Jozef Nagy
Kronecker index in abstract dynamical systems, II

Persistent URL: http://dml.cz/dmlcz/100715

KRONECKER INDEX IN ABSTRACT DYNAMICAL SYSTEMS, II

Jozef Nagy, Praha

(Received February 11, 1965)

In [6] there was defined the index of a point or of a simple loop in R^2, and then applied to the investigation of qualitative properties of critical points of dynamical systems in R^2. Now we will attempt to extend this definition to the case of a local dynamical system defined in p-dimensional Euclidean space R^p. In [6] it was shown that the index of the boundary of a Jordan region in R^2, under certain suppositions, yields considerable information about the behaviour of the dynamical system within this domain. Particularly interesting are the cases when this boundary is a closed trajectory or transversal. Of course, these results are characteristic for the case $p = 2$ and it seems at a first sight that for $p > 2$ the use of indices would not lead to such elegant results. However, it will be shown in the present paper that one may obtain, for general p, several interesting results, which are generalisations of results of [6]. These include e.g. the theorem on the index of the boundary of an invariant domain in R^p, the theorem on the expression of the index of the boundary of a domain by means of the sum of indices of enclosed critical points, etc. The main results of this paper are contained in theorems 2.10, 2.12, 2.16, 2.17 and 2.18.

2.1. Let us recall some notions and notation. We use throughout the notation introduced in [6]. The results of paper [6] will be referred to directly; e.g. theorem 1.17 is theorem 1.17 in [6]. We suppose as known the notions of an Euclidean r-dimensional simplex (r-simplex), of a simplicial complex and its subdivision, of a simplicial map, of a polyhedron and its triangulation, of an r-chain of the given complex (over the group $\mathscr{I}_0$ of integers) and of the corresponding elementary notions of algebraic topology (see e.g. [2, I, § 1]). As coefficient group we use only the group $\mathscr{I}_0$ of integers; this assumption will not be repeated.

If Π is a triangulation of a polyhedron P, we denote by $C_r(\Pi)$ the group of r-chains, ∂_r the boundary operator $C_r(\Pi)$ into $C_{r-1}(\Pi)$; $Z_r(\Pi)$ and $B_r(\Pi)$ denote the groups of r-cycles and r-boundaries respectively; $H_r(\Pi)$ denotes the factor group $Z_r(\Pi)/B_r(\Pi)$ (the r-th homology group of the triangulation Π). For a map f of a polyhedron P into a polyhedron P_1, simplicial with respect to the triangulations Π, Π_1 respectively, we denote by $f_{\#r}$ the homomorphism $C_r(\Pi) \to C_r(\Pi_1)$ induced by f. The corresponding induced homomorphism $H_r(\Pi) \to H_r(\Pi_1)$ we denote by f_{*r}.

For a polyhedron P let $H_r(P)$ be its r-th homology group, for a continuous map $f : P \to P_1$ let f_{*r} be the homomorphism $H_r(P) \to H_r(P_1)$ induced by f. This twofold use of the symbols f_* and H_r possibly will not lead to a misunderstanding. We shall write $f_\#, f_*, \partial$ in place of $f_{\#r}, f_{*r}, \partial_r$ whenever the particular value of r is not essential.

2.2. A compact triangulable subset P of the space R^p will be called an n-*pseudomanifold* $(n < p)$ if at least one its triangulations has the following five properties:

(i) Π contains at least one n-simplex but no k-simplex for $k > n$;

(ii) for each two n-simplices τ, τ' of the complex Π there exists a sequence $\tau = \tau_1, \tau_2, \ldots, \tau_s = \tau'$ of n-simplices of Π such that any two neighbouring n-simplices of this sequence have an $(n - 1)$-side in common;

(iii) each $(n - 1)$-simplex of Π is a side of precisely two n-simplices of Π;

(iv) each m-simplex $(m < n)$ of Π is a side of at least one n-simplex of Π;

(v) the pair $(\text{Int } P, P)$, where $\text{Int } P$ is a bounded component of $R^p - P$, is a triangulable pair [3].

It may be noted that the notion of a pseudomanifold used in this paper is not equivalent with that used in [1] and [2], where the property (v) is not required.

The set $\text{Int } P$ in (v) is determined uniquely, since according to the Jordan-Brouwer Theorem [1, X, § 2, theorem IV], every $(p - 1)$-pseudomanifold P in R^p has two complementary domains, of each of which it is the complete frontier, i.e.

$$R^p - P = \text{Int } P \cup \text{Ext } P,$$

where $\text{Int } P$ (the inner domain of P) and $\text{Ext } P$ (the outer domain of P) are non-void disjoint sets with common frontier P. By definition, the domain $\text{Int } P$ is bounded, $\text{Ext } P$ unbounded.

An n-pseudomanifold will be called *orientable* if $H_n(P)$ is isomorphic to $\mathscr{I}_0$. An orientable n-pseudomanifold will be called *oriented* if one of the generators of $H_n(P)$ is declared to be *orienting*. If Z is an orienting generator of $H_n(P)$, than each cycle $\zeta \in Z$ will be called an *orienting* cycle of P.

The simplest example of an n-pseudomanifold in R^p (being at the same time an n-manifold, i.e. each of its points has a neighbourhood homeomorphic to R_n) is the n-dimensional sphere S^n for $0 < n < p$. The groups $H_0(S^n)$ and $H_n(S^n)$ are isomorphic to $\mathscr{I}_0$, $H_i(S^n)$ for $0 < i < n$ are trivial [3, I, theorem 16.6]. Hence, the sphere S^n is an orientable n-pseudomanifold.

