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GENERALIZED NORMS OF MATRICES AND THE LOCATION OF THE SPECTRUM

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The authors use norms with values in an ordered vector space to obtain regularity conditions for matrices. These conditions are used further to define regions of the complex plane which contain the whole spectrum of a given matrix.

1. INTRODUCTION

In the present paper we intend to prove some general regularity conditions for matrices and to deduce location theorems for the spectrum by applying these regularity conditions to matrices $\lambda E - A$. The main idea of the present work may be sketched as follows: Consider a linear operator A in an n-dimensional complex vector space X with a direct decomposition $X = X_1 + \ldots + X_r$. The projection operators on X_i will be denoted by P_i. Choose now, in each X_i, a suitable norm g_i and consider the vector

$$p(x) = (g_1(P_1x), g_2(P_2x), \ldots, g_r(P_rx))$$

as a generalised norm on X. The operator A is partitioned into blocks by the decomposition of the identity operator $E = P_1 + P_2 + \ldots + P_r$; we associate with each block a nonnegative number and construct the matrix

$$\hat{p}(A) = \begin{pmatrix} \hat{p}_{11}, & -p_{12}, & -p_{13}, & \ldots \\ -p_{21}, & \hat{p}_{22}, & -p_{23}, & \ldots \\ \ldots & \ldots & \ldots & \ldots \end{pmatrix}.$$

We show further that A is regular if this matrix $\hat{p}(A)$ belongs to **K**, where **K** is an important class of matrices discussed by many authors.

2. TERMINOLOGY AND NOTATION

If X and Y are two finite-dimensional vector spaces over the complex field, we shall denote by $L(X, Y)$ the vector space of all linear transformations from X into Y. If $A \in L(X, Y)$ we denote by $N(A)$ the kernel of A, i.e. the set of all $x \in X$ such that $Ax = 0$. We denote by $R(A)$ the range of A, i.e. the set of all vectors of the form Ax

with $x \in X$. Both $N(A)$ and $R(A)$ are linear subspaces of X and Y. If $X = Y$, the elements of $L(X, X)$ will be called linear operators in X. An operator $P \in L(X, X)$ is called a projection if $P^2 = P$. If P is a projection in X, put $Q = E - P$ where E is the identity operator. It is easy to see that $Q^2 = Q$ and $PQ = QP = 0$. Further $R(P) = N(Q)$ and $R(Q) = N(P)$. If these subspaces are denoted by X_1 and X_2 respectively, then X_1 and X_2 form a direct decomposition of X, in other words, each $x \in X$ may be written in exactly one way in the form $x = x_1 + x_2$ with $x_i \in X_i$.

Let r be a positive integer and let V be the real linear space of all vectors of the form $x = (x_1, x_2, \ldots, x_r)$ with real coordinates x_i. A vector x is said to be nonnegative if $x_i \geqq 0$ for each i. We write then $x \geqq 0$. Similarly, a matrix $W = (w_{ik})$ of order n is said to be nonnegative if $w_{ik} \geqq 0$ for each i and k.

Further, we shall use matrices of two special classes $\boldsymbol{K}$ and $\boldsymbol{K}_0$. The class $\boldsymbol{K}$ ($\boldsymbol{K}_0$ resp.) contains all real square matrices whose off-diagonal elements are non-positive and all principal minors positive (non-negative resp.). The fundamental properties of these matrices may be found in the recent paper of the authors' [3]. This paper is in a close connection with the present one and will be here referred to several times.

Now, let us turn to the case when normed spaces are considered. Let g and h be norms in X and Y respectively, we define a norm $p = \tau(g, h)$ in $L(X, Y)$ in the following manner. If $A \in L(X, Y)$, we put

$$p(A) = \sup\left(h(Ax);\ g(x) \leqq 1\right).$$

This is the usual norm of a linear transformation. If $X = Y$, we have the case of linear operators in X; it is then customary to write simply g for $\tau(g, g)$. If Y is the real line E_1, we have the case of linear functionals on X. The norm $\tau(g, |\cdot|)$ on $L(X, E_1) = X'$ is called the *adjoint norm* of g and will be denoted by g'. Thus

$$g'(y') = \sup\left(|\langle x, y'\rangle|;\ g(x) \leqq 1\right).$$

In a recent paper [4], the authors have introduced a useful concept of a "reciprocal norm"; let us recall its definition. If X and Y are linear spaces with norms g and h, we define a function $q = \hat{\tau}(g, h)$ in the following manner: if $A \in L(X, Y)$, we put

$$q(A) = \inf\left(h(Ax);\ g(x) \geqq 1\right).$$

Clearly we have $q(A) = 0$ if A is singular. If A is regular, it is easy to show that $q(A) = (p(A^{-1}))^{-1}$, where $p = \tau(h, g)$ on $L(Y, X)$. If $X = Y$, we write simply $\hat{g}$ for $\hat{\tau}(g, g)$ in conformity with the convention already introduced for matrices.

(2,1) *Let $A, B \in L(X, X)$ and let g be a norm in X. Then*

$$g(AB) \geqq \hat{g}(A)\, g(B)\,, \quad g(AB) \geqq g(A)\, \hat{g}(B)\,.$$

Proof. If $\hat{g}(A) = 0$, the first inequality is obvious. Suppose that $\hat{g}(A) \neq 0$ so that $\hat{g}(A) = (g(A^{-1}))^{-1}$. We have then

$$g(B) = g(A^{-1}AB) \leqq g(A^{-1})\, g(AB)$$

which proves the first inequality. The proof of the other estimate is analogous.

