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NON-HOLONOMIC CONNECTIONS ON VECTOR BUNDLES I

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Linear connections on vector bundles can be treated in a way that is formally independent of the general theory of connections in principal fibre bundles developed by EHRESMANN, NOMIZU and others. There are, roughly said, two such possibilities of defining a connection on a vector bundle.

One possibility is that employed by W. GREUB in [2], where the connection is defined by means of a system of local differential forms with a priori given transformation formulae, a way similar to that used in the classical theory of linear connections. The other possibility is that of defining the connection as a linear mapping of certain vector bundles derived from the bundle in view. This method employs the theory of jets introduced by Ch. Ehresman (cf. [1]), and is applied also in this paper.

The definition of a connection on a vector bundle E given here differs slightly from that introduced by BOTT (cf. [6]). It allows to develop a formalism that is later used in the definition and study of holonomic, semi-holonomic, and non-holonomic connections of higher order on E, the semi-holonomic connections defined here being in a simple relation to those investigated by P. LIBERMANN (c.f. [3]). The main stress is laid however on the relation of non-holonomic connections to semi-holonomic connections, i.e. on the "reduction" of non-holonomic connections to semi-holonomic ones.

We start with a brief definition of a vector bundle which is slightly different from the usual one. It does not include *a priori* the notion of the structure group and a structure group appears only as a characteristic of a chosen collection of local coordinates in E, i.e. of an atlas of E. The main reason for this is to avoid the explicit use of the structure group, which may be complicated for calculations (e.g. in the case of semi-holonomic prolongations) and is not necessary for the description of problems studied below.

Next the prolongations of the vector bundle E, in the sense of Ch. Ehresmann ([1]), are investigated. The prolongations of E are compared with certain "tensor prolongations" obtained from E and the tangent bundle $T(M)$ by " tensor product" and "direct sum" operations. There is a system of local isomorphisms between each prolongation of E and the corresponding tensor prolongation. All the basic properties of the prolongations are derived – in some way – from there by "diagram chasing" methods without explicite calculations. Coordinate expressions are, but for some exceptions, avoided, and they are used only sometimes to illustrate the results obtained.

A pseudo-connection on E (of first order) is simply a bundle isomorphism of the first order prolongation of E onto $T^1(E) = E \oplus E \otimes T(M)^*$. It is a connection if it satisfies some further conditions. This definition as well as the definition of a relative connection with respect to some bundle morphism, as given in Definition 3.3, coincides in fact with that given in [6]. In the form of an illustration one shows also that it is equivalent to the definition of a connection on a vector bundle given by means of the associated principal fibre bundle. One proves some evident generalisations of facts known in the classical theory of linear connections.

Pseudo-connections of higher order are defined as bundle isomorphisms of higher order prolongations of E onto the corresponding tensor prolongations. There are some relations between these isomorphisms and first order pseudo-connections on higher order jet prolongations of E and on the corresponding tensor prolongations. These relations are relatively simple in the non-holonomic case. It seems to be advantageous to study rather sequences of pseudo-connections of subsequent orders (starting with the first order) than isolated pseudo-connections of a given order, this fact being due to the very definition of a higher order e.g. semi-holonomic connection given here, which differs from that given in [3] by a "superfluous" part including de facto pseudo-connections of lower orders.

A connection on E (of first order) together with a connection on the tangent bundle $T(M)$ give rise to canonical semi-holonomic and non-holonomic connections of any order. Especially, a semi-holonomic or non-holonomic connection of any higher order exist "almost always". Furthermore, it is shown that the sequence of canonical non-holonomic connections is reducible to the corresponding sequence of canonical semi-holonomic connections, whatever be the generating connections on E and $T(M)$.

The basic results are contained in the theorems of the last paragraph.

1. SOME REMARKS ON VECTOR BUNDLES

In the whole of this paper only real numbers are considered. A differentiable manifold, differentiable mapping etc., or simply manifold, mapping etc., means always a C^∞-differentiable manifold, mapping etc.

Let M be a manifold, $\dim M = n$. If $\mathscr{U} \subset M$ is a coordinate neighbourhood on M and $\varphi : \mathscr{U} \to \varphi(\mathscr{U}) \subset R^n$ the corresponding diffeomorphism defining the local coordinates, we call the pair $(\mathscr{U}, \varphi)$ simply a chart on the manifold M, $\mathscr{U}$ is the domain of this chart. The collection of charts defining the differentiable structure on M is called the atlas of M.

Let now E, M be differentiable manifolds and let $p = p_E : E \to M$ be a differentiable mapping which is "onto". Let $\dim E = n + m$. Further let E_0 be a fixed vector space, canonically isomorphic with R^m. Denote also $E_x = p^{-1}(x)$ for each $x \in M$.

Definition 1.1. Let $\mathcal{U} \subset M$ be open and let $r_x : E_x \to E_0$ be a one-to-one mapping for each $x \in \mathcal{U}$ with the property, that

$$\hat{r} : p^{-1}(\mathcal{U}) \to \mathcal{U} \times E_0 ,$$

given by

$$\hat{r}(y) = [py, r_{py}y] ,$$

is a diffeomorphism. Then the pair $(\mathcal{U}, r_x)$ is called a *chart* of the triple (E, p, M), $\mathcal{U}$ being the domain of $(\mathcal{U}, r_x)$.

Clearly, if $(\mathcal{U}, r_x)$ is a chart of (E, p, M), then for each $x \in \mathcal{U}$ $r_x : E_x \to E_0$ is a diffeomorphism.

Definition 1.2. A collection $\mathfrak{A}$ of charts $(\mathcal{U}_\iota, r_{\iota x})_{\iota \in I}$ is called an *atlas* of the triple (E, p, M) if

1) $\{\mathcal{U}_\iota\}_{\iota \in I}$ cover M

2) for all $\iota, \iota' \in I$ such that $\mathcal{U}_\iota \cap \mathcal{U}_{\iota'} \neq \emptyset$, the one-to-one mapping

$$\hat{r}_{\iota'}(\hat{r}_\iota)^{-1} : \mathcal{U}_\iota \cap \mathcal{U}_{\iota'} \times E_0 \to \mathcal{U}_\iota \cap \mathcal{U}_{\iota'} \times E_0$$

is a diffeomorphism, and for each $x \in \mathcal{U}_\iota \cap \mathcal{U}_{\iota'}$

$$r_{\iota' x}(r_{\iota x})^{-1} : E_0 \to E_0$$

is an isomorphism.

Two atlases $\mathfrak{A}$ and $\mathfrak{A}'$ of (E, p, M) are *equivalent* if $\mathfrak{A} \cup \mathfrak{A}'$ is again an atlas of (E, p, M). The atlas $\mathfrak{A}$ is called a *full atlas* of (E, p, M), iff it has the following property:

If $(\mathcal{V}, \varrho_x)$ is any chart of (E, p, M) such that for any $(\mathcal{U}, r_x) \in \mathfrak{A}$, $\mathcal{U} \cap \mathcal{V} \neq \emptyset$, there exists a differentiable mapping $g_{\mathcal{U}\mathcal{V}} : \mathcal{U} \cap \mathcal{V} \to GL(R, E_0) \cong GL(R, m)$ satisfying $\varrho_x r_x^{-1} = g_{\mathcal{U}\mathcal{V}}(x)$ for all $x \in \mathcal{U} \cap \mathcal{V}$, then $(\mathcal{V}, \varrho_x) \in \mathfrak{A}$.

It is evident that any atlas of (E, p, M) can be prolonged to a full atlas. Moreover, there is a one-to-one correspondence between all full atlases of (E, p, M) and the equivalence classes of atlases of (E, p, M). In fact, each equivalence class contains exactly one full atlas.

Definition 1.3. The structure defined on (E, p, M) by an equivalence class of atlases is called a vector bundle structure. Thus a vector bundle E over M is a triple (E, p, M)

provided with a vector bundle structure. The corresponding full atlas of (E, p, M) is called the full atlas of the vector bundle E and any atlas of the equivalence class defining E, i.e. any subatlas of the full atlas of E, is called an atlas of E.

Now it can be shown (see e.g. [7]), that a vector bundle structure on (E, p, M) defines an m-dimensional vector space structure on each fibre E_x such that if $(\mathscr{U}, r_x)$ is any chart of the full atlas, then $r_x : E_x \to E_0$ is an isomorphism. As usual, E_0 is called the *fibre type* of the vector bundle E.

Given a manifold M, all the vector bundles over M form a *category* $\mathscr{E}(M)$. A *morphism* $H : E \to F$ in this category is a differentiable mapping of the manifold E to F such that $p_E = p_F H$ and for any $x \in M$ the corresponding $H_x : E_x \to F_x$ is a homomorphism of the induced vector space structures. We shall call such H simply a *bundle morphism*. A bundle *isomorphism*, *projection*, *injection*, are defined in the usual way. Denote by $R = R(M)$ the trivial vector bundle corresponding to differentiable real valued functions on M.

Let E be a vector bundle over M, $\mathscr{U} \subset M$ any open subset. A differentiable mapping $f : \mathscr{U} \to E$, with the property $p_E f =$ identity, is called a (local) *section over* $\mathscr{U}$ in E. Denoting by $R(\mathscr{U})$ the ring of all differentiable real valued functions on $\mathscr{U}$, it is clear that the set of all local sections over $\mathscr{U}$ is an m-dimensional $R(\mathscr{U})$-module if and only if $\mathscr{U}$ is the underlying domain of a chart $(\mathscr{U}, r_x)$ belonging to the full atlas of E. There is a natural basis of this module given by the local sections $i_k : x \to$ $\to r_x^{-1}(i_k(0))$ $(k = 1, \ldots, m)$, where $\{i_k(0)\}$ is the canonical frame in E_0. This basis $\{i_k\}$ will be called the *frame of the chart* $(\mathscr{U}, r_x)$. It induces a frame $\{i_k(x)\}$ in E_x for each $x \in \mathscr{U}$.

Let now E be a fixed vector bundle over M and $\mathfrak{A}$ an atlas of E (not necessarily the full atlas). This atlas induces a set of isomorphisms

$$(1.1) \qquad r'_x r_x^{-1} : E_0 \to E_0$$

or, more precisely, if $(\mathscr{U}, r_x) \in \mathfrak{A}$, $(\mathscr{U}', r'_x) \in \mathfrak{A}$, $\mathscr{U} \cap \mathscr{U}' \neq \emptyset$, then $x \to r'_x r_x^{-1}$ is a differentiable mapping of $\mathscr{U} \cap \mathscr{U}'$ into $GL(R, E_0) \cong GL(R, m)$. The smallest closed subgroup $G(\mathfrak{A}) \subset GL(R, E_0)$ containing all the isomorphisms (1.1) is called the *structure group of the vector bundle E spanned by the atlas* $\mathfrak{A}$. It is well known that $G(\mathfrak{A})$ can be given the structure of a Lie group of left transformations on E_0 and thus the atlas $\mathfrak{A}$ defines on E a fibre bundle structure in the usual meaning of the word.

Two atlases $\mathfrak{A}$, $\mathfrak{A}'$ of E are called *G-equivalent* if $G(\mathfrak{A}) = G(\mathfrak{A}')$. Let G be any closed subgroup of $GL(R, E_0)$. We shall say that an atlas $\mathfrak{A}$ of E is *G-complete* if it has the following property:

If $(\mathscr{V}, \varrho_x)$ is any chart of the full atlas of E such that for any $(\mathscr{U}, r_x) \in \mathfrak{A}$, $\mathscr{U} \cap \mathscr{V} \neq$ $\neq \emptyset$ there exists a differentiable map $g_{\mathscr{U}\mathscr{V}} : \mathscr{U} \cap \mathscr{V} \to G$ satisfying $\varrho_x r_x^{-1} = g_{\mathscr{U}\mathscr{V}}(x)$ for all $x \in \mathscr{U} \cap \mathscr{V}$, then $(\mathscr{V}, \varrho_x) \in \mathfrak{A}$.

Each atlas of E can be completed in a unique way to a G-complete atlas $\overline{\mathfrak{A}}$, where $G = G(\mathfrak{A})$. $\overline{\mathfrak{A}}$ is called the *completion* of $\mathfrak{A}$. A fibre bundle structure on the vector bundle E, or briefly a G-structure, determines uniquely a complete atlas of E. An atlas is called complete, if there exists a closed subgroup $G \subset GL(R, E_0)$ such that this atlas is G-complete. Note that the full atlas of E is the only atlas that is $GL(R, E_0)$-complete. The vector bundle E is *trivial* if there exists a 1-complete atlas of E, where $1 \subset GL(R, E_0)$ is the trivial subgroup.

An atlas $\mathfrak{A}$ of E is called *semi-complete* if to each $(\mathscr{V}, \varrho_x) \in \overline{\mathfrak{A}}$ and each $x_0 \in \mathscr{V}$ there exists a chart $(\mathscr{U}, r_x) \in \mathfrak{A}$, $x_0 \in \mathscr{U}$ such that $\varrho_{x_0} = r_{x_0}$.

Remark 1. Let $\mathfrak{A}$ be the "natural" atlas of the tangent bundle $T(M)$ defined by local coordinates on M. Then $\mathfrak{A}$ is clearly semi-complete, its completion being the full atlas of $T(M)$. Let further $\mathfrak{A}'$ be a complete atlas of $T(M)$ defining a G-structure on M (i.e. on $T(M)$, in our terminology). Then this G-structure is by definition *integrable* iff there exists an atlas $\mathfrak{A}'' \subset \mathfrak{A}$ such that $\overline{\mathfrak{A}''} = \mathfrak{A}'$.