According to [1, X, § 2] for every $(p - 1)$-pseudomanifold P the group $H_{p-1}(P)$ is isomorphic to $\mathscr{I}_0$, hence, every $(p - 1)$-pseudomanifold in R^p is orientable.

According to [1, V, § 4.6, (15)] for every p-dimensional polyhedron Q in R^p the group $H_n(Q)$ is isomorphic to the direct sum $\sum\limits_{\pi_n(Q)} \mathscr{I}_0 + \Theta_n(Q)$, where $\pi_n(Q)$ and $\Theta_n(Q)$ denote n-th Betti number and n-th torsion group of Q respectively. Hence

there follows that $H_{p-1}(\text{Int } P) = 0$ for every $(p-1)$-pseudomanifold in $\mathbb{R}^p$. Indeed, according to $[1, \text{XI}, \S 3.12]$ $\Theta_{p-1}(\text{Int } P)$ is trivial and from the Alexander Duality Theorem $[1, \text{XI}, \S 4.1]$ there follows that $\pi_{p-1}(\text{Int } P) = \pi_0(\text{Ext } P) - 1 = 0$.

Definition 2.1. Let P, Q be oriented n-pseudomanifolds, Y, Z orienting generators of the groups $H_n(P)$, $H_n(Q)$ respectively, $f : P \to Q$. The *degree of the map f* is defined as the unique number $\deg(f)$ satisfying the relation $f_* Y = \deg(f) Z$.

The following three propositions are almost trivial:

(α) $\deg(f)$ *is an integer.*

(β) *If $f_1 \simeq f_2$ then* $\deg(f_1) = \deg(f_2)$.

(γ) *On changing the orientation of precisely one of the n-pseudomanifolds P, Q, the degree changes sign.*

We shall also need to choose coherent orientations for all unit $(p-1)$-spheres in $\mathbb{R}^p$. Let S_0^{p-1} be the $(p-1)$-sphere in $\mathbb{R}^p$ with center 0 and radius 1. Choosing an orienting generator of $H_{p-1}(S_0^{p-1})$, we obtain the oriented sphere S_0^{p-1}. Now define the orienting generator of an arbitrary sphere S_a^{p-1} with center a and radius 1 so that $\deg(t) = 1$ for the translation $t : S_a^{p-1} \to S_0^{p-1}$. By the *oriented sphere S_a^{p-1}* we mean throughout the sphere S_a^{p-1} oriented in this manner.

2.3. Now we shall introduce the concept of the order of a point with respect to a map in $\mathbb{R}^p$.

Definition 2.2. Let S_a^{p-1} be an oriented sphere with center in a, $\pi_a : \mathbb{R}^p - \{a\} \to S_a^{p-1}$ the projection from the point a, P an oriented $(p-1)$-pseudomanifold, $l : P \to \mathbb{R}^p - \{a\}$. The *order, $\omega(a, l)$, of a with respect to l* is defined as the number $\deg(\pi_a l)$.

In the following lemmas several properties of the order $\omega(a, l)$ will be given, which we shall often use later.

Lemma 2.3. *Let P be an oriented $(p-1)$-pseudomanifold, $l, l_1, l_2 : P \to \mathbb{R}^p - \{a\}$. Then there hold the following propositions:*

(i)   *if $\pi_a l_1 \simeq \pi_a l_2$ in S_a^{p-1}, then $\omega(a, l_1) = \omega(a, l_2)$;*

(ii) *$\omega(a, l)$ is an integer;*

(iii) *on changing the orientation of P, the order $\omega(a, l)$ changes its sign;*

(iv) *if a, a' are points in the same component of $\mathbb{R}^p - l(P)$, then $\omega(a, l) = \omega(a', l)$.*

(The p r o o f is immediate.)

Let K be an arbitrary component of the set $\mathbb{R}^p - l(P)$. Proposition (iv) allows one to define the order $\omega(K, l)$ of the component K as the number $\omega(a, l)$, where a is an arbitrary point in K. In particular, if $l(P)$ is a $(p-1)$-pseudomanifold, then $\omega(\text{Int } l(P), l)$ and $\omega(\text{Ext } l(P), l)$ are well-defined.

Lemma 2.4. *Let* P *be an oriented* $(p - 1)$*-pseudomanifold,* $a \in \operatorname{Int} P, i : P \subset$ $\subset \mathbb{R}^p - \{a\}$. *Then there hold the following propositions*:

  (i) $\omega(\operatorname{Int} P, i) = \varepsilon_P$, *where* $|\varepsilon_P| = 1$;
  (ii) $\omega(\operatorname{Ext} P, i) = 0$; *or more generally*
  (ii)' *if* $l : P \to \mathbb{R}^p - \{0\}$, K *the unbounded component of* $\mathbb{R}^p - l(P)$, *then* $\omega(K, l) = 0$.

(For the proof see [1], chap. XI, § 4, theorem III.)

In the situation of lemma 2.1, if $\varepsilon_P = 1$, then we shall say that the $(p - 1)$-pseudomanifold P is *positively oriented*; the corresponding orienting generator (or cycle) will be called a *positive orienting generator (cycle)*; in the opposite case the $(p - 1)$-pseudomanifold will be called *negatively oriented* and we shall speak about a *negative orienting generator (cycle)*.

Lemma 2.5. *Let* P *be an oriented* $(p - 1)$*-pseudomanifold,* $f : \mathbb{R}^p \approx \mathbb{R}^p$, $a \in \mathbb{R}^p$, $f_0 : \mathbb{R}^p - \{a\} \approx \mathbb{R}^p - \{f(a)\} : f_0(x) = f(x)$, $l : P \to \mathbb{R}^p - \{a\}$, $i : S_a^{p-1} \subset \mathbb{R}^p - \{a\}$ *Then* $\omega(f(a), f_0 l) = \omega(f(a), f_0 i) . \omega(a, l)$ *and* $|\omega(f(a), f_0 i)| = 1$.