(2,2) *Let $A, B \in L(X, X)$ and let g be a norm in X. Then*

$$\hat{g}(A + B) \leqq \hat{g}(A) + g(B)\,.$$

Proof. There exists a vector x_0 such that $g(x_0) = 1$ and $g(Ax_0) = \hat{g}(A)$. We have then

$$\hat{g}(A + B) \leqq g((A + B)\,x_0) \leqq g(Ax_0) + g(Bx_0) =$$
$$= \hat{g}(A) + g(Bx_0) \leqq \hat{g}(A) + g(B)\,g(x_0) = \hat{g}(A) + g(B)$$

and the proof is complete.

3. REGULARITY CONDITIONS FOR MATRICES

In this section, we intend to give some general criteria for the regularity of a matrix based on the use of generalized norms whose values are nonnegative vectors of an r-dimensional linear space. We shall also use some properties of the class $\boldsymbol{K}$ which may be found in [3].

Let n be a fixed positive integer and let X be an n-dimensional vector space over the complex field. Let r be a positive integer and let $P_1, \ldots, P_r$ be projectors in X such that $P_iP_j = 0$ for $i \neq j$ and that their sum $P_1 + \ldots + P_r$ equals the identity operator in X. Let us denote by X_i the spaces P_iX. Let g_i be a norm on X_i. Consider further the space V of all r-tuples of real numbers. The set of indices $1, 2, \ldots, r$ will be denoted by R.

With each $x \in X$, let us associate a vector $p(x) \in V$ in the following manner:

$$p(x) = (g_1(P_1x), g_2(P_2x), \ldots, g_r(P_rx))\,.$$

It is not difficult to verify that the mapping p has the following properties:

1° $p(x) \geqq 0$ for each $x \in X$ and $p(x) = 0$ if $x = 0$,

2° $p(x + y) \leqq p(x) + p(y)$,

3° $p(\lambda x) = |\lambda|\, p(x)$ for every complex λ,

4° if $p(x) = u + v$ where $u, v \in V$ and $u \geqq 0$, $v \geqq 0$, then there exist y and z in X such that $x = y + z$ and $p(y) = u$, $p(z) = v$.

It is sufficient to prove property 4° only. Take an $x \in X$ and suppose that $p(x) = u + v$ with $u \geqq 0$ and $v \geqq 0$. Consider a fixed $i \in R$. We have $g_i(P_ix) = u_i + v_i$. If $g_i(P_ix) = 0$, put $y_i = z_i = 0$. If $w_i = g_i(P_ix) > 0$, put

$$y_i = \frac{u_i}{w_i} P_ix \quad \text{and} \quad z_i = \frac{v_i}{w_i} P_ix\,.$$

Clearly $y_i + z_i = P_ix$ in both cases so that we have $y + z = x$ and $p(y) = u$, $p(z) = v$ if $y = y_1 + \ldots + y_r$ and $z = z_1 + \ldots + z_r$.

The mapping p will be called a *generalized norm* on X.

Definition. Let A be an operator $A \in L(X, X)$. A non-negative matrix G of order r is said to be an upper bound for A if

$$p(Ax) \leqq Gp(x)$$

for each $x \in X$.

We have the following result:

(3,1) Theorem. *Let $A \in L(X, X)$ and put*

$$p_{ij} = \sup \{g_i(P_iAP_jx);\ g_j(P_jx) \leqq 1\}$$

for each $i, j \in R$.

Let us denote by $p(A)$ the matrix with elements p_{ij}. Then $p(A)$ is an upper bound for A such that $p(A) \leqq G$ for every upper bound G of A.

Proof. Take an $x \in X$ and an $i \in R$. Since $x = \sum_{j \in R} P_jx$, we have

$$g_i(P_iAx) = g_i(\sum_{j \in R} P_iAP_jx) \leqq \sum_{j \in R} p_{ij}g_j(P_jx)$$

whence $p(Ax) \leqq p(A)\, p(x)$ so that $p(A)$ is an upper bound for A. Further, let $G = (g_{ij})$ be an upper bound for A and let i, j be fixed indices in R. We intend to show that $p_{ij} \leqq g_{ij}$. If $x \in X$ is arbitrary, we have, G being an upper bound for A,

$$g_i(P_iAP_jx) \leqq \sum_{k \in R} g_{ik}g_k(P_kP_jx) = g_{ij}g_j(P_jx)\,.$$

It follows that $g_i(P_iAP_jx) \leqq g_{ij}$ whenever $g_j(P_jx) \leqq 1$, so that $p_{ij} \leqq g_{ij}$ according to the definition of p_{ij}.

Definition. Let A be an operator $A \in L(X, X)$. A matrix H is said to be a lower bound for A if

$$p(Ax) \geqq Hp(x)$$

for each $x \in X$.

We shall associate with every $A \in L(X, X)$ a special lower bound which will be important in further applications.

(3,2) *Let $A \in L(X, X)$ and put*

$$\hat{p}_{ii} = \inf \{g_i(P_iAP_ix);\ g_i(P_ix) \geqq 1\}$$

for each $i \in R$. Let us denote by $\hat{p}(A)$ the matrix

$$\begin{pmatrix} \hat{p}_{11}, & -p_{12}, & -p_{13}, & \ldots, & -p_{1r} \\ -p_{21}, & \hat{p}_{22}, & -p_{23}, & \ldots, & -p_{2r} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ -p_{r1}, & -p_{r2}, & -p_{r3}, & \ldots, & \hat{p}_{rr} \end{pmatrix}.$$

Then $\hat{p}(A)$ is a lower bound for A.