Let now $H : E \to F$ be a bundle isomorphism and let the fibre types E_0 and F_0 be identified. If $(\mathscr{U}, r_x^E)$ is any chart of the full atlas of E, then clearly $(\mathscr{U}, r_x^E H_x^{-1})$ is a chart of the full atlas of F. In this way H defines a one-to-one mapping between atlases of E and F respectively.

Lemma 1.1. *If $H : E \to F$ is a bundle isomorphism, $\mathfrak{A}_E$, $\mathfrak{A}_F$ are atlases of E and F respectively such that $H(\mathfrak{A}_E) = \mathfrak{A}_F$, then $H(\overline{\mathfrak{A}}_E) = \overline{\mathfrak{A}}_F$ and $G(\mathfrak{A}_E) = G(\mathfrak{A}_F)$.*

Proof. We first show $G(\mathfrak{A}_E) = G(\mathfrak{A}_F)$. Let $(\mathscr{U}, r_x^E) \in \mathfrak{A}_E$, $(\mathscr{U}', r_x'^E) \in \mathfrak{A}_E$, $x \in \mathscr{U} \cap \mathscr{U}'$. $G(\mathfrak{A}_E)$ is the smallest closed subgroup of $GL(R, E_0)$ containing all such $r_x'^E(r_x^E)^{-1}$. But on the other hand $(\mathscr{U}, r_x^E H_x^{-1}) \in \mathfrak{A}_F$, $(\mathscr{U}', r_x'^E) \in \mathfrak{A}_F$ and thus $r_x'^E H_x^{-1} H_x (r_x^E)^{-1} \in G(\mathfrak{A}_F)$. This means that $G(\mathfrak{A}_F)$ contains all the "generators" $r_x'^E(r_x^E)^{-1}$ of $G(\mathfrak{A}_E)$ and thus $G(\mathfrak{A}_E) \subset G(\mathfrak{A}_F)$. Reverting these considerations (note that $H : \mathfrak{A}_E \to \mathfrak{A}_F$ is one-to-one) we get the converse relation and hence $G(\mathfrak{A}_E) = G(\mathfrak{A}_F)$.

Let now $(\mathscr{V}, \varrho_x^F) = (\mathscr{V}, \varrho_x^E H_x^{-1}) \in H(\overline{\mathfrak{A}}_E)$, i.e. $(\mathscr{V}, \varrho_x^E) \in \overline{\mathfrak{A}}_E$. This means, that for any $(\mathscr{U}, r_x^E) \in \mathfrak{A}_E$, $\mathscr{U} \cap \mathscr{V} \neq \emptyset$, we have the differentiable map

$$(1.2) \qquad x \in \mathscr{U} \cap \mathscr{V} \to \varrho_x^E(r_x^E)^{-1} \in G(\mathfrak{A}_E) .$$

Let $(\mathscr{U}, r_x^F) \in \mathfrak{A}_F$ and $\mathscr{U} \cap \mathscr{V} \neq \emptyset$. We wish to show that

$$(1.3) \qquad x \to \varrho_x^E H_x^{-1}(r_x^F)^{-1}$$

is a differentiable map $\mathscr{U} \cap \mathscr{V} \to G(\mathfrak{A}_F) = G(\mathfrak{A}_E)$. But since $(\mathscr{U}, r_x^F) \in \mathfrak{A}_F = H(\mathfrak{A}_E)$, we conclude that $(\mathscr{U}, r_x^F H_x) \in \mathfrak{A}_E$, and the expression in (1.3) gets the form (1.2). Thus $H(\overline{\mathfrak{A}}_E) \in \overline{\mathfrak{A}}_F$. The converse is similar and this completes the proof.

Let $\mathfrak{A}$ be any atlas of the vector bundle E. The set of all frames at the points of M, generated by all the charts of $\overline{\mathfrak{A}}$, forms a principal fibre bundle $\mathscr{P}(E, \mathfrak{A})$ associated with the fibre bundle structure on E given by $\overline{\mathfrak{A}}$, its right transformations' group being $G(\mathfrak{A})$. This acting of $G(\mathfrak{A})$ upon $\mathscr{P}(E, \mathfrak{A})$ can be expressed explicitly as follows:

Let $g \in G(\mathfrak{A})$, $\{i_k(x)\} \in \mathscr{P}(E, \mathfrak{A})_x$ and let $(\mathscr{U}, r_x)$, $(\mathscr{U}', r'_x)$ be charts of $\mathfrak{A}$ such that $x \in \mathscr{U} \cap \mathscr{U}'$, $g = r'_x r_x^{-1}$, $r_x(i_k(x)) = i_k(0)$ $(k = 1, \ldots, m)$. Then $g\{i_k(x)\} = \{(r'_x)^{-1} r_x(i_k(x))\}$.

Let again $H : E \to F$ be a bundle isomorphism, $H(\mathfrak{A}_E) = \mathfrak{A}_F$. Then H generates a one-to-one mapping $\mathscr{P}(E, \mathfrak{A}_E) \to \mathscr{P}(F, \mathfrak{A}_F)$ assigning to the frame $\{i_k(x)\} \in \mathscr{P}(E, \mathfrak{A}_E)_x$ the frame $\{H_x(i_k(x))\}$. It would not be difficult to show that this mapping is a "fibre preserving" diffeomorphism and that it commutes with the acting of the group $G(\mathfrak{A}_E) = G(\mathfrak{A}_F)$.

Lemma 1.2. *Let E, F be vector bundles and $H : E \to F$ a bundle morphism. Suppose that there exists a homomorphism $H_0 : E_0 \to F_0$ such that to each $a \in M$ there exist charts $(\mathscr{U}, r_x^E)$ and $(\mathscr{U}, r_x^F)$ $(a \in \mathscr{U})$ of the full atlases of E and F respectively, satisfying*

$$x \in \mathscr{U} \Rightarrow H_0 r_x^E = r_x^F H_x . \tag{1.4}$$

Then $\operatorname{Ker} H \subset E$ *and* $\operatorname{Im} H \subset F$ *are vector bundles and the canonical injections* $\operatorname{Ker} H \to E$, $\operatorname{Im} H \to F$ *are bundle morphisms.*

Proof. It is not difficult to see that the differentiable structure on E induces a differentiable structure on $\operatorname{Ker} H$. Now the vector bundle structure on $\operatorname{Ker} H$ is defined by the atlas consisting of all charts $(\mathscr{U}, r_x^E|_{\operatorname{Ker} H_x})$, where $(\mathscr{U}, r_x^E)$ satisfies (1.4). We have namely from (1.4)

$$r_x^E|_{\operatorname{Ker} H_x} : \operatorname{Ker} H_x \to \operatorname{Ker} H_0 .$$

A similar argument leads to the vector bundle structure on $\operatorname{Im} H$.

Let now $\mathfrak{A}$ be again a fixed atlas of E. Without loss of generality one can always suppose that if $\mathfrak{A}$ contains a chart $(\mathscr{U}, r_x)$, then it contains also all the "restrictions" of this chart. Now let $(\mathscr{U}, r_x) \in \mathfrak{A}$ be such that $\mathscr{U}$ is simultaneously a coordinate neighbourhood on M. Denote by $\mathfrak{A}' \subset \mathfrak{A}$ the atlas of all such charts. Clearly $\overline{\mathfrak{A}'} = \overline{\mathfrak{A}}$, $G(\mathfrak{A}') = G(\mathfrak{A})$ and $\mathfrak{A}'$ is semi-complete if and only if $\mathfrak{A}$ is semi-complete. An *explicite chart* $(\mathscr{U}, r_x, \varphi)$ of $\mathfrak{A}$ is defined by a chart $(\mathscr{U}, r_x) \in \mathfrak{A}'$ and a chart $(\mathscr{U}, \varphi)$ on M. Denote by $\mathring{\mathfrak{A}}$ the set of all explicite charts of $\mathfrak{A}$.

2. JET AND TENSOR PROLONGATIONS OF VECTOR BUNDLES

Let E be a vector bundle as above, i.e. $E \in \mathscr{E}(M)$. The (holonomic) jet prolongation of q-th order $(q \geqq 1)$ of E, denoted by $S^q(E)$, is the set of all jets of q-th order of differentiable local sections in E. It is well known, that $S^q(E)$ can be given a vector bundle structure with the fibre type $J_0^q(R^n, E_0)$, where $J_0^q(R^n, E_0)$ is the vector space of all jets of q-th order of local mappings from R^n into E_0 with source $0 \in R^n$ (cf. [1] and below).

We shall be mainly interested, however, in non-holonomic and semi-holonomic prolongations of the vector bundle E, and the purpose of this paragraph is to derive all the basic properties of these prolongations without using their structure groups, only from the properties of the first order prolongation $S^1(E)$.

Let $(\mathcal{U}, r_x, \varphi)$ be an explicite chart of the full atlas of E. It gives rise to an explicite chart $S^1(\mathcal{U}, r_x, \varphi) = (\mathcal{U}, r_x^S, \varphi)$ of $(S^1(E), p, M)$ such that $(\mathcal{U}, r_x^S)$ belongs by definition to the full atlas of the vector bundle $S^1(E)$. It is defined by

$$r_a^S : S^1(E)_a \to J_0^1(R^n, E_0)\,, \quad a \in \mathcal{U}$$

as

(2.1)
$$r_a^S(\dot{f}) = j_0^1(\hat{r} f \varphi^{-1} t)$$

or, respectively,

(2.2)
$$(r_a^S)^{-1}(\dot{f}_0) = j_a^1(\hat{r}^{-1} f_0 t^{-1} \varphi)\,.$$

Here we have used the following notations: $\dot{f} = j_a^1 f \in S^1(E)_a$, where f is a local section in E over a neighbourhood of a, and similarly $\dot{f}_0 = j_0^1 f_0 \in J_0^1(R^n, E_0)$, where f_0 is a differentiable map of a neighbourhood of $0 \in R^n$ into E_0. The symbol t denotes the translation in R^n taking 0 into the source of the preceding component, i.e. into $\varphi(a)$ in (2.1). Analogously t^{-1} takes the target of the succeeding component (i.e. the $\varphi(a)$ in (2.2)) into $0 \in R^n$.

If $(\mathcal{U}', r'_x, \varphi')$ is another explicite chart of the full atlas of E and $a \in \mathcal{U} \cap \mathcal{U}'$, one easily derives from (2.1) and (2.2) the expression

(2.3)
$$r_a'^S(r_a^S)^{-1}(\dot{f}_0) = j_0^1(\hat{r}'\hat{r}^{-1} f_0 t^{-1} \varphi(\varphi')^{-1}\, t)\,.$$

Now let $\mathfrak{A}$ be a fixed atlas of E (possibly, but not necessarily the full atlas of E). There exists to each explicite chart of $\mathfrak{A}$ an explicite chart of the full atlas of $S^1(E)$. In this way $\mathfrak{A}$ defines an (explicite) atlas $S^1(\mathfrak{A})$ of $S^1(E)$.

Proposition 2.1. *If $\mathfrak{A}$ is semi-complete then $S^1(\mathfrak{A})$ is also semi-complete.*

Proof. Let $a \in M$ be fixed. Clearly all the mappings of the form (2.3) belong to the group of all automorphisms of the fibre type $J_0^1(R^n, E_0)$. On the other hand, if $\dot{g} = j_0^1 g$ is a one-jet of a map $R^n \to G(\mathfrak{A})$ and $\dot{\Phi} = j_0^1 \Phi$ is a one-jet of an invertible transformation of a neighbourhood of 0 in R^n with source and target 0, then there can be found charts $(\mathcal{U}, r_x, \varphi)$, $(\mathcal{U}', r'_x, \varphi') \in \mathring{\mathfrak{A}}$, $a \in \mathcal{U} \cap \mathcal{U}'$ such that the corresponding transition formula at a given in (2.3) has the form

(2.4)
$$\dot{f}_0 \to j_0^1((g f_0)\, \Phi)\,,$$

where g acts upon f_0 "pointwise". This follows from the semi-completeness of $\mathfrak{A}$. The relation (2.4) defines clearly an automorphism of $J_0^1(R^n, E_0)$ and the set of all such automorphisms form a group $S^1(G, M)$ which is a group of linear effective right transformations of $J^1_0(R^n, E_0)$. Thus its contragradient $S^1(G, M)^*$ is a subgroup of the group of all automorphisms of $J^1(R^n, E_0)$. It is also not difficult to see that $S^1(G, M)^*$ is closed and thus $S^1(G, M)^* = G(S^1(\mathfrak{A}))$. From there we conclude easily that $S^1(\mathfrak{A})$ is semi-complete.

The fibre type $J^1_0(R^n, E_0)$ of E is canonically isomorphic with

$$E_0 \oplus E_0 \otimes R^{n*} = E_0 \otimes (R \oplus R^{n*}), \tag{2.5}$$

which is clearly also the fibre type of the bundle $T^1(E) = E \oplus E \otimes T(M)^* = E \otimes T^1(R)$. Here $T(M)$ denotes as usually the tangent bundle of M, $T(M)^*$ its dual. The atlas $\mathring{\mathfrak{A}}$ generated from the atlas $\mathfrak{A}$ of E defines in an evident way also an explicite atlas $T^1(\mathfrak{A})$ of $T^1(E)$, and if $\mathfrak{A}$ is semi-complete, then the same is true about $T^1(\mathfrak{A})$.