Proof. Let S, S_1, Z be orienting generators of $H_{p-1}(S_a^{p-1})$, $H_{p-1}(S_{f(a)}^{p-1})$, $H_{p-1}(P)$ respectively, and $\pi : \mathbb{R}^p - \{a\} \to S_a^{p-1}$, $\pi_1 : \mathbb{R}^p - \{f(a)\} \to S_{f(a)}^{p-1}$ projections. From the definition 2.1 there follow (in the $(p - 1)$-st homological groups)

$$(\pi l)_* Z = \omega(a, l) S , \quad (\pi_1 f_0 l)_* Z = \omega(f(a), f_0 l) S_1 , \quad (\pi_1 f_0 i)_* S = \omega(f(a), f_0 i) S_1 .$$

Define a homotopy $h_\lambda : i\pi l \simeq l$ in $\mathbb{R}^p - \{a\}$ as follows: $h_\lambda(x) = (1 - \lambda) i\pi l(x) + \lambda l(x)$. Clearly, $\pi_1 f_0 h_\lambda$ is then a homotopy $\pi_1 f_0 i\pi l \simeq \pi_1 f_0 l$ in $S_{f(a)}^{p-1}$, and hence $(\pi_1 f_0 i\pi l)_* Z = (\pi_1 f_0 l)_* Z$. Now, there is

$$\omega(f(a), f_0 l) S_1 = (\pi_1 f_0 l)_* Z = (\pi_1 f_0 i\pi l)_* Z = (\pi_1 f_0 i)_* (\pi l)_* Z =$$
$$= (\pi_1 f_0 i)_* \omega(a, l) S = \omega(a, l) (\pi_1 f_0 i)_* S = \omega(a, l) . \omega(f(a), f_0 i) S_1 ,$$

and thus finally

$$\omega(f(a), f_0 l) = \omega(f(a), f_0 i) . \omega(a, l) .$$

For the proof of the relation $|\omega(f(a), f_0 i)| = 1$ see [1], chap. XII, § 2, theorem VI.

The lemma 2.5 allows one to assign, to every homeomorphism $f : \mathbb{R}^p \approx \mathbb{R}^p$, the number $e_f = \omega(f(o), f_0 i)$; the homeomorphism f is said to be *orientation preserving* if $e_f = 1$, and *orientation reversing* if $e_f = -1$.

Lemma 2.6. *Let* $\operatorname{sym}_0 : \mathbb{R}^p \approx \mathbb{R}^p$ *be the symmetry of* $\mathbb{R}^p$ *relative to the origin,* *(i.e. the affine mapping given by the matrix* (a_{jk}), $a_{jj} = -1$, $a_{jk} = 0$ *for* $j \neq k$, $j, k = 1, 2, ..., n)$; *then* $e_{\operatorname{sym}_0} = (-1)^p$. *More generally, if* $\operatorname{sym} : \mathbb{R}^p \approx \mathbb{R}^p$ *is a symmetry of* $\mathbb{R}^p$ *relative to a point* x_0, *and* S *an oriented sphere with center in* x_0, $i : S \subset \mathbb{R}^p$, $f_0 : S \to \mathbb{R}^p - \{x_0\} : f_0(x) = \operatorname{sym} i(x)$, *then* $\omega(x_0, f_0) = (-1)^p . \varepsilon_S$.

Proof. Clearly $e_{sym_0} = \det(a_{jk}) = (-1)^p$ and $\omega(x_0, f_0) = \varepsilon_S \cdot e_{sym_0}$.

2.4. In this paragraph we shall introduce the notions of a local dynamical system [4] and of vector fields associated with a local dynamical system, which will play a very important role in the remaining part of the paper.

Definition 2.7. A *local dynamical system* on a topological space P is a partial mapping T out of P $\times$ R^1 into P which satisfies the following five conditions:

(1) the domain of T is open;

(2) T is continuous;

(3) to each $x \in$ P there exist α_x, β_x with $-\infty \leqq \beta_x < 0 < \alpha_x \leqq +\infty$ and such that xTθ is defined iff $\beta_x < \theta < \alpha_x$, where xTθ denotes the value of the mapping T at the point $(x, \theta) \in$ P $\times$ R^1, if this is defined;

(4) xT$0 = x$ obtains for all $x \in$ P;

(5) $(x$T$\theta_1)$T$\theta_2 = x$T$(\theta_1 + \theta_2)$ holds if both xTθ_1 and also the left or right side of this relation are defined.

A local dynamical system with P $\times$ R^1 as domain will be termed a *global dynamical system*.

For R $\subset$ P, $A \subset$ R^1 denote XT$A = \{x$T$\theta : x \in X, \theta \in A, xT\theta$ defined$\}$. Let T be a local dynamical system in P. A point $x \in P$ is said to be a *critical point* of the local dynamical system T if xT$\theta = x$ holds for all θ such that xTθ is defined. If there exists a least $\theta_0 > 0$ such that xT$\theta_0 = x$ holds, then the set xTR^1 will be called a *periodic trajectory* with the primitive period θ_0. Every number $k\theta_0$ with k an integer will be called a *period* of the periodic trajectory xTR^1. If xT$\theta \neq x$ holds for all $\theta \in R^1$ such that xTθ is defined, then the set xTR^1 will be called a (non-periodic) *trajectory*.

A set $X \subset$ P is termed a $+$*invariant* (or $-$*invariant, invariant*) set of the local dynamical system T if XT$\langle 0, +\infty) \subset X$, (or XT$(-\infty, 0\rangle \subset X$, XT$R^1 \subset X$ respectively).