Proof. Take an $x \in X$. We have for the i-th coordinate

$$[p(Ax)]_i = g_i(P_i Ax) = g_i(\sum_{j \in R} P_i A P_j x) \geqq$$

$$\geqq g_i(P_i A P_i x) - \sum_{j \neq i} g_i(P_i A P_j x) \geqq \hat{p}_{ii} g_i(P_i x) - \sum_{j \neq i} p_{ij} g_j(P_j x) = [\hat{p}(A)\, p(x)]_i$$

which completes the proof.

(3,3) Theorem. *Let $A \in L(X, X)$ and let H be a lower bound for A such that $H \in \mathbf{K}$. Then A is regular. Further, $p(A^{-1}) \leqq H^{-1}$.*

Proof. Take an $x \in X$ such that $Ax = 0$. Then $0 \geqq p(Ax) \geqq Hp(x)$. The matrix H being of class $\mathbf{K}$, we have[1]) $H^{-1} \geqq 0$ whence $0 \geqq p(x)$ so that $p(x) = 0$, and, consequently, $x = 0$. Since $p(Ax) \geqq Hp(x)$ for each $x \in X$, we have $H^{-1}p(y) \geqq p(A^{-1}y)$ for each $y \in X$ so that $H^{-1} \geqq p(A^{-1})$ according to (3,1).

(3,4) *Let $A_1, A_2 \in L(X, X)$ and $A = A_1 + A_2$. Let H be a lower bound for A_1 and G an upper bound for A_2. If $H - G \in \mathbf{K}$ then A is regular.*

Proof. According to the preceding theorem, it is sufficient to show that $H - G$ is a lower bound for A. To see that, take an arbitrary $x \in X$. Since $A = A_1 + A_2$, we have

$$p(Ax) \geqq p(A_1 x) - p(A_2 x) \geqq Hp(x) - Gp(x) = (H - G)\, p(x)$$

and the proof is complete.

(3,5) *Let $A_1, A_2 \in L(X, X)$ and $A = A_1 + A_2$. Let H be a lower bound for A_1 and G an upper bound for A_2. Suppose that $H \in \mathbf{K}$. If $E - H^{-1}G \in \mathbf{K}$ then A is regular.*

Proof. We have $H - G = H(E - H^{-1}G)$ and both H and $E - H^{-1}G$ have nonnegative inverses. It follows that $H - G$ has a nonnegative inverse as well. Since $H \in \mathbf{K}$ and $G \geqq 0$, we have[2]) $H - G \in \mathbf{K}$ and our assertion follows from the preceding theorem.

For the special lower bound $\hat{p}$, we have the following estimate:

(3,6) *Let $A, B \in L(X, X)$. Then*

$$\hat{p}(A + B) \geqq \hat{p}(A) - p(B).$$

Proof. If i is a fixed index, we have

$$\hat{p}_{ii}(A + B) \geqq \hat{p}_{ii}(A) - p_{ii}(B)$$

by (2,2). If $i \neq j$, we have clearly

$$p_{ij}(A + B) \leqq p_{ij}(A) + p_{ij}(B)$$

and the proof is complete.

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[1]) [3], th. 4,3.

[2]) [3], th. (4,3).

4. THE LOCATION OF THE SPECTRUM OF A MATRIX

In this section we use the regularity conditions for matrices to obtain estimates of the spectrum. The notation is the same as in the preceding section. First, we shall prove two lemmas.

(4,1) *Let ξ_1, ξ_2, σ, d be real numbers such that $d \geqq 0$ and $\xi_i \geqq \sigma - \sqrt{d} \geqq 0$ for $i = 1, 2$ and $\sigma \leqq \frac{1}{2}(\xi_1 + \xi_2)$. Then $\xi_1\xi_2 \geqq \sigma^2 - d$. Moreover, if $\xi_i > \sigma - \sqrt{d}$ for $i = 1, 2$, then $\xi_1\xi_2 > \sigma^2 - d$.*

Proof. We may clearly assume $\xi_1 \geqq \xi_2$. Suppose that $\xi_1\xi_2 < \sigma^2 - d$; it follows that

$$\sigma^2 - d > \xi_1\xi_2 = \tfrac{1}{4}(\xi_1 + \xi_2)^2 - \tfrac{1}{4}(\xi_1 - \xi_2)^2 \geqq \sigma^2 - \tfrac{1}{4}(\xi_1 - \xi_2)^2 .$$

Hence $\xi_1 - \xi_2 > 2\sqrt{d}$. Now,

$$\xi_1\xi_2 = [\tfrac{1}{2}(\xi_1 + \xi_2) + \tfrac{1}{2}(\xi_1 - \xi_2)]\, \xi_2 \geqq (\sigma + \sqrt{d})(\sigma - \sqrt{d}) = \sigma^2 - d > \xi_1\xi_2 .$$

This contradiction proves the first part of the lemma. The rest is obvious.

(4,2) *Let η_1, η_2, σ be non-negative real numbers such that $\sigma > \frac{1}{2}(\eta_1 + \eta_2)$. Then $\sigma^2 > \eta_1\eta_2$ and for $i = 1, 2$ $\sigma - \sqrt{\sigma^2 - \eta_1\eta_2} \leqq \eta_i \leqq \sigma + \sqrt{\sigma^2 - \eta_1\eta_2}$.*

Proof. Since $\sigma > \frac{1}{2}(\eta_1 + \eta_2) \geqq \sqrt{\eta_1\eta_2}$, we have $\sigma^2 - \eta_1\eta_2 > 0$. Let i be one of the indices 1,2. If $\eta_i = 0$, the inequalities considered are valid. Let thus $\eta_i > 0$. Then $\eta_i(2\sigma - \eta_i) > \eta_1\eta_2$, whence $(\eta_i - \sigma)^2 = \sigma^2 - \eta_i(2\sigma - \eta_i) < \sigma^2 - \eta_1\eta_2$. Consequently, $|\eta_i - \sigma| < \sqrt{\sigma^2 - \eta_1\eta_2}$ which completes the proof.

