{M} : \mu^+(R) < c\} \subset \bigcup_{\sigma \in \mathbb{S}^{\mathbb{N}}} \bigcap_{n=1}^{\infty} \{\mu \in \mathfrak{M} : \mu^+(B_{\sigma|n}) \leq c\} \subset \{\mu \in \mathfrak{M} : \mu^+(R) \leq c\}.$$

Let us show first the second inclusion. Let μ be such that there is $\sigma \in \mathbb{S}^{\mathbb{N}}$ such that for all $n \in \mathbb{N}$ we have $\mu^+(B_{\sigma|n}) \leq c$. Hence $\mu^+\left(\bigcup_{n=1}^{\infty} B_{\sigma|n}\right) \leq c$, as the latter union is by our assumptions monotone. Thus $\mu^+(R) \leq c$.

To prove the first inclusion, we use the notation

$$R_s = \bigcap_{\substack{\nu \in \mathbb{N}^{\mathbb{N}} \\ \nu|k=s}} \bigcup_{n=1}^{\infty} A_{\nu|n}$$

for $s \in \mathbb{N}^k$ and $k \in \mathbb{N}$. Moreover, we put $R_\emptyset = R$ and we have clearly $R_s = \bigcap_{n \in \mathbb{N}} R_{s^\wedge n}$, where $s^\wedge n = (s_1, \ldots, s_k, n)$ is the concatenation of s and n. Since $\mu^+(R) < c$, there is a nonempty finite set σ_1 of positive integers such that $\mu^+\left(\bigcap_{n \in \sigma_1} R_n\right) < c$. Continuing by induction, we choose an infinite sequence $\sigma = (\sigma_n)_{n=1}^{\infty} \in \mathbb{S}^{\mathbb{N}}$ such that

$$\mu^+\left(\bigcap \{R_{k_1, \ldots, k_n} : (k_1, \ldots, k_n) \in \sigma_1 \times \ldots \times \sigma_n\}\right) < c$$

for every n. As $R_s \supset A_s$ for every $s \in \mathbb{N}^n$, we have

$$\bigcap\{R_{k_1,\dots,k_n} : (k_1,\dots,k_n) \in \sigma_1 \times \dots \times \sigma_n\} \supset B_{\sigma|n},$$

and therefore $\mu^+(B_{\sigma|n}) < c$ for every n.

So, we have for all $c \geq 0$

$$\{\mu \in \mathfrak{M} : \mu^+(R) \leq c\} = \bigcap_{k=1}^{\infty} \bigcup_{\sigma \in \mathbb{S}^{\mathbb{N}}} \bigcap_{n=1}^{\infty} \left\{\mu \in \mathfrak{M} : \mu^+(B_{\sigma|n}) \leq c + \frac{1}{k}\right\},$$

which is clearly a Suslin$(\mathscr{V}^c)$ set (notice that $B_{\sigma|n} \in \mathscr{R}$). It follows that $\mathfrak{M}^+(R, c)$ is co-Suslin$(\mathscr{V}^c)$. This completes the proof. ∎

3. Descriptive properties of evaluation functions

In this section we present some results on evaluation functions on "nice" (Borel, Suslin etc.) sets. In particular we will study the behaviour of evaluation functions on the hierarchy of Borel sets. The starting point of this hierarchy is the algebra generated by the open sets of X which will be denoted $\mathfrak{A}$. It is not difficult to prove that every set of $\mathfrak{A}$ can be expressed as a finite disjoint union of intersections of closed an open sets.

Lemma 3.1. *For every $A \in \mathfrak{A}$ the evaluations Ψ_A, Ψ_A^+ and Ψ_A^- are differences of two lower semicontinuous functions on $\mathfrak{M}$.*

Proof. Observe that the differences of two lower semicontinuous functions form a vector space. It is enough to work with Ψ_A^+.

If A is open, then Ψ_A^+ is lower semicontinuous by hypothesis.

If A is the intersection of a closed and an open set, then Ψ_A^+ is the difference of two lower semicontinuous functions. Indeed, put $A = A_1 \setminus A_2$, where $A_2 \subset A_1$ are open sets. Clearly, we have $\Psi_A^+ = \Psi_{A_1}^+ - \Psi_{A_2}^+$.