Now consider a local dynamical system T on R^p, and let $K \subset R^p$, $0 < \varepsilon \in R^1$, $\vartheta : K \to (0, \varepsilon)$ any continuous mapping such that xT$\vartheta(x)$ is defined for all $x \in K$. A *vector field* of the local dynamical system T on K is the mapping $W : K \to R^p$ defined by

$$(1) \qquad\qquad W(x) = x\text{T}\vartheta(x) - x.$$

The mapping W is continuous. Clearly, W vanishes at a point x_0 if either x_0 is a critical point of the local dynamical system T, or x_0 is on a periodic trajectory with the period $\vartheta(x_0)$, or $\vartheta(x_0) = 0$ [5, VI, § 1].

If $\vartheta_1 \simeq \vartheta_2$, then, obviously, the corresponding vector fields are also homotopic.

Lemma 2.8. *Let* τ *be a local dynamical system in* R^p, $F \subset \mathsf{R}^p$ *compact non-void and containing no critical point of* τ. *Then there exists an* A, $0 < A \in \mathsf{R}^1$, *such that*

(i) $x\tau\theta$ *is defined for every* $(x, \theta) \in \mathsf{R}^p \times \langle 0, A \rangle$;

(ii) *for every mapping* $\vartheta : F \to (0, A)$, *the mapping* W *defined by* (1) *is a vector field of the local dynamical system* τ, *continuous and vanishing nowhere on* F.

Proof. Let $x \in F$. There exists an $\alpha_x > 0$ such that $x\tau\theta$ is defined for every $\theta \in \langle 0, \alpha_x)$. The domain of τ is open; hence there exists a neighbourhood G_x of x and an $a_x \in \mathsf{R}^1$ such that $0 < a_x < \alpha_x$ and $y\tau\theta$ is defined for every $(y, \theta) \in G_x \times \times \langle 0, a_x) = H_x$. The set H_x is open in $\mathsf{R}^p \times \langle 0, +\infty)$. Clearly, the system $\{H_x\}_{x \in F}$ is an open cover of $F \times \{0\}$; then there is a finite cover $\{Hx_i\}_{i=1}^r \subset \{H_x\}_{x \in F}$. Now assertion (i) holds for every A with $0 < A < \min \{a_{x_1}, a_{x_2}, \ldots, a_{x_r}\}$.

The proof of (ii) is the same as that of lemma 1.5 in [6].

Let now τ, F, A be as in lemma 2.8. Then for every two mappings $\vartheta_0, \vartheta_1 : F \to \to (0, A)$ there exists a homotopy $\vartheta_\lambda : \vartheta_0 \simeq \vartheta_1$ (e.g. $\vartheta_\lambda(x) = (1 - \lambda) \vartheta_0(x) + + \lambda\vartheta_1(x)$) such that, for every $\lambda \in I$, the mapping W_λ defined as in (1) is a vector field of the system τ, continuous and vanishing nowhere on F. Every vector field W on F defined by (1) with $\vartheta : F \to (0, A)$ defines a mapping $W_0 : F \to \mathsf{R}^p - - \{0\} : W_0(x) = W(x)$. Every such mapping W_0 will be termed a *small vector field* of the local dynamical system τ on F.

2.5. From lemma 2.8 it follows that, for every $(p - 1)$-pseudomanifold P in R^p containing no critical points of the local dynamical system τ, there exist small vector fields on P. If W_1, W_2 are two small vector fields on P, then $0 \notin W_1(P) \cup W_2(P)$, and according to lemmas 2.8 and 2.2 (i), there holds $\omega(0, W_1) = \omega(0, W_2)$. Now we can set up the following definition.

Definition 2.9. Let τ be a local dynamical system in R^p, P an oriented $(p - 1)$-pseudomanifold in R^p containing no critical points of τ, W a small vector field on P. The *Kronecker index* $\mathrm{ind}_\tau P$ of P relative to τ is defined as $\varepsilon_P . \omega(0, W)$.

Immediately from the definition it follows that $\mathrm{ind}_\tau P$ is an integer not depending on the orientation of P; thus in this definition the assumption on the orientation of P can be weakened to orientability only.

Theorem 2.10. *Let* P *be a* $(p - 1)$-*pseudomanifold in* R^p, τ *a local dynamical system in* R^p *without critical points in* Int P. *Then* $\mathrm{ind}_\tau P = 0$.

Proof. Let $Q = \overline{\mathrm{Int}\, P}$, $i : P \subset Q$, Z an orienting generator of $H_{p-1}(P)$, S an orienting generator of $H_{p-1}(S_0^{p-1})$. It is easily shown that $H_{p-1}(Q) = 0$. Hence it follows that $i_* Z$ is the zero element of $H_{p-1}(Q)$. According to lemma 2.8 there exists a small vector field W on Q, hence for $\pi W : Q \to S_0^{p-1}$ there holds

(2) $$(\pi W)_* Z = 0 .$$

119

Clearly, Wi is a small vector field on P, and from (2) and the definition 2.1 one obtains $0 = (\pi Wi)_* Z = \omega(0, Wi) S$; it follows that $\omega(0, Wi) = 0$ and thus $\text{ind}_\tau P = \varepsilon_P$. $\cdot \omega(0, Wi) = 0$.

Corollary 2.11. *If* $\text{ind}_\tau P \neq 0$, *then* $\text{Int } P$ *contains at least one critical point of the local dynamical system* τ.

Theorem 2.12. *Let* τ *be a local dynamical system on* R^p, S *a* $(p-1)$-*sphere containing no critical point of* τ. *Then*

(i) $\text{ind}_\tau S = 1$ *if* $\text{Int } S$ *is* $-$*invariant*,

(ii) $\text{ind}_\tau S = (-1)^p$ *if* $\text{Int } S$ *is* $+$*invariant*.