(4,3) Theorem. *Let $A \in L(X, X)$,*

$$p_{ij} = \sup_x \{g_i(P_iAP_jx);\ g_j(P_jx) \leqq 1\}\ , \quad (i, j \in R)\ ,$$

$$\hat{p}_{ii}(\lambda) = \inf_x \{g_i[P_i(A - \lambda E)\, P_ix];\ g_i(P_ix) \geqq 1\}$$

for every complex number λ.

Let $c_1, c_2, \ldots, c_r$ be real numbers such that the matrix

$$M = \begin{pmatrix} c_1, & -p_{12}, \ldots, & -p_{1r} \\ -p_{21}, & c_2, \quad \ldots, & -p_{2r} \\ \ldots & \ldots & \ldots \\ -p_{r1}, & -p_{r2}, \ldots, & c_r \end{pmatrix}$$

belongs to $\mathbf{K}_0$. Let $C_i\,(i \in R)$ be the set of all complex numbers z such that $\hat{p}_{ii}(z) \leqq c_i$. Then all proper values of A are contained in $\bigcup_{i \in R} C_i$.

Proof. Let λ non $\in \bigcup_{i \in R} C_i$. Then $\hat{p}_{ii}(\lambda) > c_i$ for each $i \in R$. By (4,6) of [3], the matrix

$$\hat{p}(A = \lambda E) = \begin{pmatrix} \hat{p}_{11}(\lambda), & -p_{12}, & \ldots, & p_{1r} \\ -p_{21}, & \hat{p}_{22}(\lambda), & \ldots, & p_{2r} \\ \ldots & \ldots & \ldots & \ldots \\ -p_{r1}, & -p_{r2}, & \ldots, & \hat{p}_{rr}(\lambda) \end{pmatrix}$$

belongs to $\mathbf{K}$ and, since $\hat{p}(A - \lambda E)$ is a lower bound for $A - \lambda E$ according to (3,2), $A - \lambda E$ is regular by (3,4). Thus all proper values of A are contained in $\bigcup_{i \in R} C_i$.

(4,4) Theorem. *Let* $A \in L(X, X)$,

$$p_{ij} = \sup \{g_i(P_i A P_j x);\ g_j(P_j x) \leqq 1\}$$

for $i, j \in R$. *Let* $c_1, c_2, \ldots, c_r$ *be real numbers such that the matrix*

$$M = \begin{pmatrix} c_1, & -p_{12}, & \ldots, & -p_{1r} \\ -p_{21}, & c_2, & \ldots, & -p_{2r} \\ \ldots & \ldots & \ldots & \ldots \\ -p_{r1}, & -p_{r2}, & \ldots, & c_r \end{pmatrix}$$

belongs to $\mathbf{K}_0$. *Let*

$$\hat{p}_{ii}(\lambda) = \inf \{g_i[P_i(A - \lambda E) P_i x];\ g_i(P_i x) \geqq 1\}.$$

For $i, j \in R$, $i \neq j$, *let* C_{ij} *denote the set of those complex numbers* z *which satisfy the inequality*

$$\hat{p}_{ii}(z)\, \hat{p}_{jj}(z) \leqq c_i c_j .$$

Then, each proper value of A *is contained in some* C_{ij} $(i \neq j)$.

Proof. Let λ be a complex number fulfilling the conditions λ non $\in C_{ij}$ for all $i, j \in R$, $i \neq j$. Then

$$\hat{p}_{ii}(\lambda)\, \hat{p}_{jj}(\lambda) > c_i c_j \quad (i \neq j) .$$

According to (4,10) of [3], the matrix

$$\hat{p}(A - \lambda E) = \begin{pmatrix} \hat{p}_{11}(\lambda), & -p_{12}, & \ldots, & -p_{1r} \\ -p_{21}, & \hat{p}_{22}(\lambda), & \ldots, & -p_{2r} \\ \ldots & \ldots & \ldots & \ldots \\ -p_{r1}, & -p_{r2}, & \ldots, & \hat{p}_{rr}(\lambda) \end{pmatrix}$$

belongs to $\mathbf{K}$. By (3,2) and (3,4), $A - \lambda E$ is regular. The proof is complete.

(4,5) Theorem. *Let* $A, D, B \in L(X, X)$, $A = D + B$, *and let* $P_i D P_j = 0$ *for* $i \neq j$, $i, j \in R$. *For* $i \in R$ *and* λ *complex, let us denote by* $\hat{p}_{ii}(\lambda)$ *the number* $\inf \{g_i[P_i(D - \lambda E) P_i x];\ g_i(P_i x) \geqq 1\}$, let $p_{ij} = \sup \{g_i(P_i B P_j x);\ g_j(P_j x) \leqq 1\}$, $i, j \in R$. *Let real numbers* $c_1, c_2, \ldots, c_r$ *satisfy the following conditions*:

1° *the matrix*

$$\begin{pmatrix} c_1 - p_{11}, & -p_{12}, & \ldots, & -p_{1r} \\ -p_{21}, & c_2 - p_{22}, & \ldots, & -p_{2r} \\ \ldots & \ldots & \ldots & \ldots \\ -p_{r1}, & -p_{r2}, & \ldots, & c_r - p_{rr} \end{pmatrix}$$

belongs to $\mathbf{K}_0$.