Finally the general case, if A belongs to the algebra $\mathfrak{A}$, then A is a disjoint finite union of differences of open sets, and thus Ψ_A^+ is a difference of lower semicontinuous functions. ∎

In particular we get that the evaluation Ψ_A is a Borel function on $\mathfrak{M}$ for every $A \in \mathfrak{A}$. We shall show that Ψ_A is Borel on $\mathfrak{M}$ for every Borel subset $A \subset X$. In order to give a precise estimate of measurability we shall use the following classification of Borel sets.

The family of the sets of additive class α $(\mathscr{A}_\alpha)$ and the family of the sets of multiplicative class α $(\mathscr{M}_\alpha)$ are constructed for every countable ordinal α by the following inductive process:

(i) $\mathscr{A}_0 = \mathscr{G}$ (open sets) and $\mathscr{M}_0 = \mathscr{F}$ (closed sets).

(ii) If $\alpha > 0$ then the sets of $\mathscr{A}_\alpha$ are of the form $\bigcup_{n=1}^{\infty}(A_n \cap B_n)$ and the sets of $\mathscr{M}_\alpha$ are of the form $\bigcap_{n=1}^{\infty}(A_n \cup B_n)$, where $A_n \in \bigcup_{\beta<\alpha}\mathscr{A}_\beta$ and $B_n \in \bigcup_{\beta<\alpha}\mathscr{M}_\beta$ for every $n \in \mathbb{N}$.

With the former classification every Borel set has additive and multiplicative classes. The classification coincides with the usual one in the case of metrizable spaces. Notice that $\mathscr{A}_1 = (\mathscr{F} \wedge \mathscr{G})_\sigma$. It is not difficult to show that if $A \in \mathscr{A}_\alpha$ and $\alpha > 1$, then $A = \bigcup_{n=1}^{\infty} A_n$ where $A_n \in \bigcup_{\beta<\alpha} \mathscr{M}_\beta$ for every $n \in \mathbb{N}$.

We can also define a hierarchy of "Borelian sets" in the following way:

(i) $\mathscr{G}_0^a = \mathscr{G}$ (open sets).

(ii) If $\alpha = \beta + 1$ is a successor ordinal, then the sets of $\mathscr{G}_\alpha^a$ are of the form $\bigcup_{n=1}^{\infty} \bigcap_{m=1}^{\infty} A_{n,m}$, where $A_{n,m} \in \mathscr{G}_\beta^a$.

(iii) If α is a limit ordinal then the sets of $\mathscr{G}_\alpha^a$ are of the form $\bigcup_{n=1}^{\infty} A_n$, where $A_n \in \bigcup_{\beta<\alpha} \mathscr{G}_\beta^a$.

The sets from $\mathscr{G}_\alpha^a$ are called Borelian$(\mathscr{G})$ of additive class α. Complements of sets from $\mathscr{G}_\alpha^a$ are called Borelian$(\mathscr{F})$ of multiplicative class α. Their collection is denoted by $\mathscr{F}_\alpha^m$. Similarly we can define Borelian$(\mathscr{G})$ sets of multiplicative classes and Borelian$(\mathscr{F})$ sets of additive classes but we would get no interesting results on these classes therefore we will not consider them. For metrizable spaces we have $\mathscr{G}_\alpha^a = \mathscr{A}_{2\alpha}$ and $\mathscr{F}_\alpha^m = \mathscr{M}_{2\alpha}$ for all $\alpha < \omega_1$. However, in nonmetrizable spaces there can be Borel sets which are neither Borelian$(\mathscr{G})$ not Borelian$(\mathscr{F})$.

Now we are ready to state the results on the evaluation functions for these hierarchies of sets.

Theorem 3.2. *Let $A \subset X$ be a Borel subset of additive class α. Then the sets $\{\mu \in \mathfrak{M} : \Psi_A^+(\mu) > c\}$, $\{\mu \in \mathfrak{M} : \Psi_A^-(\mu) > c\}$ and $\{\mu \in \mathfrak{M} : |\mu|(A) > c\}$ have additive Borel class α in $\mathfrak{M}$ for every $c \in \mathbb{R}$. The evaluation function Ψ_A on $\mathfrak{M}$ has Borel class $\alpha + 1$.*

Theorem 3.3.