Denote by $\{i_k(0)\}_{k=1,\ldots,m}$ the canonical basis of E_0 and by $\{\varepsilon^i(0)\}_{i=1,\ldots,n}$ the canonical basis of R^{n*}; $\varepsilon^0(0)$ let be the image of 1 under the natural injection $R \to R \oplus R^{n*}$. Thus the canonical basis of (2.5) consists of elements of the form $\varepsilon^\alpha(0) \otimes i_k(0)$ $(\alpha = 0, 1, \ldots, n;\ k = 1, \ldots, m)$. Given a chart $(\mathcal{U}, r_x, \varphi) \in \mathring{\mathfrak{A}}$, the basis $\{\varepsilon^\alpha(0) \otimes i_k(0)\}$ defines a frame $\{s^\alpha_k\}$ of the chart $S^1(\mathcal{U}, r_x, \varphi)$, and a frame $\{t^\alpha_k\} = \{\mathrm{d}x^\alpha \otimes i_k\}$ of the chart $T^1(\mathcal{U}, r_x, \varphi)$. Here again $\mathrm{d}x^0$ denotes the image of 1 under the natural injection $R(M) \to R(M) \oplus T(M)^*$. Consequently $\{s^\alpha_k(x)\}$ or $\{t^\alpha_k(x)\} = \{\mathrm{d}x^\alpha \otimes i_k(x)\}$ are frames in $S^1(E)_x$ or $T^1(E)_x$ respectively, for each $x \in \mathcal{U}$.

Using these notations we can now calculate explicitly the matrix **M** of the transition automorphism (2.3). Let $G(\mathfrak{A})$ consist of matrices $(g^{k'}_k)$. This means, that if $(\mathcal{U}, r_x, \varphi)$, $(\mathcal{U}', r'_x, \varphi') \in \mathring{\mathfrak{A}}$ and f is a local section in E over $\mathcal{U} \cap \mathcal{U}'$, then

$$f(x) = f^k(x)\, i_k(x) = f^{k'}(x)\, i_{k'}(x)\,,$$ [1]

where $f^{k'}(x) = g^{k'}_k(x) f^k(x)$ and $(g^{k'}_k) : \mathcal{U} \cap \mathcal{U}' \to G(\mathfrak{A})$. Thus $(g^{k'}_k)$ corresponds to $r'_x r_x^{-1}$.

If now $a \cap \mathcal{U} \cap \mathcal{U}'$ is fixed, we have

$$j^1_a f = f^k_\alpha s^\alpha_k(a) = f^{k'}_{\alpha'} s^{\alpha'}_{k'}(a)\,, \tag{2.6}$$

where $f^k_0 = f^k(a)$, $f^k_i = [(\partial/\partial x^i) f^k]_a$ $(i > 0)$ and similarly for $f^{k'}_{0'}, f^{k'}_{i'}$. Denoting by $A^i_{i'}$ the Jacobian $(\partial x^i/\partial x^{i'})_a$, we obtain from (2.6)

$$f^{k'}_{0'} = g^{k'}_k(a) f^k_0\,, \quad f^{k'}_{i'} = A^j_{i'} \left(\frac{\partial}{\partial x^j} g^{k'}_k\right)_a f^k_0 + A^i_{i'} g^{k'}_k(a) f^k_i\,. \tag{2.7}$$

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[1]) The usual convention of summation over repeated upper and lower indices is applied throughout the paper.

In other words, if $\mathbf{M} = (M^{k'\alpha}_{k\alpha'})$ is defined by[2] $f^{k'}_{\alpha'} = M^{k'\alpha}_{k\alpha'} f^k_\alpha$, then (2.7) yields

$$(2.8) \qquad M^{k'0}_{k0'} = g^{k'}_k(a)\,; \quad M^{k'i}_{k0'} = 0\,; \quad M^{k'0}_{ki'} = A^j_{i'}\left(\frac{\partial}{\partial x^j} g^{k'}_k\right)_a; \quad M^{k'i}_{ki'} = g^{k'}_k(a)\, A^i_{i'}\,.$$

We shall need in the next also the explicite formula for the inverse matrix $\mathbf{M}^{-1} = (M^{k\alpha'}_{k'\alpha})$

$$(2.9) \qquad M^{k0'}_{k'0} = g^k_{k'}(a)\,; \quad M^{ki'}_{k'0} = 0\,; \quad M^{k0'}_{ki'} = A^{j'}_i\left(\frac{\partial}{\partial x_{j'}} g^k_{k'}\right)_a; \quad M^{ki'}_{k'i} = g^k_{k'}(a)\, A^{i'}_i\,,$$

where $g^{k'}_h g^h_{h'} = \delta^{k'}_{h'}$.

On the other hand denoting $(\mathscr{U}, r^T_x, \varphi) = T^1(\mathscr{U}, r_x, \varphi)$, we have

$$r^T_a(y^k_\alpha t^\alpha_k(a)) = y^k_\alpha \varepsilon^\alpha(0) \otimes i_k(0)\,.$$

This gives for any point a in the intersection of the domains of two charts the transition formula

$$y^{k'}_{\alpha'} = A^\alpha_{\alpha'} g^{k'}_k y^k_\alpha\,,$$

where $A^0_{0'} = 1$ and $A^i_{0'} = A^0_{i'} = 0$ for $i, i' \neq 0$. In order to have a comparison with (2.8), represent $r'^T_a(r^T_a)^{-1}$ by a matrix $\mathbf{N} = (N^{k'\alpha}_{k\alpha'})$. Then

$$(2.10) \qquad \begin{aligned} N^{k'0}_{k0'} &= g^{k'}_k(a)\,; \quad & N^{k'i}_{k0'} &= 0\,; \\ N^{k'0}_{ki'} &= 0\,; & N^{k'i}_{ki'} &= g^{k'}_k(a)\, A^i_{i'}\,. \end{aligned}$$

The atlas $\mathfrak{A}$ defines a class of local differentiable isomorphisms $\{I_\mathscr{U}\}$ of $S^1(E)$ onto $T^1(E)$. If $(\mathscr{U}, r_x, \varphi) \in \mathring{\mathfrak{A}}$, then $I_\mathscr{U} : \overset{-1}{p}_{S^1(E)}(\mathscr{U}) \to \overset{-1}{p}_{T^1(E)}(\mathscr{U})$ is the isomorphism given by the correspondence of the frames $\{s^\alpha_k\}$ and $\{t^\alpha_k\}$. It is evident that it commutes with the operation of restriction of charts. This class $\{I_\mathscr{U}\}$, however, cannot be obtained by "restrictions" of a bundle isomorphism $I : S^1(E) \to T^1(E)$ unless $G(\mathfrak{A}) = 1$, which implies that E is trivial. In particular, the natural atlas in $R(M)$ has this property and thus $S^1(R)$ is canonically isomorphic with $T^1(R)$. We put simply $S^1(R) = T^1(R)$.

There are natural bundle projections $\Pi_S : S^1(E) \to E$ and $\Pi_T : T^1(E) \to E$, the first one being nothing but the target map. These projections clearly satisfy the conditions of Lemma 1.2, and thus $\operatorname{Ker} \Pi_S \subset S^1(E)$ and $\operatorname{Ker} \Pi_T \subset T^1(E)$ are vector bundles. One establishes easily from (2.8) and (2.10) that $\{I_\mathscr{U}\}$ gives rise to a bundle isomorphism

$$(2.11) \qquad I_0 : \operatorname{Ker} \Pi_S \to \operatorname{Ker} \Pi_T$$

observing that in a given chart of $\mathfrak{A}$ the element $f = f^k_\alpha s^\alpha_k(x) \in S^1(E)_x$ ($y = y^k_\alpha t^\alpha_k(x) \in$

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[2]) For the sake of simplicity we sometimes omit in the whole paper the argument x, a, etc. in expressions "over a point x, a, etc. of M" if this cannot lead to confusion.

$\in T^1(E)_x$) belongs to $\operatorname{Ker} \Pi_S$ (to $\operatorname{Ker} \Pi_T$) if $f_0^k = 0$ ($y_0^k = 0$). Further denote by $j_T^1 : E \to T^1(E) = E \oplus E \otimes T(M)^*$ the natural injection.

Let $F \in \mathscr{E}(M)$ be another vector bundle and $\Phi : E \to F$ be a bundle morphism. Define

$$T^1(\Phi) = \Phi \otimes 1_{T^1(R)} : T^1(E) \to T^1(F)\,,$$

and $S^1(\Phi) : S^1(E) \to S^1(F)$ by

(2.12) $$S^1(\Phi)\, j_a^1 f = j_a^1(\Phi f)\,.$$

It is not difficult to see that S^1 and T^1 thus defined are *covariant functors* from the category $\mathscr{E}(M)$ into itself.

Remark 2. For the sake of simplicity we shall not indicate the vector bundle E in the symbols Π_S, Π_T, j_T^1, etc., so that we shall use the same symbols for these projections or injections connected with any bundle of $\mathscr{E}(M)$.

Now it is not difficult to verify the following properties of the functors S^1 and T^1:

(2.13) $$\Pi_S\, S^1(\Phi) = \Phi \Pi_S$$

(2.14) $$\Pi_T\, T^1(\Phi) = \Phi \Pi_T$$

(2.15) $$T^1(\Phi)\, j_T^1 = j_T^1 \Phi\,.$$

The non-holonomic jet prolongation $\tilde{S}^q(E)$ of E and the non-holonomic tensor prolongation $\tilde{T}^q(E)$ of E or order $q > 1$ are defined recurrently as $\tilde{S}^q(E) = S^1(\tilde{S}^{q-1}(E))$ and $\tilde{T}^q(E) = T^1(\tilde{T}^{q-1}(E))$. A non-holonomic jet of order q is thus an element of $\tilde{S}^q(E)$. Our task will be to derive the basic properties of the jet prolongations of E by comparing them with the tensor prolongations which we are now going to introduce.

Let $q \geqq 1$ and define recurrently the *holonomic tensor prolongation*

(2.16) $$T^q(E) = T^{q-1}(E) \oplus E \otimes (\overset{q}{\odot}\, T(M)^*) = \sum_{k=0}^{q} E \otimes (\overset{k}{\odot}\, T(M)^*)\,,$$

the *semi-holonomic tensor prolongation*

(2.17) $$\bar{T}^q(E) = \bar{T}^{q-1}(E) \oplus E \otimes (\overset{q}{\otimes}\, T(M)^*) = \sum_{k=0}^{q} E \otimes (\overset{k}{\otimes}\, T(M)^*)\,,$$

and the *non-holonomic tensor prolongation*

(2.18) $$\tilde{T}^q(E) = \tilde{T}^{q-1}(E) \oplus \tilde{T}^{q-1}(E) \otimes T(M)^* = E \otimes (\overset{q}{\otimes}\, (R \oplus T(M)^*))$$

putting $T^0(E) = \bar{T}^0(E) = \tilde{T}^0(E) = E$. The atlas $\mathfrak{A}$ of E generates canonically atlases $T^q(\mathfrak{A})$, $\bar{T}^q(\mathfrak{A})$, $\tilde{T}^q(\mathfrak{A})$ of the vector bundles $T^q(E)$, $\bar{T}^q(E)$, $\tilde{T}^q(E)$ respectively.

We shall use three kinds of multiindices in the explicite formulae below:

1) $\sum_{|p|=k}$ consists of all indices $p = (p_1, \ldots, p_k)$, where each p_i runs from 1 to n and $p = 0$ if $k = 0$;

2) $\bigcirc \sum_{|p|=k}$ consists of all ordered indices $p = (p_1, \ldots, p_k)$, p_i running from 1 to n. In other words, $p \in \sum_{|p|=k}$ belongs to $\bigcirc \sum_{|p|=k}$ iff $i < j \Rightarrow p_i \leqq p_j$;

3) $\bullet \sum_{|p|=q}$ consists of all indices $p = (p_1, \ldots, p_q)$, where each p_i runs from 0 to n.

The same symbols are thus applied to indicate the way of summation as well as the set of the indices.