2° *for an index* $i \in R$, *for all* $l \neq i$ $(l \in R)$ *and all complex* λ *there exist numbers* h_{il} *such that*

$$c_i + c_l < h_{il} \leqq \hat{p}_{ii}(\lambda) + \hat{p}_{ll}(\lambda) .$$

If C_i^* is the set of all complex numbers z such that

$$\hat{p}_{ii}(z) \leqq \tfrac{1}{2} \max_{l \neq i} \left[h_{il} + p_{ii} - p_{ll} - \sqrt{(h_{il} - p_{ii} - p_{ll})^2 - 4(c_i - p_{ii})(c_l - p_{ll})}\right],$$

$C_j(j \in R)$ the set of those complex z satisfying $\hat{p}_{jj}(z) \leqq c_j$ and $C = \bigcup_{i \neq l} C_l$, then

1° $C_i^* \subset C_i$; 2° $C_i \cap C = \emptyset$;

3° *if d_i is the dimension of $P_i X$ then exactly d_i proper values (with corresponding multiplicities) of A are contained in C_i^*, all remaining proper values of A being contained in C.*

Proof. If we put $\sigma = h_{il} - p_{ii} - p_{ll}$, $\eta_1 = c_i - p_{ii}$, $\eta_2 = c_l - p_{ll}$ in (4,2), we obtain that

$$\tfrac{1}{2}\left[h_{il} + p_{ii} - p_{ll} - \sqrt{(h_{il} - p_{ii} - p_{ll})^2 - 4(c_i - p_{ii})(c_l - p_{ll})}\right] \leqq c_i .$$

Thus, $C_i^* \subset C_i$. Suppose that $\lambda \in C_i \cap C$. Then there exists an index $l \neq i$ such that $\lambda \in C_l$. According to assumption 2°,

$$c_i + c_l < \hat{p}_{ii}(\lambda) + \hat{p}_{ll}(\lambda) \leqq c_i + c_l$$

which is a contradiction. Thus, $C_i \cap C = \emptyset$. It remains to prove 3°. Let λ non $\in$ non $\in C_i^* \cup C$. Let $j, k \in R, j \neq k$. We shall prove that

$$(\hat{p}_{jj}(\lambda) - p_{jj})(\hat{p}_{kk}(\lambda) - p_{kk}) > (c_j - p_{jj})(c_k - p_{kk}) .$$

Let us distinguish two cases:

a) Let $\hat{p}_{jj}(\lambda) > c_j$, $\hat{p}_{kk}(\lambda) > c_k$; then the inequality considered is fulfilled since $c_m - p_{mm} \geqq 0$ for $m \in R$ according to assumption 1°.

b) Let one of the indices j, k, say j, satisfy $\hat{p}_{jj}(\lambda) \leqq c_j$. Then, since λ non $\in C$, $j = i$ and (λ non $\in C_i^*$)

$$\tfrac{1}{2}\left[h_{ik} + p_{ii} - p_{kk} - \sqrt{(h_{ik} - p_{ii} - p_{kk})^2 - 4(c_i - p_{ii})(c_k - p_{kk})}\right] < \\ < p_{ii}(\lambda) \leqq c_i .$$

Now, put for a moment,

$$\hat{p}_{ii}(\lambda) - p_{ii} = \xi_1, \quad \hat{p}_{kk}(\lambda) - p_{kk} = \xi_2, \quad c_i - p_{ii} = \eta_1, \quad c_k - p_{kk} = \eta_2,$$
$$\tfrac{1}{2}(h_{ik} - p_{ii} - p_{kk}) = \sigma, \quad \tfrac{1}{4}(h_{ik} - p_{ii} - p_{kk})^2 - (c_i - p_{ii})(c_k - p_{kk}) = d .$$

Thus, $\eta_1 \geqq \xi_1 > \sigma - \sqrt{d}$. By (4,2), $\eta_2 \geqq \sigma - \sqrt{d}$ since $\sigma > \frac{1}{2}(\eta, + \eta_2)$. If follows that $\xi_2 \neq \sigma - \sqrt{d}$. But

$$\sigma = \tfrac{1}{2}(h_{ik} - p_{ii} - p_{kk}) \leqq \tfrac{1}{2}(\hat{p}_{ii}(\lambda) - p_{ii}) + \tfrac{1}{2}(\hat{p}_{kk}(\lambda) - p_{kk}) = \tfrac{1}{2}(\xi_1 + \xi_2)$$

according to our assumption. By (4,1),

$$(\hat{p}_{ii}(\lambda) - p_{ii})(\hat{p}_{kk}(\lambda) - p_{kk}) = \xi_1 \xi_2 > \sigma^2 - d = (c_i - p_{ii})(c_k - p_{kk})$$

and the inequality considered is valid, too.

By theorem (4,10) of [3], the matrix

$$M_1 = \begin{pmatrix} \hat{p}_{11}(\lambda) - p_{11}, & -p_{12}, & \dots, & -p_{1r} \\ -p_{21}, & \hat{p}_{22}(\lambda) - p_{22}, \dots, & & -p_{2r} \\ \dots & \dots & \dots & \dots \\ -p_{r1}, & -p_{r2}, & \dots, & \hat{p}_{rr}(\lambda) - p_{rr} \end{pmatrix}$$

belongs to $\boldsymbol{K}$. From (3,6), applied for $D - \lambda E$ and B, it follows that $\hat{p}(A - \lambda E) \geqq \geqq M_1$. Consequently,[3]) $\hat{p}(A - \lambda E) \in \boldsymbol{K}$. By (3,2) and (3,4), $A - \lambda E$ is regular. Thus, all proper values of A are contained in $C_i^* \cup C$, the sets C_i^* and C being disjoint.

