Czechoslovak Mathematical Journal

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Czechoslovak Mathematical Journal, Vol. 11 (1961), No. 4, 629–631

Persistent URL: `http://dml.cz/dmlcz/100488`

ON BORELIAN AND BIANALYTIC SPACES

(*Preliminary communication*)

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(Received September 20, 1961)

Spaces which are Borel subsets of some complete metrizable separable space posses important properties; they will be termed Borelian spaces in the classical sense here, and their continuous images, if metrizable, will be called analytic spaces in the classical sense. In the present note the definition of Borelian spaces is extended to the class of all completely regular spaces in such a way that certain deeper theorems remain true.

1. NOTATION AND TERMINOLOGY

All spaces under consideration are supposed to be completely regular. Letters X, Y, T, Z always denote spaces. For any X, $\mathbf{F}(X)$, $\mathbf{Z}(X)$, $\mathbf{K}(X)$ denote, respectively, the family of all closed sets, zero-sets, compact sets contained in X. With a given family $\mathscr{M}$ of subsets of X there are associated the following families of sets: $\mathscr{B}(\mathscr{M})$, the smallest family containing $\mathscr{M}$ and closed under countable unions and countable intersections, and $\mathscr{B}_*(\mathscr{M})$, the smallest family of sets containing $\mathscr{M}$ and closed under countable intersections and countable disjoint unions.

It is clear that, for any X, the family $\mathscr{B}(\mathbf{Z}(X))$ is closed under complementation. By M. KATĚTOV the elements of $\mathscr{B}(\mathbf{Z}(X))$ will be called *Baire sets* of X. If X is metrizable (more generally, if X is perfectly normal), then $\mathbf{F}(X) = \mathbf{Z}(X)$, in particular, $\mathscr{B}(\mathbf{Z}(X) = \mathscr{B}(\mathbf{F}(X))$. Thus for metrizable spaces, Borel sets and Baire sets coincide. In general, the family $\mathscr{B}(\mathbf{F}(X))$ is not closed under complementation.

2. BORELIAN SPACES

Definition 1. A *Borelian structure* in X is a sequence $\{\mathscr{M}_n\}$ of countable disjoint coverings of X (not necessarily open or closed) such that

(a) if $M_n, N_n \in \mathscr{M}_n$ and $M_k \neq N_k$ for some k, then $\bigcap_{n=1}^{\infty} \overline{M}_n \cap \bigcap_{n=1}^{\infty} \overline{N}_n = \emptyset$;

(b) the sequence $\{\mathscr{M}_n\}$ is complete.

Definition 2. A space X will be called *Borelian* if there exists a Borelian structure in X. The family of all Borelian subspaces of a given space X will be denoted by $\mathbf{B}(X)$.

The following assertions (1)—(4) follow directly from the definition.

(1) If X is Borelian, then $\mathbf{F}(X) \subset \mathbf{B}(X)$.

(2) One-to-one continuous images of Borelian spaces are Borelian.

(3) The inverse image under a perfect mapping[1]) of a Borelian space is Borelian.

(4) $$\mathscr{B}_*(\mathbf{B}(X)) = \mathbf{B}(X).$$

Indeed, if $\{\mathscr{M}_n\}$ is a Borelian structure in X and Y is closed in X, then $\{Y \cap \mathscr{M}_n\}$ is a Borelian structure in Y. The proof of (2)—(4) is quite analoguous. If $\{\mathscr{M}_n\}$ is a Borelian structure in X and $X \subset Y$, then for every M in $\mathscr{M}_n$ there exists a $Z(M) \in \in \mathscr{B}(\mathbf{Z}(X))$ with $Z(M) \supset M$ such that the family $\{Z(M); M \in \mathscr{M}_n\}$ is disjoint. From this fact it follows at once

(5) $$\mathbf{B}(X) \subset \mathscr{B}^*(\mathbf{F}(X) \cup \mathscr{B}(\mathbf{Z}(X))).$$

It is easy to see that the space of all rational numbers and the space of all irrational numbers are Borelian spaces. By preceding results from this fact it follows at once that the complement of a zero-set of a compact space is a Borelian space, and finally, we obtain

(6) Every Baire set of a compact space is a Borelian space.