Denote by $\lambda : \sum \to \bigcirc\sum$ the rule of ordering the components. Note that if $p \in \bigcirc \sum_{|p|=k}$ and the integer i ($i = 1, \ldots, n$), occurs in p μ_i-times, then

$$\text{card}\, \lambda^{-1}(p) = \frac{k!}{\mu_1! \ldots \mu_n!}\,. \tag{2.19}$$

Further $\omega : \bullet\sum \to \sum$ denotes the rule of dropping all the zero components in a multi-index (but $\omega(p) = 0$ if p consists of zeros only). Note again that if $\omega : \bullet \sum_{|p|=q} \to \sum_{0 \leqq |p| \leqq q}$ and $p \in \sum_{|p|=k}$, then

$$\text{card}\, \omega^{-1}(p) = \binom{q}{k} = C^q_k\,. \tag{2.20}$$

Now we can pass to coordinate expressions in the bundles (2.16)–(2.18). A chart $(\mathscr{U}, r_x, \varphi) \in \mathring{\mathfrak{A}}$ gives rise to the frames

$$\{\bigcirc t^p_k\} = \{i_k \otimes (\bigcirc t^p)\}\,, \quad k = 1, \ldots, m\,; \quad p \in \bigcirc \sum_{0 \leqq |p| \leqq q}\,, \tag{2.21}$$

$$\{\otimes t^p_k\} = \{i_k \otimes (\otimes t^p)\}\,, \quad k = 1, \ldots, m\,; \quad p \in \sum_{0 \leqq |p| \leqq q}\,, \tag{2.22}$$

$$\{\bullet\, t^p_k\} = \{i_k \otimes (\bullet\, t^p)\,,\} \quad k = 1, \ldots, m\,; \quad p \in \bullet \sum_{|p|=q} \tag{2.23}$$

of the charts $T^q(\mathscr{U}, r_x, \varphi)$, $\bar{T}^q(\mathscr{U}, r_x, \varphi)$ and $\tilde{T}^q(\mathscr{U}, r_x, \varphi)$ respectively. Here we have used the abbreviations

$$p \in \bigcirc\sum \Rightarrow \bigcirc t^p = \mathrm{d}x^{p_1} \bigcirc \ldots \bigcirc \mathrm{d}x^{p_k},$$
$$p \in \sum \Rightarrow \otimes t^p = \mathrm{d}x^{p_1} \otimes \ldots \otimes \mathrm{d}x^{p_k},$$
$$p \in \bullet\sum \Rightarrow \bullet\, t^p = \mathrm{d}x^{p_1} \otimes \ldots \otimes \mathrm{d}x^{p_q}$$

and $\bigcirc t^0 = \otimes t^0 = 1$.

There are natural bundle injections $\iota^q_T : T^q(E) \to \bar{T}^q(E)$, $\bar{\iota}^q_T : \bar{T}^q(E) \to \tilde{T}^q(E)$ defined locally as follows: If $(\mathscr{U}, r_x, \varphi) \in \mathring{\mathfrak{A}}$ is a fixed chart, then

$$\xi = \bigcirc \sum_{|p|=q} \xi^k_p \bigcirc t^p_k \in T^q(E)_x\,, \quad x \in \mathscr{U} \Rightarrow$$

$$\iota^q_T \xi = \sum_{|p| \leqq q} \frac{1}{\text{card}\, \lambda^{-1}(\lambda(p))}\, \xi^k_{\lambda(p)} \otimes t^p_k \in \bar{T}^q(E)_x \tag{2.24}$$

and

$$\eta = \sum_{|p|\leq q} \eta^k_p \otimes t^k_p \in T^q(E)_x\,, \quad x \in \mathscr{U} \Rightarrow$$

(2.25)
$$\bar\imath^q_T \eta = \bullet \sum_{|p|=q} \eta^k_{\omega(p)} \bullet t^p_k \in \tilde{T}^q(E)_x\,.$$

Note that $\eta \in \bar{T}^q(E)_x$ belongs to Im ι^q_T if $\lambda(p) = \lambda(p')$ implies $\eta^k_p = \eta^k_{p'}$ for all $k = 1, \ldots, m$; $p, p' \in \sum_{|p|\leq q}$, and $\zeta = \bullet \sum_{|p|=q} \zeta^k_p \bullet t^p_k \in \tilde{T}^q(E)_x$, $x \in \mathscr{U}$ belongs to Im $\bar\imath^q_T$ if $\omega(p) = \omega(p')$ implies $\zeta^k_p = \zeta^k_{p'}$ for all $k = 1, \ldots, m$; $p, p' \in \bullet \sum_{|p|=q}$.

The direct sum decompositions in (2.16), (2.17) and (2.18) give rise to the diagrams

(2.26)
$$T^{q-1}(E) \underset{j^q_T}{\overset{\Pi^q_T}{\rightleftarrows}} T^q(E) \underset{j^{q*}_T}{\overset{\Pi^{q*}_T}{\rightleftarrows}} E \otimes (\overset{q}{\odot}\, T(M)^*)\,,$$

(2.27)
$$\bar{T}^{q-1}(E) \underset{\bar{j}^q_T}{\overset{\bar\Pi^q_T}{\rightleftarrows}} \bar{T}^q(E) \underset{\bar{j}^{q*}_T}{\overset{\bar\Pi^{q*}_T}{\rightleftarrows}} E \otimes (\overset{q}{\otimes}\, T(M)^*)\,,$$

and

(2.28)
$$\tilde{T}^{q-1}(E) \underset{j^1_T}{\overset{\tilde\Pi^q_T = \Pi_T}{\rightleftarrows}} \tilde{T}^q(E) \underset{j^{1*}_T}{\overset{\tilde\Pi^{q*}_T = \Pi^*_T}{\rightleftarrows}} \tilde{T}^{q-1}(E) \otimes T(M)^*$$

which, together with evident relations between the corresponding projections and injections, totally characterize the direct sum structures of $T^q(E)$, $\bar{T}^q(E)$ and $\tilde{T}^q(E)$. Note that in particular Ker $\bar\Pi^q_T$ is canonically isomorphic with $E \otimes (\overset{q}{\otimes}\, T(M)^*)$.

Given a bundle morphism $\Phi : E \to F$, the formulae (2.16)–(2.18) suggest in a natural manner the morphisms

(2.29)
$$T^q(\Phi) : T^q(E) \to T^q(F)\,, \quad \bar{T}^q(\Phi) : \bar{T}^q(E) \to \bar{T}^q(F)\,, \quad \tilde{T}^q(\Phi) : \tilde{T}^q(E) \to \tilde{T}^q(F)\,,$$

and it is not difficult to see that T^q, $\bar{T}^q$ and $\tilde{T}^q$ are again covariant functors from the category $\mathscr{E}(M)$ into itself.

The following lemmas are either evident or can be aesily verified by direct calculations with local coordinate expressions.

Lemma 2.1. *The diagram* $(\mathbf{D}^q_E)$:

$$\begin{array}{ccc}
T^q(E) & \xrightarrow{\ \Pi^q_T\ } & T^{q-1}(E) \\
\downarrow{\scriptstyle \iota^q_T} & & \downarrow{\scriptstyle \iota^{q-1}_T} \\
\bar{T}^q(E) & \xrightarrow{\ \bar\Pi^q_T\ } & \bar{T}^{q-1}(E) \\
\downarrow{\scriptstyle \bar\imath^q_T} & & \downarrow{\scriptstyle \bar\imath^{q-1}_T} \\
\tilde{T}^q(E) & \xrightarrow{\ \tilde\Pi^q_T\ } & \tilde{T}^{q-1}(E)
\end{array}$$

is commutative for any $q \geq 1$.

Lemma 2.2. *Let $\Phi : E \to F$ be a bundle morphism, $q \geqq 1$. Then the "three-dimensional" diagram, obtained by connecting* $(\mathbf{D}_E^q)$ *with* $(\mathbf{D}_F^q)$ *through bundle morphisms of the form* (2.29), *is commutative.*

The non-holonomic jet prolongation $\tilde{S}^q(E)$ is a vector bundle with the fibre type $E_0 \otimes (\overset{q}{\otimes} (R \oplus R^{n*}))$ and the atlas $\mathfrak{A}$ of E defines by recurrence an atlas $\tilde{S}^q(\mathfrak{A})$ of $\tilde{S}^q(E)$.

Let $(\mathcal{U}, r_x, \varphi)$ be a chart of $\mathfrak{A}$ and let $\{\bullet s_k^p\}$, $k = 1, \ldots, m$; $p \in \bullet \sum_{|p|=q}$ be the frame of the corresponding chart $\tilde{S}^q(\mathcal{U}, r_x, \varphi)$. The atlas $\mathfrak{A}$ defines a family $\{\tilde{I}_{\mathcal{U}}^q\}$ of local isomorphisms

$$\tilde{I}_{\mathcal{U}}^q : p_{\tilde{S}^q(E)}^{-1}(\mathcal{U}) \to p_{\tilde{T}^q(E)}^{-1}(\mathcal{U})$$

given by the correspondence of the frames $\{\bullet s_k^p\}$ and $\{\bullet t_k^p\}$.

The target mapping $\tilde{\Pi}_S^q : \tilde{S}^q(E) \to \tilde{S}^{q-1}(E)$ is a bundle projection and takes each element $\tilde{f} \in \tilde{S}^q(E)_a$ with local expression

$$\tilde{f} = \bullet \sum_{|p|=q} \tilde{f}_p^k \bullet s_k^p(a) \tag{2.30}$$

into the element with local expression

$$\tilde{\Pi}_S^q \tilde{f} = \bullet \sum_{|p|=q-1} \tilde{f}_{p,0}^k \bullet s_k^p(a) .$$

Hence we have a commutative "local" diagram

$$\begin{array}{ccc} \tilde{S}^q(E) & \xrightarrow{\tilde{I}_{\mathcal{U}}^q} & \tilde{T}^q(E) \\ \Big\downarrow \tilde{\Pi}_S^q & & \Big\downarrow \tilde{\Pi}_T^q \\ \tilde{S}^{q-1}(E) & \xrightarrow{\tilde{I}_{\mathcal{U}}^{q-1}} & \tilde{T}^{q-1}(E) \end{array} . \tag{2.31}$$

The dashed arrow means that the morphism is only local, depending on the choice of the chart.

Now let f be any local section in E over a neighbourhood, say $\mathcal{V}(a)$, of $a \in M$. The section $x \to j_x^1 f$ $(x \in \mathcal{V}(a))$ in $S^1(E)$ over $\mathcal{V}(a)$ is called the *flot of* f. The flot of this section is again a section in $\tilde{S}^2(E)$, and repeating this procedure we get a local section $x \to j_x^{q-1} f$ in $\tilde{S}^{q-1}(E)$ which finally gives rise to an element $j_a^q f$ of $\tilde{S}^q(E)_a$. An arbitrary element of $\tilde{S}^q(E)$ which can be obtained from a local section in E in this way is called a *holonomic jet* of q-th order. If $\{i_k\}$ is the frame of a chart $(\mathcal{U}, r_x, \varphi) \in \mathfrak{A}$ and $f(x) = f^k(x)\, i_k(x)$, then the corresponding $j_a^q f (a \in \mathcal{U})$, being by definition an element of $\tilde{S}^q(E)$, can be expressed as

$$j_a^q f = \bullet \sum_{|p|=q} \partial_p f^k \bullet s_k^p(a) , \tag{2.32}$$

where $$\partial_p = \frac{\partial}{\partial x^{p_1}} \frac{\partial}{\partial x^{p_2}} \cdots \frac{\partial}{\partial x^{p_q}} \quad \text{and} \quad \frac{\partial}{\partial x^0} = \text{identity} .$$

However, the set of all holonomic jets is nothing but the holonomic jet prolongation $S^q(E)$ of E, and we shall show below explicitly that the atlas $\mathfrak{A}$ together with the family of local isomorphisms $\{\tilde{I}^q_{\mathcal{U}}\}$, defines an atlas $S^q(\mathfrak{A})$ of $S^q(E)$. But before that define semi-holonomic jets.

Definition 2.1. (cf. [1]). Each element of $S^1(E)$ is a semi-holonomic jet. An element $\tilde{f} \in \tilde{S}^q(E)_a$ — written in the form $\tilde{f} = j^1_a g$, where $x \to g(x)$ is a local section in $\tilde{S}^{q-1}(E)$ — is a semi-holonomic jet if $g(x)$ is a semi-holonomic jet of order $q - 1$ for all x in a neighbourhood of a and

$$j^1_a(\tilde{\Pi}^{q-1}_S g) = g(a) . \tag{2.33}$$

Remark 3. The non-holonomic jets or prolongations are introduced relatively simply by "iteration" of the first order prolongations, and therefore some of their properties are evident. This is the main reason for starting with non-holonomic jets and introducing not only semi-holonomic but also holonomic jets a priori as subsets of the set of non-holonomic jets.

Proposition 2.2. **A)** *If $\tilde{f} \in \tilde{S}^q(E)_a$ is semi-holonomic and $(\mathcal{U}, r_x, \varphi) \in \mathring{\mathfrak{A}}$, $a \in \mathcal{U}$, then the local expression* (2.30) *satisfies*

$$\omega(p) = \omega(p') \Rightarrow \tilde{f}^k_p = \tilde{f}^k_{p'} \quad \textit{for} \quad k = 1, \ldots, m \quad \textit{and} \quad p, p' \in \bullet \sum_{|p|=q} . \tag{2.34}$$

B) *If $\tilde{f} \in \tilde{S}^q(E)_a$ in* (2.30) *satisfis* (2.34) *for some chart $(\mathcal{U}, r_x, \varphi) \in \mathring{\mathfrak{A}}$, $a \in \mathcal{U}$ then $\tilde{f}$ is semi-holonomic.*

Proof. **A)** We shall proceed by induction. Suppose that **A)** is true for all semi-holonomic jets of orders less than q. If $\tilde{f} = j^1_a g$, where $g(x) \in \tilde{S}^{q-1}(E)$ is semi-holonomic for each x in a neighbourhood of a, then comparing (2.30) with

$$g(x) = \bullet \sum_{|\beta|=q-1} g^k_\beta(x) \bullet s^\beta_k(x) \tag{2.35}$$

we get

$$f^k_{\beta 0} = g^k_\beta(a) , \tag{2.36}$$

$$f^k_{\beta r} = \left(\frac{\partial}{\partial x^r} g^k_\beta\right)_a \tag{2.37}$$

and (2.33) yields

$$g^k_{\gamma s}(a) = \left(\frac{\partial}{\partial x^s} g^k_{\gamma 0}\right)_a . \tag{2.38}$$

If $\omega(p) = \omega(p') = 0$, then (2.34) is trivial. Thus suppose $\omega(p) = \omega(p') \neq 0$ ($|p| = q$). If 0 is on the end of both p and p', one easily concludes by recurrence from (2.36) that $\tilde{f}^k_p = \tilde{f}^k_{p'}$. Let $p = (\beta 0)$, $p' = (\beta' r)$, $r \neq 0$. Then

$$\tilde{f}^k_p = g^k_\beta(a) ; \quad \tilde{f}^k_{p'} = \left(\frac{\partial}{\partial x^r} g^k_{\beta'}\right)_a . \tag{2.39}$$

Since $g(x)$ is semi-holonomic and $\omega(p) = \omega(p')$, we have $g^k_{\beta'} = g^k_{\beta''0}$, $g^k_\beta = g^k_{\beta''r}$, these relations being valid in a neighbourhood of a. But then (2.39) yields

$$\tilde{f}^k_{p'} = \left(\frac{\partial}{\partial x^r} g^k_{\beta''0}\right)_a ; \quad \tilde{f}^k_p = g^k_{\beta''r}(a) = \left(\frac{\partial}{\partial x^r} g^k_{\beta''0}\right)_a$$

and hence $\tilde{f}^k_{p'} = \tilde{f}^k_p$. Finally let $p = (\beta r)$, $p' = (\beta' s)$, $r, s \neq 0$. But this is not possible unless $r = s$ which implies $\omega(\beta) = \omega(\beta')$ and the rest is now evident.