Now, let $0 \leqq \xi \leqq 1$ and let us denote by $A(\xi)$ the matrix $D + \xi B$. Since the matrix

$$M(\xi) = \begin{pmatrix} c_1 - \xi p_{11}, & -\xi p_{12}, & \dots, & -\xi p_{1r} \\ -\xi p_{21}, & c_2 - \xi p_{22}, & \dots, & -\xi p_{2r} \\ \dots & \dots & \dots & \dots \\ -\xi p_{r1}, & -\xi p_{r2}, & \dots, & c_r - \xi p_{rr} \end{pmatrix}$$

belongs to K_0, the assumptions of our theorem are valid for $A(\xi) = D + \xi B$ as well. The sets C_j and C being independent of ξ, all proper values of $A(\xi)$ are contained in $C_i \cup C$. Since C_i is disjoint from C, a continuity argument yields that C_i contains the same number of proper values of $A(0) = D$ and $A(1) = A$. Since $P_u D P_v = 0$ for $u \neq v$, it follows that $D - \lambda E$ singular if and only if $P_j D P_j - \lambda P_j$ is singular in $P_j X$ for some index $j \in R$. Now, $P_i D P_i - \lambda P_i$ is singular in $P_i X$ for exactly d_i numbers λ (each considered with its multiplicity); all of them are contained in C_i since $\hat{p}_{ii}(\lambda) = 0$ for these values of λ. If $j \neq i$, $P_j D P_j - \lambda P_j$ is regular in $P_j X$ for λ non $\in C_j$. But C_i is disjoint from C_j, and consequently, exactly d_i proper values of D are contained in C_i. Since $C_i - C_i^*$ contains no proper values of A, the proof is complete.

In [4], the notions of tensor products have been used to obtain more convenient estimates. We shall recall the most important properties specialized for matrices and l_p-norms. These norms can be easily evaluated if $p = 1, 2$, and ∞.

If A and $B = (b_{ij})$ are matrices, then $A \otimes B$ denotes the partitioned matrix

$$\begin{pmatrix} Ab_{11}, & Ab_{12}, & \dots \\ Ab_{21}, & Ab_{22}, & \dots \\ \dots & \dots & \dots \end{pmatrix}.$$

Especially, if a, $b = (b_i)$ are column vectors with n_1, n_2 respectively rows, i.e. in linear spaces X_1, X_2 of dimensions n_1, n_2 respectively, then all column vectors spanned by tensor products

$$a \otimes b = \begin{pmatrix} ab_1 \\ ab_2 \\ \dots \end{pmatrix}$$

[3]) [3], th. 4,6.

with $a \in X_1$ and $b \in X_2$ form a linear space X_3 of dimension $n_3 = n_1 n_2$. If $x \in X_i$ $(i = 1, 2, 3)$ with coordinates $x_1, x_2, \ldots, x_{n_i}$ and $p \geqq 1$ then the l_p-norm of x is

$$g_{i,p}(x) = \Big(\sum_{j=1}^{n_i} |x_j|^p\Big)^{1/p}.$$

It is easy to see that if A and B are square matrices of orders n_1, n_2 respectively, $a \in X_1$ and $b \in X_2$ vectors, then

$$(A \otimes B)(a \otimes b) = Aa \otimes Bb\,, \quad g_{3,p}(a \otimes b) = g_{1,p}(a)\, g_{2,p}(b)\,.$$

If we define as usually the norm of a matrix $C \in L(X_i, X_i)$

$$g_{i,p}(C) = \sup_{x \in X_i} \{g_{i,p}(Cx)\,; \quad g_{i,p}(x) \leqq 1\}$$

and the "reciprocal norm"

$$\hat{g}_{i,p}(C) = \inf_{x \in X_i} \{g_{i,p}(Cx)\,; \quad g_{i,p}(x) \geqq 1\}\,,$$

$i = 1, 2, 3$, then the following theorem follows from theorem (6, 2) of [4]:

(4,6) *If A_i $(i = 1, 2)$ are square matrices of orders n_i and E_i identity matrices of orders n_i, then*

$$\hat{g}_{3,p}(A_1 \otimes E_2 - E_1 \otimes A_2) \leqq \hat{g}_{1,p}(A_1 - \lambda E_1) + \hat{g}_{2,p}(A_2 - \lambda E_2)$$

for every complex number λ.

Remark. This inequality can be used in theorem (4, 5) specialized for matrices and l_p-norms since it is possible to choose

$$h_{il} = \hat{p}(D_i \otimes E_l - E_i \otimes D_l)$$

where D_k is the corresponding matrix of $P_k D P_k$.

(4,7) *If in (4, 6) $A_2 = c_2 E_2$, then*

$$\hat{g}_{3,p}(A_1 \otimes E_2 - E_1 \otimes A_2) = \hat{g}_{1,p}(A_1 - c_2 E_1)\,;$$

if $A_1 = c_1 E_1$, then

$$\hat{g}_{3,p}(A_1 \otimes E_2 - E_1 \otimes A_2) = \hat{g}_{2,p}(A_2 - c_1 E_2)\,.$$

Proof. Let $A_2 = c_2 E_2$. Then

$$\hat{g}_{3,p}(A_1 \otimes E_2 - E_1 \otimes A_2) = \hat{g}_{3,p}[(A_1 - c_2 E_1) \otimes E_2]\,.$$

But $g_{3,p}(B_1 \otimes E_2) = g_{1,p}(B_1)$ since the l_p-norm has the property that the norm of a direct sum $g \begin{pmatrix} C_1, & 0 \\ 0, & C_2 \end{pmatrix}$ with C_i square is equal to $\max[g(C_1), g(C_2)]$. Thus, $\hat{g}_{3,p}(B_1 \otimes E_2) = \hat{g}_{1,p}(B_1)$ as well and the first part is proved. The proof of the second part is analogous.