1) *Let $A \subset X$ be a Borelian$(\mathscr{G})$ set of additive class α. Then for every $c \geq 0$ the sets $\{\mu \in \mathfrak{M} : \Psi_A^+(\mu) > c\}$, $\{\mu \in \mathfrak{M} : \Psi_A^-(\mu) > c\}$ and $\{\mu \in \mathfrak{M} : |\mu|(A) > c\}$ are Borelian$(\mathscr{G})$ sets of additive class α in $\mathfrak{M}$.*

2) *There is a compact space X and a closed subset $A \subset X$ such that, for all $c \geq 0$, neither of the sets $\{\mu \in \mathfrak{M}_t(X) : \Psi_A^+(\mu) > c\}$, $\{\mu \in \mathfrak{M}_t(X) : \Psi_A^-(\mu) > c\}$ and $\{\mu \in \mathfrak{M}_t(X) : |\mu|(A) > c\}$ is Borelian$(\mathscr{F})$. These sets are even not* Suslin$(\mathscr{F})$.

To prove Theorem 3.2 we shall introduce a generalization of semicontinuity which was already considered in [4] in the frame of Polish spaces. We say that $f : X \to \mathbb{R}$ is α-lower semicontinuous if the set $\{x \in X : f(x) > c\}$ is of additive Borel class α for every $c \in \mathbb{R}$. We will need the following obvious lemma.

Lemma 3.4. *Let f_1 and f_2 be real functions defined on a space X. Then we have*

$$\{x \in X : f_1(x) + f_2(x) > c\} = \bigcup_{r \in \mathbb{Q}} \{x \in X : f_1(x) > c - r\} \cap \{x \in X : f_2(x) > r\}$$

and

$$\{x \in X : f_1(x) - f_2(x) > c\} = \bigcup_{r \in \mathbb{Q}} \{x \in X : f_1(x) > c - r\} \cap \{x \in X : f_2(x) \leq r\}$$

In particular, if f_1 and f_2 are α-lower semicontinuous, then $f_1 + f_2$ is α-lower semicontinuous and $f_1 - f_2$ is $(\alpha + 1)$-lower semicontinuous.

Proposition 3.5. *If A has additive Borel class α in X, then Ψ_A^+ and Ψ_A^- are α-lower semicontinuous on $\mathfrak{M}$. In particular, these functions are Borel of class $\alpha + 1$.*

Proof. It is enough to prove the result for Ψ_A^+. For $\alpha = 0$ the set A is open. In this case Ψ_A^+ is lower semicontinuous by the assumption (*).

For ordinals $\alpha \geq 1$ we shall use induction. In case $\alpha = 1$ we can write $A = \bigcup_{n=1}^{\infty} A_n$ where (A_n) is an increasing sequence of sets from $\mathfrak{A}$. Every $\Psi_{A_n}^+$ is 1-lower semicontinuous by Lemma 3.1 and Lemma 3.4, and so $\mathfrak{M}^+(A_n, c)$ is $(\mathscr{F} \wedge \mathscr{G})_\sigma$. Then by Lemma 2.1 we get that $\mathfrak{M}^+(A, c)$ is $(\mathscr{F} \wedge \mathscr{G})_\sigma$.

For $\alpha > 1$ the result now easily follows by transfinite induction using Lemma 2.1 and Lemma 3.4. ∎

Proof of Theorem 3.2. The statement for the sets $\{\mu \in \mathfrak{M} : \Psi_A^+(\mu) > c\}$ and $\{\mu \in \mathfrak{M} : \Psi_A^-(\mu) > c\}$ follows from Proposition 3.5. For the set $\{\mu \in \mathfrak{M} : |\mu|(A) > c\}$ just observe that $\Psi_A^+ + \Psi_A^-$ is α-lower semicontinuous by Lemma 3.4. Finally notice that $\Psi_A = \Psi_A^+ - \Psi_A^-$, thus (again by Lemma 3.4) Ψ_A and $-\Psi_A$ are $(\alpha + 1)$-lower semicontinuous, and thus Ψ_A is clearly Borel of class $\alpha + 1$. ∎

Proof of Theorem 3.3. 1) The statement for sets $\mathfrak{M}^+(A, c)$ and $\mathfrak{M}^-(A, c)$ follows easily from (*) using Lemma 2.1 and transfinite induction. For the remaining set we can use Lemma 3.4.

2) Put $X = [0, \omega_1]$ and $A = \{\omega_1\}$.