Proof. Denote by x_0 and r the center and the radius of the sphere S, $i : S \subset R^p - \{x_0\}$.

Ad (i). From the assumptions it follows that there exists an $A > 0$ such that $|x\tau\theta - x_0| \geq r$ and $x\tau A \neq x$ for all $(x, \theta) \in S \times \langle 0, A \rangle$. Define mappings $W_0, W_1 : S \to R^p - \{0\}$ and a homotopy $W_\lambda : W_0 \simeq W_1$ as follows:

$$W_0(x) = x - x_0, \quad W_1(x) = x\tau A - x,$$
$$W_\lambda(x) = x\tau(\lambda A) - x_0 - \lambda(x - x_0).$$

Obviously, W_λ is continuous and vanishes nowhere on S, since $|x\tau(\lambda A) - x_0| \geq r > \lambda|x - x_0|$ for $0 \leq \lambda < 1$, $W_1(x) = x\tau A - x \neq 0$. Thus

(3) $$\omega(0, W_0) = \omega(0, W_1) = \omega(x_0, i) = \varepsilon_S,$$

and hence $\text{ind}_\tau S = \varepsilon_S \cdot \omega(0, W_1) = \varepsilon_S \cdot \varepsilon_S = 1$.

Ad (ii): From the assumptions it follows that $\text{Int } S$ is also $+$invariant, so that there exists an $A > 0$ such that $|x\tau\theta - x_0| \leq r$ for every $(x, \theta) \in S \times \langle 0, A \rangle$. Define mappings $W_0, W_1 : S \to R^p - \{0\}$ and a homotopy $W_\lambda : W_0 \simeq W_1$ as follows: $W_0(x) = x_0 - x$, $W_1(x) = x\tau A - x$, $W_\lambda(x) = \lambda(x\tau A - x_0) + (x_0 - x)$. Obviously, W_λ is continuous and vanishes nowhere on S, since $\lambda|x\tau A - x| < r = |x_0 - x|$ for $0 \leq \lambda < 1$, $W_1(x) = x\tau A - x \neq 0$. According to lemmas 2.3 and 2.6

(4) $$\omega(0, W_0) = \omega(0, W_1) = \omega(x_0, f_0) = \varepsilon_S \cdot (-1)^p, \quad (f_0 \text{ from } 2.6),$$

and hence

$$\text{ind}_\tau S = \varepsilon_S \cdot \varepsilon_S \cdot (-1)^p = (-1)^p.$$

The theorem 2.12 is thus proved.

From (3) and (4) it may be noted that $\text{ind}_\tau S$ in the theorem is expressed by topological invariants $(\omega(x_0, i)$ and $\omega(x_0, sym))$ respectively, see lemma 2.6). The assumption about S in the theorem may now be weakened thus: S is an arbitrary $(p-1)$-manifold in R^p, homeomorphic to a $(p-1)$-sphere.

Corollary 2.13. *Let R^p be odd-dimensional, let a $(p - 1)$-manifold S be homeomorphic to a $(p - 1)$-sphere. Then* Int S *is not an invariant set of any local dynamical system without critical points on S.*

Proof. Let Int S be the invariant set of a local dynamical system τ. Then Int S is also $-$invariant. From the proposition 2.12 (i) there follows $\mathrm{ind}_\tau S = 1$, from (ii) there follows $\mathrm{ind}_\tau S = -1$; this is a contradiction.

Corollary 2.14. *If the $(p - 1)$-sphere S satisfies the assumptions of the theorem 2.12, then* Int S *contains at least one critical point of the local dynamical system τ.*

For proof see corollary 2.11.

2.6. Definition 2.9 generalises the definition of the index of a simple loop [6, def. 1.6], and also allows one to generalise in the same manner the notion of the index of a critical point. We shall prove that this generalised index of a critical point is also a topological invariant.

Definition 2.15. Let τ be a local dynamical system, $x \in R^p$ not an accumulation point of critical points of τ, S an oriented $(p - 1)$-sphere with center x and small enough to have Int $S - \{x\}$ contain no critical points of τ. The *Kronecker index* $\mathrm{ind}_\tau x$ of the point x relative to τ is defined as the number $\mathrm{ind}_\tau S$.

It is easily shown that $\mathrm{ind}_\tau x$ does not depend on the choice of S.

By a generalisation of theorem 1.17 we now obtain an important result which also enables us to replace a $(p - 1)$-sphere S in definition 2.15 by an arbitrary $(p - 1)$-pseudomanifold.

Theorem 2.16. *Let the inner domain of a $(p - 1)$-pseudomanifold P contain only a finite number $x_1, x_2, \ldots, x_n$ of critical points of a local dynamical system τ, and let there be none on P. Then*

$$\mathrm{ind}_\tau P = \sum_{j=1}^{n} \mathrm{ind}_\tau x_j \,.$$

Proof. For $j = 1, 2, \ldots, n$ let S_j be a $(p - 1)$-sphere with center in x_j, such that $S_k \cap S_j = \emptyset$ holds for $1 \leqq k < j \leqq n$. Denote $Q = \mathrm{Int}\, P - \bigcup_{j=1}^{n} \mathrm{Int}\, S_j$, $i_0 : P \subset Q$, $i_j : S_j \subset Q$. Let $Z_0, Z_1, \ldots, Z_n$ be positive orienting generators of $H_{p-1}(P), H_{p-1}(S_1), \ldots, H_{p-1}(S_n)$, $Z'_l = (i_l)_* Z_l$ for $l = 0, 1, \ldots, n$.