5. SPECIAL CASES

In this section we shall specialize the results obtained in section 4 and we intend to show that the well known theorems of S. A. GERSHGORIN, A. OSTROWSKI, KY FAN and A. BRAUER are contained therein.

The first special case is obtained if $r = n$, i.e. if the dimensions of $P_i X$ are all equal to 1. Then, for $A = (a_{ij})$, $p_{ij} = |a_{ij}|$, $\hat{p}_{ii}(\lambda) = |a_{ii} - \lambda|$.

Put $p_i = \sum_{j \neq i} |a_{ij}|$, $q_i = \sum_{j \neq i} |a_{ji}|$ and consider the matrix

$$W = \begin{pmatrix} c_1, & -|a_{12}|, & \ldots, & -|a_{1n}| \\ -|a_{21}|, & c_2, & \ldots, & -|a_{2n}| \\ \ldots & \ldots & \ldots & \ldots \\ -|a_{n1}|, & -|a_{n2}|, & \ldots, & c_n \end{pmatrix}.$$

We have shown in (6,8) of [3] that $W \in \mathbf{K}_0$ if $c_i = p_i$ for all $i \in \{1, 2, \ldots, n\} = N$ or if $c_i = q_i$ for all $i \in N$. Similarly, $W \in \mathbf{K}_0$ if $c_i = p_i^\alpha q_i^{1-\alpha}$ for some $0 < \alpha < 1$. The preceding remarks and the theorems (4,3), (4,4) and (4,6) of the present paper yield immediately the well-known estimates of S. A. Gershgorin [5], A. Ostrowski [6] and A. Brauer [1]:

(5,1) *Let A be a matrix and let $p_i = \sum_{j \neq i} |a_{ij}|$, $q_i = \sum_{j \neq i} |a_{ji}|$. Then the spectrum of A is contained in each of the following subsets of the complex plane:*

(1) *the union of the circular disks $|a_{ii} - \lambda| \leqq p_i$ for $i \in N$;*

(2) *the union of the circular disks $|a_{ii} - \lambda| \leqq q_i$ for $i \in N$;*

(3) *the union of the circular disks $|a_{ii} - \lambda| \leqq p_i^\alpha q_i^{1-\alpha}$ for $i \in N$ and any (fixed) α, $0 < \alpha < 1$;*

(4) *the union of all ovals*

$$|a_{ii} - \lambda| \, |a_{jj} - \lambda| \leqq p_i^\alpha p_j^\alpha q_i^{1-\alpha} q_j^{1-\alpha}$$

for $i, j \in N$ and $i \neq j$, where α is a number, $0 \leqq \alpha \leqq 1$.

Moreover, if $|a_{ii} - a_{jj}| > p_i + p_j$ for all $i, j \in N$, $i \neq j$, then each circular region $|a_{ii} - \lambda| \leqq p_i$ contains exactly one proper value of A.

Further, a combination of theorem (4, 3) of the present paper and of theorem (6,4) of [3] yields immediately the following estimate which is a generalization of a result of A. Ostrowski [6].

(5,2) *Let $A = (a_{ij})$ be a matrix, let $p > 1$ and put $g_i = \sum_{j \neq i} |a_{ij}|$, $g_i(p) = \left(\sum_{j \neq i} |a_{ij}|^p\right)^{1/p}$ for $i \in N$. Further, let $k_1, \ldots, k_n$ be positive numbers. Denote by W the set of those $j \in N$ for which $[g_j(p)]^{-1} g_j > k_j$. If $W \neq \emptyset$, put*

$$m = \max_{j \in W} (1 + k_j^q)^{-1/q},$$

the number q being connected with p by the relation $p + q = pq$. Let M_0 be the set of those $j \in N$ for which $(1 + k_j^q)^{-1} < m^q \sigma_j k_j^{-q}$ where

$$\sigma_j = \left(\frac{g_j}{g_j(p)}\right)^q .$$

Put

$$\sigma = \sum_{i \in M_0} (1 + k_i^q)^{-1} + m^q \sum_{i \in N - M_0} \sigma_i k_i^{-q} .$$

If either $W = \emptyset$ or $\sigma \leqq 1$ then the spectrum of A is contained in the union of the circular disks $|a_{ii} - \lambda| \leqq k_i g_i(p)$ for $i \in N$.

Analogously, from the theorem (6,5) of [3] follows the following theorem due to K. Fan and A. J. Hoffman ([2], the 1,5):

(5,3) *If, in the notation of the preceding theorem, there exists a positive α such that*

$$\sum_{i \in N} \left(\frac{g_i}{g_i(p)}\right)^q \leqq \alpha^q(1 + \alpha^q) ,$$

then the spectrum of A is contained in the union of the circular disks

$$|a_{ii} - \lambda| \leqq \alpha g_i(p) , \quad i \in N .$$

Let us turn now our attention to another important special case when $r = 2$. We shall formulate these theorems in terms of matrices, the norms being specialized for the case of l_p-norms (indices of $g_{i,p}$ will be omitted).

(5,4) Theorem. *Let $A = D + B$ be a partitioned matrix*

$$A = \begin{pmatrix} D_1 + B_{11}, & B_{12} \\ B_{21}, & D_2 + B_{22} \end{pmatrix}$$

where D_1, D_2 (and B_{11}, B_{22}) are square matrices of orders d_1, d_2 resp. Let us denote by p_{ij} the l_p-norms $(i, j = 1, 2)$

$$p_{ij} = \sup_{x \in X_i} \{g(B_{ij}x) ; \quad g(x) \leqq 1\}$$

where X_j is the space of d_j-rowed column vectors, and by E_j $(j = 1, 2)$ the identity matrix in $L(X_j, X_j)$.