Let us first show that the set $\{\mu \in \mathfrak{M}_t(X) : |\mu|(A) > 0\}$ is not Suslin$(\mathscr{F})$. Suppose, on the contrary it is Suslin$(\mathscr{F})$ in $\mathfrak{M}_t(X)$. Then the set

$$M = \{\mu \in \mathfrak{M}_t(X) : \ \|\mu\| \leq 1, \mu(\{\omega_1\}) \neq 0\}$$

is Suslin$(\mathscr{F})$ in the compact set

$$B = \{\mu \in M_t(X) : \|\mu\| \leq 1\}.$$

So there are closed sets $F_s \subset B$ indexed by finite sequences of positive integers such that $F_s \subset F_t$ whenever t is a beginning of s and

$$M = \bigcup_{\nu \in \mathbb{N}^{\mathbb{N}}} \bigcap_{n=1}^{\infty} F_{\nu|n}.$$

Fix arbitrary $\nu \in \mathbb{N}^{\mathbb{N}}$. Then $\bigcap_{n=1}^{\infty} F_{\nu|n} \subset M$. We claim that $F_{\nu|n} \subset M$ for some $n \in \mathbb{N}$. Suppose not. Choose $\mu_n \in F_{\nu|n} \setminus M$. Then $\mu_n(\{\omega_1\}) = 0$ for all n. Let μ be a cluster point of the sequence μ_n. Then it can be easily checked (using the uncountability of ω_1) that $\mu(\{\omega_1\}) = 0$, hence $\mu \in \bigcap_{n=1}^{\infty} F_{\nu|n} \setminus M$, a contradiction. It follows that M is an $\mathscr{F}_\sigma$-set. But this contradicts [3, Example 6.5].

Next suppose that $\{\mu \in \mathfrak{M}_t(X) : |\mu|(\{\omega_1\}) > c\}$ is Suslin($\mathscr{F}$) for some $c > 0$. But as all these sets are homeomorphic, they are Suslin($\mathscr{F}$) for each $c > 0$. Hence

$$\{\mu \in \mathfrak{M}_t(X) : |\mu|(\{\omega_1\}) > 0\} = \bigcup_{\nu=1}^{\infty} \left\{\mu \in \mathfrak{M}_t(X) : |\mu|(\{\omega_1\}) > \frac{1}{n}\right\}$$

is Suslin($\mathscr{F}$) as well, a contradiction with the already proved case.

Further, if $\mathfrak{M}^+(A, c)$ is Suslin($\mathscr{F}$) for some $c \geq 0$, then $\mathfrak{M}^+(A, 0)$ is clearly Suslin($\mathscr{F}$), too. This set is homeomorphic to $\mathfrak{M}^-(A, 0)$ and hence the latter set is Suslin($\mathscr{F}$) as well. Therefore,

$$\{\mu \in \mathfrak{M}_t(X) : |\mu|(A) > 0\} = \mathfrak{M}^+(A, 0) \cup \mathfrak{M}^-(A, 0)$$

is Suslin($\mathscr{F}$), a contradiction.

Finally note that $\mathfrak{M}^-(A, c)$ is homeomorphic to $\mathfrak{M}^+(A, c)$ and thus it is not Suslin($\mathscr{F}$). ∎

Next we collect results on measurability of evaluation functions on Suslin and co-Suslin sets. By $\mathscr{F}$ we denote, as above, the collection of closed sets, by $\mathscr{B}$ the σ-algebra of Borel sets.

Theorem 3.6. *Let $A \subset X$ be a* co-Suslin($\mathscr{F}$) *(*Suslin($\mathscr{B}$), co-Suslin($\mathscr{B}$)*) set. Then the sets $\{\mu \in \mathfrak{M} : \Psi_A^+(\mu) > c\}$, $\{\mu \in \mathfrak{M} : \Psi_A^-(\mu) > c\}$ and $\{\mu \in \mathfrak{M} : |\mu|(A) > c\}$ are* co-Suslin($\mathscr{F}$) *(*Suslin($\mathscr{B}$), co-Suslin($\mathscr{B}$)*, respectively) for each $c \geq 0$.*

The analogous statement for Suslin($\mathscr{F}$) sets is not valid, see Theorem 3.3.

Proof. The case of Suslin($\mathscr{B}$) follows immediately from Theorem 3.2 and Proposition 2.3 applied for $\mathscr{R}$ and $\mathscr{V}$ being collections of Borel sets. The case of co-Suslin($\mathscr{B}$) follows then from Proposition 2.4 (applied again to collections of Borel sets).

The remaing case of co-Suslin($\mathscr{F}$) sets follows from Proposition 2.4 applied for $\mathscr{R}$ and $\mathscr{V}$ being collections of open sets. ∎

4. Properties of spaces of signed measures

In the following theorem we collect consequences of the results of the previous section for spaces of measures.