B) Suppose again that **B**) is true for all jets of orders less than q and let $\tilde{f} \in \tilde{S}^q(E)_a$ satisfy the condition in a given chart. One can find functions $g^k_\beta(x)$ defined in a neighbourhood of a such that $g^k_\beta(x) = g^k_{\beta'}(x)$ if $\omega(\beta) = \omega(\beta')$ and such that (2.36) and (2.37) hold. From the recurrence assumption we conclude that these functions define a local semi-holonomic section g in $\tilde{S}^{q-1}(E)$ such that $\tilde{f} = j^1_a g$. It remains to show (2.33) or (2.38). But this is evident since $[(\partial/\partial x^s)\, g^k_{\gamma 0}]_a = \tilde{f}^k_{\gamma 0 s} = \tilde{f}^k_{\gamma s 0} = g^k_{\gamma s}(a)$. This completes the proof.

Corollary. *Each $\tilde{I}^q_{\mathscr{U}}$ induces an isomorphism $\bar{I}^q_{\mathscr{U}}$ of the set of all semi-holonomic jets with source contained in $\mathscr{U}$ onto $p^{-1}_{T^q(E)}(\mathscr{U})$.*

Proof. In fact we have seen that $\zeta = \tilde{I}^q_{\mathscr{U}} \tilde{f}$ belongs to Im $\bar{\imath}^q_T$ iff (2.34) holds.

It is not difficult to see that any holonomic jet is necessarily a semi-holonomic one. Moreover we have the

Proposition 2.3. **A**) *If $\tilde{f} \in \tilde{S}^q(E)_a$ is holonomic and $(\mathscr{U}, r_x, \varphi) \in \mathring{\mathfrak{A}}$, $a \in \mathscr{U}$, then the local expression* (2.30) *satisfies*

(2.40) $\lambda\omega(p) = \lambda\omega(p') \Rightarrow \tilde{f}^k_p = \tilde{f}^k_{p'}$, *for* $k = 1, \ldots, m$ *and* $p, p' \in \bullet \sum_{|p|=q}$.

B) *If $\tilde{f} \in \tilde{S}^q(E)_a$ in* (2.30) *satisfies* (2.40) *for some chart $(\mathscr{U}, r_x, \varphi) \in \mathring{\mathfrak{A}}$, $a \in \mathscr{U}$, then $\tilde{f}$ is holonomic.*

Proof. **A**) follows immediately from (2.32) and **B**) is a consequence of a well known fact in classical analysis.

Corollary. *Each $\tilde{I}^q_{\mathscr{U}}$ induces an isomorphism $I^q_{\mathscr{U}}$ of the set of all holonomic jets with source contained in $\mathscr{U}$ onto $p^{-1}_{T^q(E)}(\mathscr{U})$.*

Proof. In fact, $\zeta = \tilde{I}^q_{\mathscr{U}} \tilde{f}$ belongs to Im $(\bar{\imath}^q_T \imath^q_T)$ iff (2.40) holds.

Denote by $S^q(E) \subset \tilde{S}^q(E)$ the set of all holonomic jets and by $\bar{S}^q(E) \subset \tilde{S}^q(E)$ the set of all semi-holonomic jets. Note that $S^q(E) \subset \bar{S}^q(E)$. We have just seen that the atlas $\mathfrak{A}$ induces an atlas $S^q(\mathfrak{A})$ and $\bar{S}^q(\mathfrak{A})$ of $S^q(E)$ and $\bar{S}^q(E)$ respectively. In fact if $(\mathscr{U}, r_x, \varphi) \in \mathring{\mathfrak{A}}$, then the frame of the chart $S^q(\mathscr{U}, r_x, \varphi)$ or $\bar{S}^q(\mathscr{U}, r_x, \varphi)$ is given by $\{\circ s^p_k\}$ or $\{\otimes s^p_k\}$, where

$$\circ s^p_k = (I^q_{\mathscr{U}})^{-1}(\circ t^p_k), \quad k = 1, \ldots, m ; \quad p \in \circ \sum_{|p| \leqq q}$$

or

$$\otimes s_k^p = (\bar{I}_k^p)^{-1}(\otimes t_k^p)\,, \quad k = 1, \ldots, m\,; \quad p \in \sum_{|p| \leqq q}$$

respectively. It is not difficult to verify that they really define vector bundle structures on $S^q(E)$ and $\bar{S}^q(E)$ respectively. Denote by $\iota_S^q : S^q(E) \to \bar{S}^q(E)$ and $\bar{\iota}_S^q : \bar{S}^q(E) \to \tilde{S}^q(E)$ the natural bundle injections and by $\Pi_S^q : S^q(E) \to S^{q-1}(E)$, $\bar{\Pi}_S^q : \bar{S}^q(E) \to \bar{S}^{q-1}(E)$ the projections induced by $\tilde{\Pi}_S^q$. Note that Π_S^q and $\bar{\Pi}_S^q$ satisfy the conditions of Lemma 1.2 and consequently $\operatorname{Ker} \Pi_S^q$ and $\operatorname{Ker} \bar{\Pi}_S^q$ are also vector bundles.

The following lemma can be again easily verified using local coordinate expressions

Lemma 2.3. *Let $(\mathscr{U}, r_x, \varphi) \in \mathfrak{A}$. Then the "local three-dimensional" diagram obtained by connecting the diagram*

$$\begin{array}{ccc}
S^q(E) & \xrightarrow{\Pi_S^q} & S^{q-1}(E) \\
\downarrow{\scriptstyle \iota_S^q} & & \downarrow{\scriptstyle \iota_S^{q-1}} \\
\bar{S}^q(E) & \xrightarrow{\bar{\Pi}_S^q} & \bar{S}^{q-1}(E) \\
\downarrow{\scriptstyle \bar{\iota}_S^q} & & \downarrow{\scriptstyle \bar{\iota}_S^{q-1}} \\
\tilde{S}^q(E) & \xrightarrow{\tilde{\Pi}_S^q} & \tilde{S}^{q-1}(E)
\end{array}$$

with $(\mathbf{D}_E^q)$ through suitable local isomorphisms induced by $\tilde{I}_{\mathscr{U}}^q$ and $\tilde{I}_{\mathscr{U}}^{q-1}$ is commutative.

Let $q > r \geqq 0$ and $(\mathscr{U}, r_x, \varphi) \in \mathring{\mathfrak{A}}$. The chart $\tilde{S}^r(\mathscr{U}, r_x, \varphi)$ induces a local isomorphism

(2.41) $$I_{\mathscr{U},r} : \tilde{S}^{r+1}(E) = S^1(\tilde{S}^r(E)) \to T^1(\tilde{S}^r(E))\,.$$

Let us define the local isomorphism

(2.42) $$A_{\mathscr{U}}^{r,q} : \tilde{T}^r(\tilde{S}^{q-r}(E)) \to \tilde{T}^{r+1}(\tilde{S}^{q-r-1}(E))$$

as $A_{\mathscr{U}}^{r,q} = \tilde{T}^r(I_{\mathscr{U},q-r-1})$. An element $\tilde{f} \in \tilde{T}^r(\tilde{S}^{q-r}(E))_x$, $x \in \mathscr{U}$, can be expressed in the form

(2.43) $$\tilde{f} = \bullet \sum_{|p|=q} \tilde{f}^k_{p_1,\ldots,p_q} \bullet s_k^{p_1,\ldots,p_{q-r}} \otimes (\bullet t^{p_{q-r+1},\ldots,p_q})$$

and (2.42) takes it into

(2.44) $$A_{\mathscr{U}}^{r,q}\tilde{f} = \bullet \sum_{|p|=q} \tilde{f}^k_{p_1,\ldots,p_q} \bullet s_k^{p_1,\ldots,p_{q-r-1}} \otimes (\bullet t^{p_{q-r},\ldots,p_q})\,.$$

Hence the local isomorphisms $A_{\mathscr{U}}^{r,q}$ $(r = 0, \ldots, q-1)$ determine a decomposition of the local isomorphism $\tilde{I}_{\mathscr{U}}^q : \tilde{S}^q(E) \to \tilde{T}^q(E)$ into the sequence

(2.45)

$$\tilde{S}^q(E) \xrightarrow{A_{\mathscr{U}}^{0,q}} T^1(\tilde{S}^{q-1}(E)) \xrightarrow{A_{\mathscr{U}}^{1,q}} \tilde{T}^2(\tilde{S}^{q-2}(E)) \xrightarrow{A_{\mathscr{U}}^{2,q}} \ldots \xrightarrow{A_{\mathscr{U}}^{q-2,q}} \tilde{T}^{q-1}(S^1(E)) \xrightarrow{A_{\mathscr{U}}^{q-1,q}} \tilde{T}^q(E)$$

in such a way that the diagram (2.46) is commutative.

(2.46)

$$
\begin{array}{ccccccccccccc}
\tilde{S}^q(E) & \xrightarrow{A^{0,q}_{\mathcal{U}}} & T^1(\tilde{S}^{q-1}(E)) & \overset{A^{1,q}_{\mathcal{U}}}{\dashrightarrow} & \cdots & \xrightarrow{A^{r-1,q}_{\mathcal{U}}} & \tilde{T}^r(\tilde{S}^{q-r}(E)) & \xrightarrow{A^{r,q}_{\mathcal{U}}} & \tilde{T}^{r+1}(\tilde{S}^{q-r-1}(E)) & \xrightarrow{A^{r+1,q}_{\mathcal{U}}} & \cdots & \xrightarrow{A^{q-1,q}_{\mathcal{U}}} & \tilde{T}^q(E) \\
& \searrow{\scriptstyle \tilde{\Pi}^q_S} & \downarrow{\scriptstyle \Pi^1_T} & & & & \downarrow{\scriptstyle \tilde{\Pi}^r_T} & & \downarrow{\scriptstyle \tilde{\Pi}^{r+1}_T} & & & & \downarrow{\scriptstyle \tilde{\Pi}^q_T} \\
& & \tilde{S}^{q-1}(E) & \xrightarrow{A^{0,q-1}_{\mathcal{U}}} & \cdots & \xrightarrow{A^{r-1,q-1}_{\mathcal{U}}} & T^{r-1}(\tilde{S}^{q-r}(E)) & \xrightarrow{A^{r-1,q-1}_{\mathcal{U}}} & \tilde{T}^r(\tilde{S}^{q-r-1}(E)) & \xrightarrow{A^{r,q-1}_{\mathcal{U}}} & \cdots & \xrightarrow{A^{r-2,q-1}_{\mathcal{U}}} & \tilde{T}^q(E)
\end{array}
$$

(2.47)

$$
\begin{array}{ccccccccccccccc}
\bar{S}^q(E) & \xrightarrow{i^q_S} & \tilde{S}^q(E) & \xrightarrow{A^{0,q}_{\mathcal{U}}} & T^1(\tilde{S}^{q-1}(E)) & \xrightarrow{A^{1,q}_{\mathcal{U}}} & \cdots & \xrightarrow{A^{r+1,q}_{\mathcal{U}}} & \tilde{T}^r(\tilde{S}^{q-r}(E)) & \xrightarrow{A^{r,q}_{\mathcal{U}}} & \tilde{T}^{r+1}(\tilde{S}^{q-r-1}(E)) & \xrightarrow{A^{r+1,q}_{\mathcal{U}}} & \cdots & \xrightarrow{A^{q-1,q}_{\mathcal{U}}} & \tilde{T}^q(E) \\
& & & \searrow{\scriptstyle \tilde{\Pi}^q_S} & \downarrow{\scriptstyle \Pi^1_T} & & & & \downarrow{\scriptstyle \tilde{\Pi}^r_T} & \searrow{\scriptstyle \tilde{T}^r(\tilde{\Pi}^{q-r}_S)} & \downarrow{\scriptstyle \tilde{\Pi}^{r+1}_T} & & & & \downarrow \\
& & & & \tilde{S}^{q-1}(E) & \overset{A^{0,q-1}_{\mathcal{U}}}{\dashrightarrow} & \cdots & \xleftarrow{A^{r-2,q-1}_{\mathcal{U}}} & \tilde{T}^{r-1}(\tilde{S}^{q-r}(E)) & \xrightarrow{A^{r-1,q-1}_{\mathcal{U}}} & \tilde{T}^r(\tilde{S}^{q-r-1}(E)) & \xrightarrow{A^{r,q-1}_{\mathcal{U}}} & \cdots & \xrightarrow{A^{q-2,q-1}_{\mathcal{U}}} & \tilde{T}^q(E)
\end{array}
$$