If

$$h = \hat{g}(D_1 \otimes E_2 - E_1 \otimes D_2) < p_{11} + p_{22} + 2\sqrt{p_{12}p_{21}} ,$$

then the regions C_1^*, C_2^* resp. of those complex z numbers fulfilling the inequalities

$$C_1^*: \ \hat{g}(D_1 - zE_1) \leqq \tfrac{1}{2}[h + p_{11} - p_{22} - \sqrt{((h - p_{11} - p_{22})^2 - 4p_{12}p_{21})}] ,$$

$$C_2^*: \ \hat{g}(D_2 - zE_2) \leqq \tfrac{1}{2}[h - p_{11} + p_{22} - \sqrt{((h - p_{11} - p_{22})^2 - 4p_{12}p_{21})}]$$

respectively, contain d_1, d_2 resp. proper values of A, each considered with its multiplicity.

Proof. Follows immediately from theorem (4,5) if we choose $c_1 = p_{11} + \sqrt{(p_{12}p_{21})}$, $c_2 = p_{22} + \sqrt{(p_{12}p_{21})}$.

Remark 1. If, in the notation of the preceding theorem, $D_1 = c_1 E_1$, the lemma (4,7) enables us to simplify the estimate since then $h = \hat{g}(D_2 - c_1 E_2)$ and $\hat{p}(D_1 - zE_1) = |z - c_1|$.

Remark 2. If $p = 2$ and if D_1, D_2 are both normal, then C_i^* $(i = 1, 2)$ is the spherical neighborhood of the set of all proper values of D_i and h is the distance of these sets (cf. [4], section 8).

(5,5) Theorem. *Let the matrix $A = (a_{ij})$ be partitioned in the following manner:*

$$A = \begin{pmatrix} a_{11}, & a_1 \\ a_2', & A_{22} \end{pmatrix}.$$

If $h = \hat{g}(a_{11}E_2 - A_{22}) < 2\sqrt{p_1 p_2}$ where $p_1 = g(a_1)$, $p_2 = g'(a_2')$, then the circular disk

$$|z - a_{11}| \leqq \tfrac{1}{2}[h - \sqrt{(h^2 - 4p_1p_2)}]$$

contains exactly one proper value of A; all remaining proper values are contained in the region

$$\hat{g}(A_{22} - zE_2) \leqq \tfrac{1}{2}[h - \sqrt{(h^2 - 4p_1p_2)}]\,.$$

Bibliography

[1] *A. Brauer:* Limits for the characteristic roots of a matrix I—V. Duke Math. Journal *13* (1946), 387—395; *14* (1947), 21—26; *15* (1948), 871—877; *19* (1952), 75—91; *19* (1952), 553—562.

[2] *K. Fan, A. J. Hoffman:* Lower bounds for the rank and location of the eigenvalues of a matrix. Nat. Bur. Stand. Appl. Math. Ser. *39,* 117—130.

[3] *M. Fiedler, V. Pták:* On matrices with non-positive off-diagonal elements and positive principal minors. Czech. Math. Journal *87* (1962), 382—400.

[4] *M. Fiedler, V. Pták:* Some inequalities for the spectrum of a matrix. Matem.-fyz. časopis SAV *10* (1960), 148—166.

[5] *S. Geršgorin:* Über die Abgrenzung der Eigenwerte einer Matrix. Izv. Akad. Nauk SSSR *7* (1931), 749—754.

[6] *A. Ostrowski:* Sur les conditions générales pour la régularité des matrices. Rend. di Matem. e delle sue appl. *10* (1951), 156—168.

Резюме

ОБОБЩЕННЫЕ НОРМЫ МАТРИЦ И РАСПОЛОЖЕНИЕ ИХ СПЕКТРА

МИРОСЛАВ ФИДЛЕР и ВЛАСТИМИЛ ПТАК (Miroslav Fiedler a Vlastimil Pták), Прага

Пусть X n-мерное комплексное векторное пространство. Под обобщенной нормой подразумевается отображение p пространства X в неотрицательную часть r-мерного действительного линейного пространства, определенное следующим образом:

Пусть дано прямое разложение пространства $X = X_1 + \ldots + X_r$ и в каждом из пространств X_i какая-нибудь норма g_i. Если $I = P_1 + \ldots + P_r$ — соответствующее разложение единичного оператора в сумму проекционных операторов (так что $X_i = P_i X$), то обобщенная норма $p(x)$ определена формулой

$$p(x) = (g_1(P_1 x), \ldots, g_r(P_r x)) .$$

В настоящей работе понятие обобщенной нормы используется для вывода условий регулярности матриц и для исследований расположения спектра матриц в комплексной плоскости.

В параграфе 3 доказываются некоторые достаточные условия для регулярности матрицы, использующие определенную обобщенную норму матриц, которая получается из обобщенной нормы вектора аналогичным образом как в классическом случае норма линейного оператора получена из нормы вектора.

Основные результаты содержатся в теоремах (3,3), (3,4) и (3,5). В параграфе 4 применяются эти теоремы к матрицам вида $\lambda E - A$ и доказываются [именно в (4,3) и (4,4)] теоремы о расположению спектра матрицы. В последнем параграфе специализируются результаты параграфа 4 (напр. для $r = n$ и $r = 2$) и указывается, что они обобщают известные теоремы Гершгорина, Брауера, Островского и др.