Theorem 4.1. *Let K be a compact space and $Y \subset X \subset K$. Consider spaces $\mathfrak{M}_t(Y)$ and $\mathfrak{M}_t(X)$ as topological subspaces of $(C(K)^*, w^*)$.*

(i) *If Y is Borelian($\mathscr{F}$) of multiplicative class α in X, then $\mathfrak{M}_t(Y)$ is Borelian($\mathscr{F}$) of multiplicative class α in $\mathfrak{M}_t(X)$. In particular, if Y is closed or $\mathscr{F}_{\sigma\delta}$ in X, then $\mathfrak{M}_t(Y)$ has the same property in $\mathfrak{M}_t(Y)$.*

(ii) *If Y is a Borel subset of X of multiplicative class α, then $\mathfrak{M}_t(Y)$ has the same property in $\mathfrak{M}_t(X)$. In particular, if Y is $(\mathscr{F} \vee \mathscr{G})_\delta$ in X, then $\mathfrak{M}_t(Y)$ is $(\mathscr{F} \vee \mathscr{G})_\delta$ in $\mathfrak{M}_t(X)$.*

(iii) *If Y is* Suslin$(\mathscr{F})$ *(*Suslin$(\mathscr{B})$ *or* co-Suslin$(\mathscr{B})$*) in X, then $\mathfrak{M}_t(Y)$ has the respective property in $\mathfrak{M}_t(X)$.*

Proof. All the statements follow from results of previous section using the equality

$$\mathfrak{M}_t(Y) = \{\mu \in \mathfrak{M}_t(X) : |\mu|(X \setminus Y) = 0\}. \qquad \blacksquare$$

The previous theorem is an analogue of [1, Theorem 1]. However, the analogy is not complete, as we have the following.

Theorem 4.2. *There is a compact space K and an open subset $X \subset K$ such that $\mathfrak{M}_t(X)$ is not* co-Suslin$(\mathscr{F})$ *in $\mathfrak{M}_t(K)$.*

Proof. This follows from the second part of Theorem 3.3. Recall that we can take $K = [0, \omega_1]$ and $X = [0, \omega_1)$. $\blacksquare$

We can also study absolute descriptive properties. Recall that a space X is absolute Borel (of multiplicative class α) if it is of this kind in every Hausdorff superspace. It is proved in [5] and [2] that a Tychonoff space is absolute Borel of multiplicative class α if it is of this kind in some compactification. Further, a Tychonoff space X is $\mathscr{K}$-analytic (Čech-analytic) if it is Suslin$(\mathscr{F})$ (Suslin$(\mathscr{B})$) in some, or equivalently in any, compactification.

Theorem 4.3. *Let X be a Tychonoff space, K any compactification of X. Consider $\mathfrak{M}_t(X)$ as a topological subspace of $(C(K)^*, w^*)$.*

- *If X is absolute Borel space of multiplicative class α for some $\alpha > 1$, then so is $\mathfrak{M}_t(X)$.*
- *If X is $\mathscr{K}$-analytic, then so is $\mathfrak{M}_t(X)$.*
- *If X is Čech-analytic, then so is $\mathfrak{M}_t(X)$.*

This result is immediate consequence of Theorem 4.1 using the fact that $\mathfrak{M}_t(K)$ is $\mathscr{K}_\sigma$ for K compact. The case of $\mathscr{K}$-analytic spaces was proved already in [1] by another method. Notice that we do not formulate such a result for Borelian$(\mathscr{F})$ sets as these are not an absolute notion by [6] (i.e., there are Tychonoff spaces which are Borelian$(\mathscr{F})$ in some, but not in every, compactification).

We could also study further descriptive properties (similarly as in [1]). If we denote by $\mathscr{H}$ the family of all scattered unions of $(\mathscr{F} \wedge \mathscr{G})$ sets, then $\mathscr{H}$ is an algebra. We can define a hierarchy of additive and multiplicative classes of sets from $\sigma(\mathscr{H})$. Using the natural analogue of [1, Proposition 3] one can show that $\mathfrak{M}^+(A, c) \in \mathscr{H}_\sigma$ whenever $A \in \mathscr{H}$, $c \geq 0$ and all measures from $\mathfrak{M}$ are Radon. Again, we get that $\mathfrak{M}_t(Y)$ is "of multiplicative $\mathscr{H}$-class α" (Suslin$(\mathscr{H})$, co-Suslin$(\mathscr{H})$) in $\mathfrak{M}_t(X)$ whenever Y has the respective property in X. However we will not state these results in detail.

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