Let $\pi : R^p - \{0\} \to S_0^{p-1}$ be the projection, S an orienting generator of S_0^{p-1}. Then it is easily shown that

(8)
$$Z'_0 - \sum_{j=1}^{n} Z'_j = 0 \,.$$

Let W be a small vector field on Q. Then from (8) there follows $(\pi W)_* \left(Z_0' - \sum_{j=1}^n Z_j' \right) =$
$= 0$; substituting $Z_j' = (i_j)_* Z_j$, one obtains $(\pi W)_* (i_0)_* Z_0 = \sum_{j=1}^n (\pi W)_* (i_j)_* Z_j$, or,
equivalently, $\omega(0, Wi_0) S = \sum_{j=1}^n \omega(0, Wi_j) S$; hence there follows $\omega(0, Wi_0) =$
$= \sum_{j=1}^n \omega(0, Wi_j)$. According to the assumption, $P, S_1, \ldots, S_n$ are positively oriented,
i.e. $\varepsilon_P = \varepsilon_{S_1} = \ldots = \varepsilon_{S_n} = 1$, so that

$$\text{ind}_\mathsf{T} \, P = \varepsilon_P \cdot \omega(0, Wi_0) = \sum_{j=1}^n \varepsilon_{S_j} \cdot \omega(0, Wi_j) = \sum_{j=1}^n \text{ind}_\mathsf{T} \, x_j \, .$$

The theorem 2.16 is proved.

From theorem 2.16 it follows immediately that if $x_0 \in \mathsf{R}^p$ is not an accumulation point of critical points of T and P_1, P_2 are $(p-1)$-pseudomanifolds such that $x_0 \in \text{Int } P_1 \cap \text{Int } P_2$, $\overline{\text{Int } P_1} \cup \overline{\text{Int } P_2} - \{x_0\}$ contains no critical point of T, then $\text{ind}_\mathsf{T} \, P_1 = \text{ind}_\mathsf{T} \, P_2 = \text{ind}_\mathsf{T} \, x_0$. Hence the sphere S in definition 2.15 may be replaced by an arbitrary $(p-1)$-pseudomanifold P having the properties described in this definition.

From the definition of local dynamical systems one has directly the following proposition: If T is a local dynamical system in R^p, $f : \mathsf{R}^p \approx \mathsf{R}^p$, then the relation $f(x\mathsf{T}\theta) = f(x)\mathsf{T}_f \, \theta$ defines a local dynamical system T_f in R^p. Naturally there arises the question as to the relation between $\text{ind}_\mathsf{T} \, x$ and $\text{ind}_{\mathsf{T}_f} f(x)$. The answer is the following theorem on the topological invariance of the index of a point.

Theorem 2.17. *Let $x_0 \in \mathsf{R}^p$ not be an accumulation point of critical points of a local dynamical system T in R^p, $f : \mathsf{R}^p \approx \mathsf{R}^p$. Then $\text{ind}_\mathsf{T} x_0 = \text{ind}_{\mathsf{T}_f} f(x_0)$.*

Proof. Let S be as in definition 2.15. Let $\theta \in \mathsf{R}^1$ be such that the mapping $W_0 : S \to \mathsf{R}^p - \{0\}$ given by the relation $W_0(x) = x\mathsf{T}\theta - x$ is a small vector field on S. Let $f_0 : S \to \mathsf{R}^p - \{0\}$ be given by the relation $f_0(x) = x\mathsf{T}\theta$. The proof of the theorem will be divided into two parts.

I. Let $f_0(S) \subset \text{Ext } S$.

Define a mapping $W_1 : S \to \mathsf{R}^p - \{0\}$ and a homotopy $W_\lambda : W_0 \simeq W_1$ in $\mathsf{R}^p - \{0\}$ by the relations (also see fig. 1) $W_1(x) = f_0(x) - x_0$, $W_\lambda(x) = f_0(x) - x + \lambda(x - x_0)$. W_λ is continuous, and vanishes nowhere on S, as $W_\lambda(x) = f_0(x) - x_0 + (1 - \lambda) \cdot (x_0 - x)$, $(1 - \lambda) |x - x_0| = (1 - \lambda) r \leqq r < |f_0(x) - x_0|$. Thus according to lemma 2.3, $\omega(0, W_0) = \omega(0, W_1)$. From the definition of the mapping W_1 it follows easily that $\omega(x_0, f_0) = \omega(0, W_1)$, so that

$$(6) \qquad \qquad \omega(0, W_0) = \omega(x_0, f_0) \, ,$$

and hence $\text{ind}_\mathsf{T} \, x_0 = \varepsilon_S \cdot \omega(x_0, f_0)$. From lemma 2.5 one obtains $\varepsilon_{f(S)} = e_f \cdot \varepsilon_S$, $\omega(f(x_0), ff_0) = e_f \cdot \omega(x_0, f_0)$ and thus $\text{ind}_{\mathsf{T}_f} f(x_0) = \varepsilon_{f(S)} \cdot \omega(f(x_0), ff_0) = \varepsilon_S \cdot e_f \cdot e_f \cdot \omega(x_0, f_0) = \text{ind}_\mathsf{T} \, x_0$. The topological invariance of the index in case I is proved.

II. Let $f_0(S) \not\subset$ Ext S.

First we shall prove one proposition which will make it possible to carry over the proof of case I to the present case.

Proposition. There exist $f_1 : S \to R^p - \{0\}$, $f_1(S) \subset$ Ext S and a homotopy $h_\lambda : f_0 \simeq f_1$ such that $h_\lambda(x) \neq x$ for every $\lambda \in I$.