The bundle projection

$$\tilde{T}^r(\tilde{\Pi}_S^{q-r}) : \tilde{T}^r(\tilde{S}^{q-r}(E)) \to \tilde{T}^r(\tilde{S}^{q-r-1}(E)) \tag{2.48}$$

takes the element $\tilde{f} \in \tilde{T}^r(\tilde{S}^{q-r}(E))$ in (2.43) into

$$\tilde{T}^r(\tilde{\Pi}_S^{q-r})\tilde{f} = \bullet \sum_{|p|=q} \tilde{f}^k_{p_1,\dots,p_{q-r-1},0,p_{q-r+1},\dots,p_q} \bullet s_k^{p_1,\dots,p_{q-r-1}} \otimes (\bullet t^{p_{q-r+1},\dots,p_q}) . \tag{2.49}$$

Lemma 2.4. *Let $q > 0$, $0 \leqq r \leqq q - 1$. If $\tilde{f} \in \operatorname{Ker} \tilde{T}^r(\tilde{\Pi}_S^{q-r})$, then $A_{\mathcal{U}}^{r,q}\tilde{f} \in \tilde{T}^{r+1}(\tilde{S}^{q-r-1}(E))$ does not depend on the chart $(\mathcal{U}, r_x, \varphi)$ defining the local isomorphisms $A_{\mathcal{U}}^{r,q}$.*

Proof. Write $\tilde{T}^r(\tilde{S}^{q-r}(E)) = \tilde{S}^{q-r}(E) \otimes \tilde{T}^r(R)$ and suppose $\tilde{f} = f_1 \otimes f_2$, where $f_1 \in \tilde{S}^{q-r}(E)$, $f_2 \in \tilde{T}^r(R)$. Then $\tilde{T}^r(\tilde{\Pi}_S^{q-r})\ \tilde{f} = 0$ implies $f_1 \in \operatorname{Ker} \tilde{\Pi}_S^{q-r}$. But $A_{\mathcal{U}}^{r,q}\tilde{f} = I_{\mathcal{U},q-r-1} f_1 \otimes f_2$ and since $I_{\mathcal{U},q-r-1}|_{\operatorname{Ker}\tilde{\Pi}^{q-r}_S}$ is a local isomorphism $I_{\mathcal{U}}$ – connected with the first order prolongations of the bundle $\tilde{S}^{q-r-1}(E)$ – restricted to the kernel of the corresponding projection, we see that $I_{\mathcal{U},q-r-1} f_1$ does not depend on the chart. From there we conclude the same property for $A_{\mathcal{U}}^{r,q}\tilde{f}$ and this proves the lemma.

The integers q and r being given as in the lemma, consider now the diagram

$$\begin{array}{ccccccc}
\tilde{S}^q(E) & \xrightarrow{A_{\mathcal{U}}^{0,q}} & \dots & \xrightarrow{A_{\mathcal{U}}^{r-1,q}} & \tilde{T}^r(\tilde{S}^{q-r}(E)) & \xrightarrow{A_{\mathcal{U}}^{r,q}} & \tilde{T}^{r+1}(\tilde{S}^{q-r-1}(E)) \\
 & & & & & \searrow^{\tilde{T}^r(\tilde{\Pi}_S^{q-r})} & \downarrow{\scriptstyle \tilde{\Pi}_T^{r+1}} \\
 & & & & & & \tilde{T}^r(\tilde{S}^{q-r-1}(E))
\end{array} \tag{2.50}$$

It is not commutative. In fact, if $\tilde{f} \in \tilde{T}^r(\tilde{S}^{q-r}(E))$, $p\tilde{f} \in \mathcal{U}$, is given as in (2.43), then $\tilde{T}^r(\tilde{\Pi}_S^{q-r})$ has the form (2.49) and from (2.44) we have

$$(\tilde{\Pi}_T^{r+1} A_{\mathcal{U}}^{r,q})\tilde{f} = \bullet \sum_{|p|=q} \tilde{f}^k_{p_1,\dots,p_{q-1},0} \bullet s^{p_1,\dots,p_{q-r-1}} \otimes (\bullet t^{p_{q-r},\dots,p_{q-1}}) = \tag{2.51}$$

$$= \bullet \sum_{|p|=q} \tilde{f}^k_{p_1,\dots,p_{q-r-1},p_{q-r+1},0} \bullet s^{p_1,\dots,p_{q-r-1}} \otimes (\bullet t^{p_{q-r+1},\dots,p_q}) .$$

Comparing (2.51) with (2.49) we see that they are equal if and only if $\tilde{f}$ has the property (2.34), i.e. if $\tilde{f}$ is semi-holonomic. In other words, the diagram (2.50) will be commutative if and only if one starts in $\tilde{S}^q(E)$ with a semi-holonomic jet. In this sense we can say that the diagram (2.47) is commutative "as a whole", i.e. each possible "path" starting in $\bar{S}^q(E)$ and ending in the same space gives the same result.

In particular, if we start with $\bar{f} \in \operatorname{Ker} \bar{\Pi}_S^q$, i.e. $\tilde{\iota}_S^q \bar{f} \in \operatorname{Ker} \tilde{\Pi}_S^q$, we pass each $\tilde{T}^r(\tilde{S}^{q-r}(E))$ through $\operatorname{Ker} \tilde{T}^r(\tilde{\Pi}_S^{q-r})$ and consequently we conclude from Lemma 2.4, that for such $\bar{f}$ the image $\tilde{I}_{\mathcal{U}}^q \tilde{\iota}_S^q \bar{f} \in \tilde{T}^q(E)$ is independent of the chart $(\mathcal{U}, r_x, \varphi)$. An easy application of Lemma 2.3 yields now the

Proposition 2.4. *The family of local isomorphisms* $\{\bar I^{\mathfrak{U}}_q\}$ *corresponding to the atlas* $\mathfrak{A}$ *induces by restriction a bundle isomorphism*

$$\bar I^q_0 : \operatorname{Ker} \bar\Pi^q_S \to \operatorname{Ker} \bar\Pi^q_T = E \otimes (\overset{q}{\otimes} T(M)^*) .$$

Similarly the family of local isomophisms $\{I^q_{\mathfrak{U}}\}$ *induces by restriction a bundle isomorphism*

$$I^q_0 : \operatorname{Ker} \Pi^q_S \to \operatorname{Ker} \Pi^q_T = E \otimes (\overset{q}{\odot} T(M)^*) .$$

This may be also expressed in a more usual form as

Corollary. *The short sequences of bundle morphisms*

$$0 \to E \otimes (\overset{p}{\odot} T(M)^*) \xrightarrow{(I^q_0)^{-1} j^{q*}_T} S^q(E) \xrightarrow{\Pi^q_S} S^{q-1}(E) \to 0$$

$$0 \to E \otimes (\overset{q}{\otimes} T(M)^*) \xrightarrow{(\bar I^q_0)^{-1} j^{q*}_T} \bar S^q(E) \xrightarrow{\bar\Pi^q_S} \bar S^{q-1}(E) \to 0$$

are exact.

Suppose $q > 1$. The definition of semi-holonomic jets yields a decomposition of the injection $\bar i^q_S : \bar S^q(E) \to \tilde S^q(E)$ into the sequence

$$\bar S^q(E) \xrightarrow{\bar i^{q\prime}_S} S^1(\bar S^{q-1}(E)) \xrightarrow{S^1(\bar i^{q-1}_S)} \tilde S^q(E) . \tag{2.52}$$

Moreover we conclude from this definition, Lemma 2.3 and (2.13) that

1) the diagram

$$\begin{array}{ccccc} \bar S^q(E) & \xrightarrow{\bar i^{q\prime}_S} & S^1(\bar S^{q-1}(E)) & \xrightarrow{S^1(\bar i^{q-1}_S)} & \tilde S^q(E) \\ & \searrow^{\bar\Pi^q_S} & \downarrow{\scriptstyle \Pi_S} & & \downarrow{\scriptstyle \tilde\Pi^q_S} \\ & & \bar S^{q-1}(E) & \xrightarrow{\bar i^{q-1}_S} & \tilde S^{q-1}(E) \end{array} \tag{2.53}$$

is commutative. We define $\bar i^{1\prime}_S$ as the identity.

2) an element $X \in S^1(\bar S^{q-1}(E))$ belongs to Im $\bar i^{q\prime}_S$ if and only if

$$S^1(\bar i^{q-2}_S \bar\Pi^{q-1}_S)\, X = \bar i^{q-1}_S \Pi_S X , \tag{2.54}$$

where $\bar i^0_S : E \to E$ is the identity.

Define now the mapping $\Theta^q_S : S^1(\bar S^{q-1}(E)) \to S^1(\bar S^{q-2}(E))$ by

$$\Theta^q_S = \bar i^{q-1\prime}_S \Pi_S - S^1(\bar\Pi^{q-1}_S) \tag{2.55}$$

for each $q > 1$. We get from (2.13) and the commutativity of (2.53) $\Pi_S \Theta^q_S = \bar\Pi^{q-1}_S \Pi_S -$

$- \Pi_S S^1(\bar{\Pi}_S^{q-1}) = 0$ and thus $\mathrm{Im}\, \Theta_S^q \subset \mathrm{Ker}\, \Pi_S$. Therefore one can define the mapping $I_0 \Theta_S^q : S^1(\bar{S}^{q-1}(E)) \to \mathrm{Ker}\, \Pi_T (\subset T^1(\bar{S}^{q-2}(E)) \cong \bar{S}^{q-2}(E) \otimes T(M)^*$.

Lemma 2.5. $\mathrm{Im}\, \Theta_S^q = \mathrm{Ker}\, \Pi_S \subset S^1(\bar{S}^{q-2}(E))$.

Proof. It remains to show $\mathrm{Ker}\, \Pi_S \subset \mathrm{Im}\, \Theta_S^q$. This means: Given a local section $x \to u(x)$ in $\bar{S}^{q-2}(E)$ such that $u(a) = 0$, find a local section $x \to z(x)$ in $\bar{S}^{q-1}(E)$ such that $j_a^1 u = \bar{i}_S^{q-1\prime} \Pi_S j_a^1 z - S^1(\bar{\Pi}_S^{q-1}) j_a^1 z = \bar{i}_S^{q-1\prime}\, z(a) - j_a^1(\bar{\Pi}_S^{q-1} z)$. Applying $S^1(\bar{i}_S^{q-2})$ to this relation we get an equivalent formula $0 = \bar{i}_S^{q-1}\, z(a) - j_a^1(\bar{i}_S^{q-2} \bar{\Pi}_S^{q-1} z - \bar{i}_S^{q-2} u)$. In a fixed chart $(\mathscr{U}, r_x, \varphi)$ this means

$$0 = z^k_{p_1,\dots,p_{q-1}}(a) - \frac{\partial}{\partial x^{p_{q-1}}} (z^k_{p_1,\dots,p_{q-2},0} - u^k_{p_1,\dots,p_{q-2}})_a$$

for any $(p_1, \dots, p_{q-1})$, the components running from 0 to n. Since $u(a) = 0$, this equation is satisfied in the case $p_{q-1} = 0$ automatically and the rest is a simple differential equation.

Lemma 2.6. *$\Theta_S^q X = 0$ if and only if $X \in \mathrm{Im}\, \bar{i}_S^{q\prime}$.*

Proof. $S^1(\bar{i}_S^{q-2})$ is an injection. Therefore $\Theta_S^q X = 0$ is equivalent with $S^1(\bar{i}_S^{q-2})\, \Theta_S^q X = 0$, but $S^1(\bar{i}_S^{q-2})\, \Theta_S^q X = \bar{i}_S^{q-1} \Pi_S X - S^1(\bar{i}_S^{q-2} \bar{\Pi}_S^{q-1})\, X$ and a comparison with (2.54) yields the result.

Combining these two lemmas we get the

Proposition 2.5. *The short sequence*

$$(2.56) \qquad 0 \to \bar{S}^q(E) \xrightarrow{\bar{i}_S^{q\prime}} S^1(\bar{S}^{q-1}(E)) \xrightarrow{\Pi_T^* I_0 \Theta_S^q} \bar{S}^{q-2}(E) \otimes T(M)^* \to 0$$

is exact for each $q > 1$. (We put $\bar{S}^0(E) = E$.)

The relation $\Theta_S^q \bar{i}_S^{q\prime} = 0$ can be written in the form $\bar{i}_S^{q-1\prime} \Pi_S \bar{i}_S^{q\prime} = S^1(\bar{\Pi}_S^{q-1})\, \bar{i}_S^{q\prime}$, i.e., using the commutativity of the diagram (2.53),

$$(2.57) \qquad \bar{i}_S^{q-1\prime} \bar{\Pi}_S^q = S^1(\bar{\Pi}_S^{q-1})\, \bar{i}_S^{q\prime}\,.$$

The following lemma is evident and we bring it without proof.