Now it is easy to prove

(7) If X is Borelian, then $\mathbf{B}(X) = \mathscr{B}_*(\mathbf{F}(X) \cup \mathscr{B}(\mathbf{Z}(X)))$.

In general the family $\mathbf{B}(X)$ is not closed under countable unions, moreover, σ-compact space may fail to be a Borelian space. For example, let X be an uncountable compact space with only one accumulation point, say x, and let Y be the discrete space of positive integers. Identifying the points of the subset $(x) \times Y$ of the product space $X \times Y$ we obtain a σ-compact space which is not Borelian.

It is easy to prove that every Borelian space is a Lindelöf space (and hence, a normal one). By (7) from this fact it follows at once:

(8) A metrizable space X is Borelian if and only if X is a Borelian space in the classical sense.

The preceding assertion shows that our definition of Borelian spaces is an extension of the classical one. In the following assertion all Borelian spaces are described in terms of classical Borelian spaces and one-to-one perfect mappings.

(9) A space X is Borelian if and only if X is a one-to-one continuous image of

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[1]) By a perfect mapping is meant a continuous and closed mapping such that the preimages of points are compact.

a space which admits of a perfect mapping onto a closed subspace of the space of all irrational numbers.

The first part of (9) follows from the preceding assertions. To prove the second part, let $\{\mathscr{M}_n\}$ be a Borelian structure in X. Declaring all sets from $\mathscr{M}_n$, $n = 1, 2, \ldots$, open we obtain a space Y and $\{\mathscr{M}_n\}$ remains a Borelian structure in Y. Now identifying the points of sets of the form $\bigcap_{n=1}^{\infty} \overline{M}_n$, where $M_n \in \mathscr{M}_n$, we obtain a closed subspace of the space of all irrational numbers.

3. BIANALYTIC SPACES

In the classical descriptive theory of sets the proofs of certain theorems concerning Borel sets depend on the theory of analytic spaces (in the classical sense). These applications of analytic spaces make use of the fact that a subspace M of a complete metrizable separable space X is a Borelian space if and only if both M and $X - M$ are analytic spaces.

By V. E. Šneider and G. Choquet a space X will be called analytic if X is a continuous image of a space which belongs to $\mathscr{B}(\mathbf{K}(Y))$ for some Y.

Definition 3. A space X will be called bianalytic if both X and $K - X$ are analytic for some compact space $K \supset X$.

The following assertion has been proved by the author:

(10) If X and Y are disjoint analytic subspaces of a space T, then there exists a $B \in \mathscr{B}(\mathbf{Z}(T))$ with $X \subset B \subset T - Y$.

From this result and from the properties of analytic spaces it follows at once:

(11) If K is a compact space then $X \subset K$ is a Baire set in K if and only if both X and $K - X$ are analytic spaces.

It follows:

(12) A space X is bianalytic if and only if X is a Baire set of some compact spaces.

By (7) we have that every bianalytic space is a Borelian one. The class of all bianalytic spaces is not closed under one-to-one continuous mapping. For example, if X is the discrete space of positive integers and if $Y = X \cup (x) \subset \beta(X)$ where $x \in \beta(X) - X$, then Y is a one-to-one continuous image of X; however, X is bianalytic but Y is not. It is easy to show:

(13) A space X is Borelian if and only if X is a one-to-one continuous image of a bianalytic space.

From (10) we can deduce the following internal characterization of Borelian spaces:

(14) A space X is Borelian if and only if there exists a complete sequence $\{\mathscr{M}_n\}$ of countable disjoint coverings of X such that the elements of $\mathscr{M}_n$, $n = 1, 2, \ldots$, are analytic spaces.