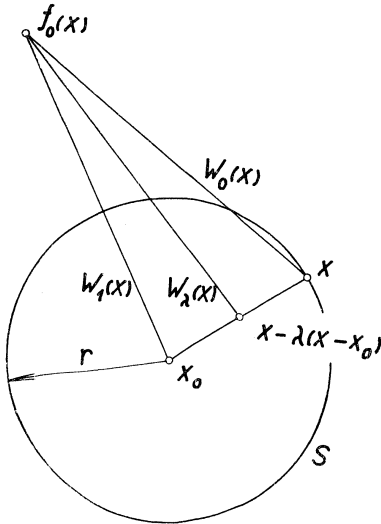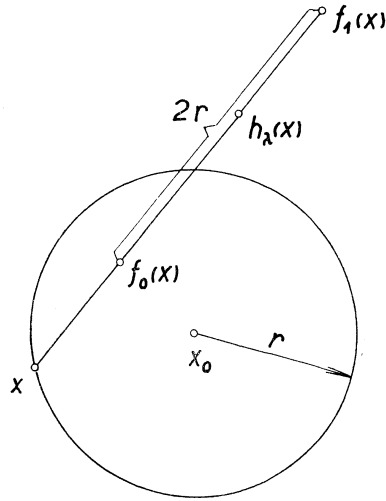Fig. 1.   Fig. 2.

Proof of proposition. Define f_1 and h_λ as follows (see fig. 2):

$$f_1(x) = x + \left(|f_0(x) - x| + 2r\right) \frac{f_0(x) - x}{|f_0(x) - x|},$$

$$h_\lambda(x) = x + \left(|f_0(x) - x| + \lambda \cdot 2r\right) \frac{f_0(x) - x}{|f_0(x) - x|}.$$

From the relation $|f_1(x) - x| = |f_0(x) - x| + 2r > 2r$ there follows $f_1(S) \subset$ Ext S, and from the relation $|h_\lambda(x) - x| = |f_0(x) - x| + \lambda \cdot 2r > 0$ for $\lambda \in I$ there follows $h_\lambda(x) \neq x$ for every $\lambda \in I$.

The proposition is proved.

Returning to the proof of the theorem, we shall show that the mapping f_1 from the proposition has the properties of the mapping f_0 from case I. The relation $f_1(S) \subset$ Ext S is evident. Defining the mapping $W_0' : S \to R^p - \{0\}$ and the homotopy

123

$W_\lambda : W_0 \simeq W_0'$ by $W_0'(x) = f_1(x) - x$, $W_\lambda(x) = h_\lambda(x) - x$, we obtain $\omega(0, W_0) = \omega(0, W_0')$. Furthermore, from $f_1(S) \subset \mathrm{Ext}\, S$ there follows (see (6)) $\omega(0, W_0') = \omega(x_0, f_1)$, hence

$$(7) \qquad\qquad \omega(0, W_0) = \omega(0, W_0') = \omega(x_0, f_1) \,.$$

This shows that f_1 has the property of f_0 from (6). Now we may apply the first part of the proof which concludes the proof of theorem 2.16.

Now we shall give two results which may be useful for the calculation of the index.

Corollary 2.12. *Let the inner domain of a $(p-1)$-pseudomanifold P contain only a finite number of critical points of a local dynamical system* T, *and let there be none on P; let $f : \mathrm{R}^p \approx \mathrm{R}^p$. Then* $\mathrm{ind}_\mathsf{T}\, P = \mathrm{ind}_{\mathsf{T}_f}\, f(P)$.

Proof. If $x_1, x_2, \ldots, x_n$ are all the critical points of T in $\mathrm{Int}\, P$, then $f(x_1), f(x_2), \ldots, f(x_n)$ are all the critical points of T_f in $\mathrm{Int}\, f(P)$. From theorems 2.16 and 2.17 there follows $\mathrm{ind}_\mathsf{T}\, P = \sum_{j=1}^n \mathrm{ind}_\mathsf{T}\, x_j = \sum_{j=1}^n \mathrm{ind}_{\mathsf{T}_f}\, f(x_j) = \mathrm{ind}_{\mathsf{T}_f}\, f(P)$.

Theorem 2.19. *Let the inner domain of a $(p-1)$-pseudomanifold P contain only a finite number of critical points of a local dynamical system* T *and let there be none on P. Let W_0 be a small vector field on P and let $f : P \to \mathrm{R}^p$ be such that the mapping $W_1 : P \to \mathrm{R}^p - \{0\}$ defined by $W_1(x) = f(x) - x$ satisfies on P the relation $|W_0(x) - W_1(x)| < |W_0(x)|$. Then* $\mathrm{ind}_\mathsf{T}\, P = \varepsilon_P \cdot \omega(0, W_1)$.

Proof. We shall prove that W_0 and W_1 are homotopic in $\mathrm{R}^p - \{0\}$. Define a homotopy $W_\lambda : W_0 \simeq W_1$ in $\mathrm{R}^p - \{0\}$ by $W_\lambda(x) = (1 - \lambda)\, W_0(x) + \lambda\, W_1(x)$. W_λ is evidently continuous. We shall prove that it vanishes nowhere on $\mathrm{R}^p - \{0\}$. Suppose that there exists a point $(\lambda_0, x_0) \in \langle 0, 1 \rangle \times P$ such that $W_{\lambda_0}(x_0) = 0$. Then $(1 - \lambda_0)\, W_0(x_0) + \lambda_0\, W_1(x_0) = 0$, hence $W_0(x_0) = \lambda_0(W_0(x_0) - W_1(x_0))$, and thus $|W_0(x_0)| \leq |W_0(x_0) - W_1(x_0)|$; this contradicts the assumption of the theorem. Thus W_λ is indeed a homotopy $W_0 \simeq W_1$ in $\mathrm{R}^p - \{0\}$, and then the theorem follows immediately.