Lemma 2.7. *Let E_i $(i = 1, 2, 3, 4)$ be trivial vector bundles over a trivial manifold M, i.e. admitting "global" explicite charts*

$$(2.58) \qquad (M, r_x^{(i)}, \varphi)\,, \quad (i = 1, 2, 3, 4)\,.$$

Let $\dim E_1 = \dim E_2$ and $\dim E_3 = \dim E_4$. *Denote by $I_{1,2} : E_1 \to E_2$, $I_{3,4} : E_3 \to E_4$, $I^{1,2} : S^1(E_1) \to T^1(E_2)$, $I^{3,4} : S^1(E_3) \to T^1(E_4)$ the natural isomorphisms connected with the charts* (2.58). *Let $\Phi : E_1 \to E_3$ and $\psi : E_2 \to E_4$ be bundle* *morphisms which have constant coefficients with respect to the charts* (2.58), *and such that the diagram*

$$\begin{array}{ccc} E_1 & \xrightarrow{\quad I_{1,2}\quad} & E_2 \\ {\scriptstyle\Phi}\big\downarrow & & \big\downarrow{\scriptstyle\Psi} \\ E_3 & \xrightarrow{\quad I_{3,4}\quad} & E_4 \end{array}$$

is commutative. Then also the diagram

$$\begin{array}{ccc} S^1(E_1) & \xrightarrow{\quad I^{1,2}\quad} & T^1(E_2) \\ {\scriptstyle S^1(\Phi)}\big\downarrow & & \big\downarrow{\scriptstyle T^1(\Psi)} \\ S^1(E_3) & \xrightarrow{\quad I^{3,4}\quad} & T^1(E_4) \end{array} \tag{2.59}$$

is commutative.

Let $q > 1$, $(\mathscr{U}, r_x, \varphi) \in \mathring{\mathfrak{A}}$ be fixed. Applying Lemma 2.7 to the commutative diagram

$$\begin{array}{ccc} \bar{S}^{q-1}(E) & \overset{\bar{I}^{q-1}_{\mathscr{U}}}{\dashrightarrow} & \bar{T}^{q-1}(E) \\ {\scriptstyle \bar{i}^{q-1}_S}\big\downarrow & & \big\downarrow{\scriptstyle \bar{i}^{q-1}_T} \\ \tilde{S}^{q-1}(E) & \overset{\tilde{I}^{q-1}_{\mathscr{U}}}{\dashrightarrow} & \tilde{T}^{q-1}(E)\,, \end{array}$$

we get the commutative (local) diagram

$$\begin{array}{ccc} S^1(\bar{S}^{q-1}(E)) & \overset{T^1(\bar{I}^{q-1}_{\mathscr{U}})I_{\mathscr{U}}}{\dashrightarrow} & T^1(\bar{T}^{q-1}(E)) \\ {\scriptstyle S^1(\bar{i}^{q-1}_S)}\big\downarrow & & \big\downarrow{\scriptstyle T^1(\bar{i}^{q-1}_T)} \\ S^1(\tilde{S}^{q-1}(E)) & \overset{\tilde{I}^{q}_{\mathscr{U}}}{\dashrightarrow} & T^1(\tilde{T}^{q-1}(E))\,. \end{array} \tag{2.60}$$

Define now

$$(\bar{i}^{q'}_T)_{\mathscr{U}} = T^1(\bar{I}^{q-1}_{\mathscr{U}})\, I_{\mathscr{U}} \bar{i}^{q'}_S (\bar{I}^q_{\mathscr{U}})^{-1}\,. \tag{2.61}$$

We see immediately from the commutativity of (2.60) that $T^1(\bar{i}^{q-1}_T)\,(\bar{i}^{q'}_T)_{\mathscr{U}} = \bar{i}^q_T$. But $T^1(\bar{i}^{q-1}_T)_{\mathscr{U}}$ is a bundle injection and it is not difficult to deduce from there that $(\bar{i}^{q'}_T)_{\mathscr{U}}$ is a restriction onto $\mathscr{U}$ of a bundle injection $\bar{i}^{q'}_T : \bar{T}^q(E) \to T^1(\bar{T}^{q-1}(E))$. Consequently we have the decomposition

$$\bar{T}^q(E) \xrightarrow{\;\bar{i}^{q'}_T\;} T^1(\bar{T}^{q-1}(E)) \xrightarrow{\;T^1(\bar{i}^{q-1}_T)\;} \tilde{T}^q(E)$$

analogously as in (2.52). Define also $\bar{i}_T^{1'}$ = identity. The diagram

$$
\text{(2.62)} \qquad
\begin{array}{ccccc}
\bar{T}^q(E) & \xrightarrow{\ \bar{i}_T^{q'}\ } & T^1(\bar{T}^{q-1}(E)) & \xrightarrow{\ T^1(\bar{i}_T^{q-1})\ } & \tilde{T}^q(E) \\
 & \searrow^{\bar{\Pi}_T^q} & \Big\downarrow{\scriptstyle \Pi_T} & & \Big\downarrow{\scriptstyle \tilde{\Pi}_T^q} \\
 & & \bar{T}^{q-1}(E) & \xrightarrow{\ \bar{i}_T^{q-1}\ } & \tilde{T}^{q-1}(E)
\end{array}
$$

is commutative again and

$$
\text{(2.63)} \qquad T^1(\bar{i}_T^{q-2}\bar{\Pi}_T^{q-1})\,X = \bar{i}_T^{q-1}\Pi_T X \Leftrightarrow X \in \operatorname{Im} \bar{i}_T^{q'} \subset T^1(\bar{T}^{q-1}(E))\,,
$$

or equivalently, $\Theta_T^q X \equiv \bar{i}_T^{q-1'}\Pi_T X - T^1(\bar{\Pi}_T^{q-1})\,X = 0 \Leftrightarrow X \in \operatorname{Im} \bar{i}_T^{q'}$. This can be again expressed by the formula

$$
\text{(2.64)} \qquad \bar{i}_T^{q-1'}\,\bar{\Pi}_T^q = T^1(\bar{\Pi}_T^{q-1'})\,\bar{i}_T^{q'}\,.
$$

We need not prove explicitly these relations, since their local expressions connected with a chart $(\mathscr{U}, r_x, \varphi) \in \mathring{\mathfrak{A}}$ are formally identical with the local expressions of the corresponding (established) relations in jet prolongations connected with the same chart $(\mathscr{U}, r_x, \varphi)$.

Without running the risk of confusion we shall write in the next $j_T^1, \Pi_T, j_T^{1*}, \Pi_T^*$ instead of $\bar{j}_T^q, \bar{\Pi}_T^q, \bar{j}_T^{q*}, \bar{\Pi}_T^{q*}$ respectively for each $q \geqq 1$.

If $E' \subset E$, then $T^1(E')$ can be naturally injected into $T^1(E)$ and we shall write for simplicity $T^1(E') \subset T^1(E)$. According to this convention if $\Phi : E \to F$ is a bundle morphism then $\operatorname{Ker} T^1(\Phi) = T^1(\operatorname{Ker} \Phi) = \operatorname{Ker} \Phi \otimes T^1(R)$. Moreover, using local coordinate expressions one easily verifies

$$
\text{(2.65)} \qquad X \in \operatorname{Ker} \Pi_S \subset S^1(E) \Rightarrow T^1(\Phi)\,I_0 X = I_0 S^1(\Phi)\,X\,.
$$

Now if we write (2.61) in the form

$$
\bar{i}_T^{q'}(\bar{I}_{\mathscr{U}}^q)\,X = T^1(\bar{I}_{\mathscr{U}}^{q-1})\,I_{\mathscr{U}}\bar{i}_S^{q'}X
$$

and suppose $X \in \operatorname{Ker} \bar{\Pi}_S^q$, using (2.53) we get $\bar{i}_S^{q'}X \in \operatorname{Ker} \Pi_S$ and consequently $I_{\mathscr{U}}\bar{i}_S^{q'}X = I_0\bar{i}_S^{q'}X \in \operatorname{Ker} \Pi_T$. Further (2.56) and (2.57) give $T^1(\bar{\Pi}_S^{q-1})\,I_0\bar{i}_S^{q'}X = I_0 S^1(\bar{\Pi}_S^{q-1})\,.\,\bar{i}_S^{q'}X = I_0\bar{i}_S^{q-1'}\bar{\Pi}_S^q X = 0$. Hence $I_0\bar{i}_S^{q'}X \in T^1(\operatorname{Ker} \bar{\Pi}_S^{q-1})$ and the above local relation can now be written as

$$
\text{(2.66)} \qquad X \in \operatorname{Ker} \bar{\Pi}_S^q \Rightarrow (\bar{i}_T^{q'})\,X = T^1(\bar{I}_0^{q-1})\,I_0\bar{i}_S^{q'}X\,.
$$

There is another simple property of the functor T^1. If $\Phi : E \to F$ is a bundle morphism, we can also write

$$
T^1(\Phi) = j_T^1\Phi\Pi_T + j_T^{1*}(\Phi \otimes 1)\,\Pi_T^*
$$

and hence

$$(2.67) \qquad \Pi_T^* T^1(\Phi) = (\Phi \otimes 1)\Pi_T^* ,$$

$$(2.68) \qquad T^1(\Phi) j_T^{1*} = j_T^{1*}(\Phi \otimes 1) .$$

Here and in what follows we denote by 1 any identity in the category $\mathscr{E}(M)$.

Next we bring a lemma which will be of use in the following paragraphs.

Lemma 2.8. *Let* $q > 2$. *Then*

$$(2.69) \qquad \Pi_T^* \bar{\imath}_T^{q-1\prime} \bar{j}_T^{q-1} = (\bar{j}_T^{q-2} \otimes 1)\Pi_T^* \bar{\imath}_T^{q-2\prime}$$

and

$$(2.70) \qquad \Pi_T^* \bar{\imath}_T^{q-1} \bar{j}_T^{q-1*} = \bar{\imath}_T^{q-2} \bar{j}_T^{q-2*} \otimes 1 .$$

Proof. It suffices to prove the relations but locally. Thus suppose a fixed $(\mathscr{U}, r_x, \varphi) \in \mathring{\mathfrak{A}}$ is given and let $a \in \mathscr{U}$. Let

$$\xi = \sum_{|p| \leqq q-2} \xi_p^k \otimes t_k^p(a) \in \bar{T}^{q-2}(E)_a .$$

Then

$$\bar{j}_T^{q-1} \xi = \sum_{|p| \leqq q-1} \xi_p^k \otimes t_k^p(a) \in \bar{T}^{q-1}(E)_a ,$$

where we have put $\xi_p^k = 0$ if $|p| = q - 1$. From there

$$(2.71) \qquad \Pi_T^* \bar{\imath}_T^{q-1\prime} \bar{j}_T^{q-1} \xi = \sum_{r=1}^{n} \sum_{|s| \leqq q-2} \xi_{sr}^k \otimes t_k^s(a) \otimes \mathrm{d}x^r ,$$

where $\xi_{sr}^k = 0$ if $|s| = q - 2$.

On the other hand

$$\Pi_T^* \bar{\imath}_T^{q-2\prime} \xi = \sum_{r=1}^{n} \sum_{|u| \leqq q-3} \xi_{ur}^k \otimes t_k^u(a) \otimes \mathrm{d}x^r ,$$

and

$$(\bar{j}_T^{q-2} \otimes 1)\Pi_T^* \bar{\imath}_T^{q-2\prime} \xi = \sum_{r=1}^{n} \sum_{|s| \leqq q-2} \xi_{sr}^k \otimes t_k^s(a) \otimes \mathrm{d}x^r ,$$

where we must put $\xi_{sr}^k = 0$ if $|s| = q - 2$. A comparison with (2.71) immediately yields (2.69).

Let now

$$\eta = \sum_{|p| = q-1} \eta_p^k \otimes t_k^p(a) \in E \otimes (\overset{q-1}{\otimes} T(M)^*)_a .$$

Then

$$\bar{j}_T^{q-1*} \eta = \sum_{|p| \leqq q-1} \eta_p^k \otimes t_k^p(a) ,$$

with $\eta^k_p = 0$ if $|p| < q - 1$. Further

$$\bar{\imath}^{q-1}_T\, j^{q-1*}_T \eta = \bullet \sum_{|u|=q-1} \eta^k_{\omega(u)} \bullet t^u_k(a)$$

and, noticing that $\omega(sr) = \omega(s)\, r$ if $r \neq 0$,

$$\Pi^*_T \bar{\imath}^{q-1}_T j^{q-1*}_T \eta = \sum_{r=1}^{n} \bullet \sum_{|s|=q-2} \eta^k_{\omega(s)r} \bullet t^s_k(a) \otimes \mathrm{d}x^r\,, \tag{2.72}$$

where we have put $\eta^k_{\omega(s)r} = 0$ if $|\omega(s)| < q - 2$.