2.7. In this last paragraph we make one remark concerning the rôle of the unicity of trajectories assumption in our investigation. If T is a local dynamical system, then the relation $x\mathsf{T}\theta = y\mathsf{T}\theta$ implies $x = y$. It may be noted that we have not exploited this property of T in our treatment. Thus it is very natural to generalise, in a certain sense, our results to the category of the so-called local semi-dynamical systems on R^p [4]. Let us recall the definition of a local semi-dynamical system (R^+ denotes the set of all non-negative reals).

Definition 2.20. A *local semi-dynamical system* on a topological space P is a partial mapping T out of P $\times$ R$^+$ into P which satisfies the following five conditions:

(1) the domain of T is open;

(2) T is continuous;

(3) to each $x \in$ P there exists an α_x with $0 < \alpha_x \leq +\infty$ and such that xTθ is defined iff $0 \leq \theta < \alpha_x$; here xTθ denotes, whenever defined, the value of the mapping T at the point $(x, \theta) \in$ P $\times$ R$^+$;

(4) xT$0 = x$ obtains for all $x \in$ P;

(5) $(x$T$\theta_1)$T$\theta_2 = x$T$(\theta_1 + \theta_2)$ holds iff the left or right side of this relation is defined.

A local semi-dynamical system with P $\times$ R$^+$ as domain will be termed a *global semi-dynamical system*.

It is clear that for a local semi-dynamical system -unicity need not hold, i.e. from the relation xT$\theta = y$Tθ there need not follow $x = y$.

The main apparatus in the present paper were small vector fields of local dynamical systems T on compact sets in R^p. In the definition of these fields only the values of $\theta \in \langle 0, \varepsilon \rangle$, for some $\varepsilon > 0$, were used. Hence it is clear that one can define the notion of a small vector field and also the notions of the Kronecker index of a $(p - 1)$-pseudomanifold or of a point in R^p in the same manner also for a local semi-dynamical system. It is easily verified that all the results holding for local dynamical systems also hold for local semi-dynamical systems.

Note. I have been advised that reference [2] of [6] will not be published as a separate paper, but will appear as part of chap. VI, section 3 in [5].

References

[1] *Alexandroff P., Hopf H.:* Topologie, Springer Verlag, Berlin 1935.
[2] *Болтянский В. Г.:* Гомотопическая теория непрерывных отображений и векторных полей, Труды матем. инст. имени Стеклова *XLVII*, Издат. АН СССР, Москва 1955.
[3] *Eilenberg S., Steenrod N. E.:* Foundations of Algebraic Topology, Princeton University Press, 1952.
[4] *Hájek O.:* Structure of dynamical system, CMUC *6* (1965) No. 1., 53—72.
[5] *Hájek O.:* Dynamical Systems in the Plane. To be published by Academic Press, London.
[6] *Nagy J.:* Kronecker index in abstract dynamical systems, I, Czech. math. journal *15* (1965), No. *3*, 394—405.

Author's address: Praha 1 - Loretánské nám. 3, ČSSR (Výzkumný ústav matematických strojů).

Резюме

ИНДЕКС КРОНЕККЕРА В АБСТРАКТНЫХ ДИНАМИЧЕСКИХ СИСТЕМАХ, II

ЙОСЕФ НАДЬ, (Jozef Nagy),Прага

В работе определяется *локальная динамическая система* т в p-мерном евклидовом пространстве R^p (определение 2.7). Потом определяется (определение 2.9 и 2.15) *индекс Кронеккера* $\mathrm{ind}_\mathsf{T} P$ $(p - 1)$-мерного псевдомногообразия $P \subset \mathsf{R}^p$, не содержащего критических точек динамической системы т, и *индекс Кронеккера* $\mathrm{ind}_\mathsf{T} x$ точки $x \in \mathsf{R}^p$ относительно локальной динамической системы т. Доказывается (теорема 2.17), что *индекс* $\mathrm{ind}_\mathsf{T} x$ *топологически инвариантен*.

Дальнейшие важные результаты этой работы содержатся в следующих теоремах:

Теорема 2.10. *Если* P — $(p - 1)$-*мерное псевдомногообразие в* R^p *и* $\overline{\mathrm{Int}\,P}$ (*замыкание внутренней области* P) *не содержит критических точек локальной динамической системы* т, *то* $\mathrm{ind}_\mathsf{T} P = 0$.

Следствие 2.11. *Если* $\mathrm{ind}_\mathsf{T} P \neq 0$, *то* $\mathrm{Int}\,P$ *содержит по крайней мере одну критическую точку системы* т.

Теорема 2.12. *Пусть* $(p - 1)$-*мерная сфера* S *в* R^p *не содержит критических точек локальной динамической системы* т. *Тогда*

(i) $\mathrm{ind}_\mathsf{T} S = 1$, *если* $\mathrm{Int}\,S$ *является* −*инвариантной областью*,

(ii) $\mathrm{ind}_\mathsf{T} S = (-1)^p$, *если* $\mathrm{Int}\,S$ *является* +*инвариантной областью*.

Следствие 2.14. *Если* S — $(p - 1)$-*мерная сфера из теоремы* 2.12., *то* $\mathrm{Int}\,S$ *содержит по крайней мере одну критическую точку системы* т.

Теорема 2.16. *Пусть* $(p - 1)$-*мерное псевдомногообразие в* R^p *не содержит критических точек локальной динамической системы* т *и пусть* $\mathrm{Int}\,P$ *содержит только конечное число критических точек* $x_1, x_2, \ldots, x_n$ *системы* т. *Тогда имеет место соотношение*

$$\mathrm{ind}_\mathsf{T} P = \sum_{j=1}^{n} \mathrm{ind}_\mathsf{T} x_j .$$