On the other hand

$$(\bar{\imath}^{q-2*}_T \otimes 1)\, \eta = \sum_{r=1}^{n} \sum_{|u| \leqq q-2} \eta^k_{ur} \otimes t^u_k(a) \otimes \mathrm{d}x^r\,,$$

where we put $\eta^k_{ur} = 0$ if $|u| < q - 2$. Further

$$(\bar{\imath}^{q-2}_T \otimes 1)\,(j^{q-2*}_T \otimes 1)\, \eta = \sum_{r=1}^{n} \bullet \sum_{|s|=q-2} \eta^k_{\omega(s)r} \bullet t^s_k(a) \otimes \mathrm{d}x^r\,,$$

where $\eta^k_{\omega(s)r} = 0$ if $|\omega(s)| < q - 2$. Comparing again this result with (2.72), we get (2.70) and this completes the proof.

Corollary. *If $q > 2$, then*

$$j^{1*}_T(\bar{\imath}^{q-2}_T\, j^{q-2*}_T \otimes 1) = \bar{\imath}^{q-1}_T j^{q-1*}_T\,, \tag{2.73}$$

$$j^{q-2*}_T \otimes 1 = \Pi^*_T \bar{\imath}^{q-1'}_T j^{q-1*}_T\,, \tag{2.74}$$

$$(\overline{\Pi}^{q-2*}_T \otimes 1)\, \Pi^*_T \bar{\imath}^{q-1'}_T = \overline{\Pi}^{q-1*}_T\,, \tag{2.75}$$

$$\Pi^*_T\, \bar{\imath}^{q-1}_T\, j^{q-1}_T = (\bar{\imath}^{q-2}_T\, j^{q-2}_T \otimes 1)\, \Pi^*_T\, \bar{\imath}^{q-2'}_T \tag{2.76}$$

$$T^1(j^{q-2}_T) \bar{\imath}^{q-2'}_T + T^1(j^{q-2*}_T)\, j^1_T \overline{\Pi}^{q-2*}_T = \bar{\imath}^{q-1'}_T\, j^{q-1}_T \tag{2.77}$$

$$\bar{\imath}^{q-1}_T j^{q-1}_T - j^1_T \bar{\imath}^{q-2}_T = j^{1*}_T(\bar{\imath}^{q-2}_T j^{q-2}_T \otimes 1)\, \Pi^*_T \bar{\imath}^{q-2'}_T\,. \tag{2.78}$$

Proof. "Multiplying" (2.70) by j^{1*}_T we get from Lemma 2.3 $j^{1*}_T(\bar{\imath}^{q-2}_T\, j^{q-2}_T \otimes 1) = \bar{\imath}^{q-1}_T\, j^{q-1*}_T - j^1_T \Pi_T \bar{\imath}^{q-1}_T\, j^{q-1*}_T = \bar{\imath}^{q-1}_T\, j^{q-1*}_T - j^1_T \bar{\imath}^{q-2}_T \overline{\Pi}^{q-1}_T\, j^{q-1*}_T = \bar{\imath}^{q-1'}_T\, j^{q-1T}_T$ and this is (2.73). Using (2.67) we give (2.70) the form $(\bar{\imath}^{q-2}_T \otimes 1)\,(j^{q-2*}_T \otimes 1) = \Pi^*_T\,.$ $.\, T^1(\bar{\imath}^{q-2}_T) \bar{\imath}^{q-1'}_T\, j^{q-1*}_T = (\bar{\imath}^{q-2}_T \otimes 1) \Pi^*_T \bar{\imath}^{q-1'}_T\, j^{q-1*}_T$ and since $(\bar{\imath}^{q-2}_T \otimes 1)$ is an injection, we have (2.74). Now applying $(\overline{\Pi}^{q-2*}_T \otimes 1)$ to (2.74) we get $1 = (\overline{\Pi}^{q-2*}_T \otimes 1)\, \Pi^*_T \bar{\imath}^{q-1'}_T\, j^{q-1*}_T$ and from there $\overline{\Pi}^{q-1*}_T = (\overline{\Pi}^{q-2*}_T \otimes 1)\, \Pi^*_T \bar{\imath}^{q-1'}_T\, j^{q-1*}_T \overline{\Pi}^{q-1*}_T$, or $\overline{\Pi}^{q-1*}_T = (\overline{\Pi}^{q-2*}_T \otimes 1)\, \Pi^*_T \bar{\imath}^{q-1'}_T - (\overline{\Pi}^{q-2*}_T \otimes 1)\, \Pi^*_T \bar{\imath}^{q-1'}_T\, j^{q-1}_T \overline{\Pi}^{q-1}_T$. But the last term here gives according to (2.69) $(\overline{\Pi}^{q-2*}_T \otimes 1)\,(j^{q-2}_T \otimes 1)\, \Pi^*_T \bar{\imath}^{q-2'}_T \overline{\Pi}^{q-1}_T = 0$ and this proves (2.75). Now we have from (2.67) and (2.69) $\Pi^*_T \bar{\imath}^{q-1}_T\, j^{q-1}_T = \Pi^*_T T^1(\bar{\imath}^{q-2}_T)\, \bar{\imath}^{q-1'}_T\, j^{q-1}_T = (\bar{\imath}^{q-2}_T \otimes 1)\, \Pi^*_T \bar{\imath}^{q-1'}_T j^{q-1}_T = (\bar{\imath}^{q-2}_T\, j^{q-2}_T \otimes 1)\, \Pi^*_T \bar{\imath}^{q-2'}_T$ and this is (2.76).

Further (2.53), (2,69), (2.67) and (2.14) give $\bar{i}_T^{q-1'}\bar{j}_T^{q-1} = j_T^1 \Pi_T \bar{i}_T^{q-1'}\bar{j}_T^{q-1} + j_T^{1*}\Pi_T^* \bar{i}_T^{q-1'}\bar{j}_T^{q-1} = j_T^1 \overline{\Pi}_T^{q-1}\bar{j}_T^{q-1} + j_T^{1*}\Pi_T^* T^1(\bar{j}_T^{q-2})\,\bar{i}_T^{q-2'} = j_T^1 + T^1(\bar{j}_T^{q-2})\,\bar{i}_T^{q-2'} - j_T^1 \bar{j}_T^{q-2}\overline{\Pi}_T^{q-2} = T^1(\bar{j}_T^{q-2})\,\bar{i}_T^{q-2'} - j_T^1 \bar{j}_T^{q-2*}\overline{\Pi}_T^{q-2*}$ and this verifies (2.77).

Finally from (2.67), (2.14), (2.53), (2.15) and (2.77) we get subsequently $j_T^{1*}(\bar{i}_T^{q-2}\ \bar{j}_T^{q-2} \otimes 1)\,\Pi_T^* \bar{i}_T^{q-2'} = j_T^{1*}\Pi_T^* T^1(\bar{i}_T^{q-2}\bar{j}_T^{q-2})\,\bar{i}_T^{q-2'} = T^1(\bar{i}_T^{q-2}\bar{j}_T^{q-2})\,\bar{i}_T^{q-2'} - j_T^1 \bar{i}_T^{q-2}\,\bar{j}_T^{q-2}\overline{\Pi}_T^{q-2} = T^1(\bar{i}_T^{q-2}\,\bar{j}_T^{q-2})\,\bar{i}_T^{q-2'} + j_T^1 \bar{i}_T^{q-2}\,\bar{j}_T^{q-2*}\overline{\Pi}_T^{q-2*} - j_T^1 \bar{i}_T^{q-2} = T^1(\bar{i}_T^{q-2})\,.\,[T^1(\bar{j}_T^{q-2})\,\bar{i}_T^{q-2'} + T^1(\bar{j}_T^{q-2*})\,j_T^1 \overline{\Pi}_T^{q-2*}] - j_T^1 \bar{i}_T^{q-2} = T^1(\bar{i}_T^{q-2})\,\bar{i}_1^{q-1'}\bar{j}_T^{q-1} - j_T^1 \bar{i}_T^{q-2}$ and this completes the proof of (2.78).

3. FIRST ORDER CONNECTIONS AND PSEUDO-CONNECTIONS ON VECTOR BUNDLES

Let E be a vector bundle over M and $\mathfrak{A}$ be a chosen atlas of E. As we have seen, there is a natural family $\{I_{\mathscr{U}}\}$ of local isomorphisms

$$(3.1)\qquad (\mathscr{U}, r_x, \varphi) \in \mathring{\mathfrak{A}} \Rightarrow I_{\mathscr{U}} : S^1(E) \to T^1(E)\,.$$

However, the fibre bundle structures $(S^1(E), S^1(\mathfrak{A}))$ and $(T^1(E), T^1(\mathfrak{A}))$ are not isomorphic in general, i.e. there is no bundle isomorphism $S^1(E) \to T^1(E)$ connecting $S^1(\mathfrak{A})$ with $T^1(\mathfrak{A})$ since $G(S^1(\mathfrak{A})) \neq G(T^1(\mathfrak{A}))$ unless $\mathfrak{A}$ defines a trivial fibre bundle structure. In other words, in general, there is no isomorphism of the corresponding principal fibre bundles $\mathscr{P}(S^1(E), S^1(\mathfrak{A}))$ and $\mathscr{P}(T^1(E), T^1(\mathfrak{A}))$. Therefore a bundle isomorphism of the vector bundles $S^1(E)$ and $T^1(E)$ may exist only a posteriori, defining in this way an additional structure on the vector bundle E. Note that it follows from what we have just said, that this isomorphism cannot take $S^1(\mathfrak{A})$ into $T^1(\mathfrak{A})$ but for the trivial case. In fact, we shall see that each connection on E can be interpreted as some bundle isomorphism $H : S^1(E) \to T^1(E)$.

Definition 3.1. A *pseudo-connection* H (of first order) on the vector bundle E is a bundle isomorphism $H : S^1(E) \to T^1(E)$.

Given a pseudo-connection H on E, the local isomorphisms (3.1) define local isomorphisms

$$(3.2)\qquad (\mathscr{U}, r_x, \varphi) \in \mathring{\mathfrak{A}} \Rightarrow \Gamma_{\mathscr{U}} : \overset{-1}{p}_{T^1(E)}(\mathscr{U}) \to \overset{-1}{p}_{T^1(E)}(\mathscr{U})$$

subject to

$$(3.3)\qquad (\mathscr{U}, r_x, \varphi) \in \mathring{\mathfrak{A}} \Rightarrow \Gamma_{\mathscr{U}} I_{\mathscr{U}} = H\,.$$

In this way the pseudo-connection H connects with the atlas $\mathfrak{A}$ a family $\{\Gamma_{\mathscr{U}}\}$ of local isomorphisms.

Let $(\mathscr{U}, r_x, \varphi) \in \mathring{\mathfrak{A}}$. Then for each $a \in \mathscr{U}$

$$\Gamma_{\mathscr{U}}(y^k_\alpha\, \mathrm{d}x^\alpha \otimes i_k(a)) = \hat{y}^h_\beta\, \mathrm{d}x^\beta \otimes i_h(a) \tag{3.4}$$

and $\Gamma_{\mathscr{U}}$ can be expressed in coordinates as an element of $[R \oplus T(M)^*]^*_a \otimes E^*_a \otimes \otimes [R \oplus T(M)^*]_a \otimes E_a$, i.e. as a tensor at $a \in \mathscr{U}$ with components $\Gamma^{h\alpha}_{k\beta} = \Gamma^{h\alpha}_{k\beta}(a)$, $(h, k = 1, \dots, m;\ \alpha, \beta = 0, 1, \dots, n)$. The relation (3.4) is then to be written as

$$\hat{y}^h_\beta = \Gamma^{h\alpha}_{k\beta} y^k_\alpha\,. \tag{3.5}$$

Suppose now $(\mathscr{U}', r'_x, \varphi') \in \mathring{\mathfrak{A}}$ and $a \in \mathscr{U} \cap \mathscr{U}'$. Then together with (3.5) we have

$$\hat{y}^{h'}_{\beta'} = \Gamma^{h'\alpha'}_{k'\beta'} y^{k'}_{\alpha'}\,. \tag{3.6}$$

The condition (3.3) determines the transition formulae for $\{\Gamma_{\mathscr{U}}\}$. In fact, we have

$$j^1_a f = f^k_\alpha s^\alpha_k(a) \in \overset{-1}{p}_{S^1(E)}(\mathscr{U} \cap \mathscr{U}') \Rightarrow \Gamma^{h'\alpha'}_{k'\beta'} M^{k'\alpha'}_{k\alpha'} f^k_\alpha = f^h_\beta N^{h'\beta}_{h\beta'}$$

and from there we get the required relation

$$\Gamma^{h'\alpha'}_{k'\beta'} = M^{k\alpha'}_{k'\alpha} N^{h'\beta}_{h\beta'} \Gamma^{h\alpha}_{k\beta}\,. \tag{3.7}$$

This can be written explicitly using (2.9) and (2.10) as

$$\Gamma^{h'0'}_{k'0'} = g^k_{k'} g^{h'}_h \Gamma^{h0}_{k0} + A^{r'}_i \left(\frac{\partial}{\partial x^{r'}} g^k_{k'}\right) g^{h'}_h \Gamma^{hi}_{k0}\,, \tag{3.8}$$

$$\Gamma^{h'0'}_{k'j'} = g^k_{k'} g^{h'}_h A^j_{j'} \Gamma^{h0}_{kj} + A^{r'}_i \left(\frac{\partial}{\partial x^{r'}} g^k_{k'}\right) g^{h'}_h A^j_{j'} \Gamma^{hi}_{kj}\,,$$

$$\Gamma^{h'i'}_{k'0'} = g^k_{k'} g^{h'}_h A^{i'}_i \Gamma^{hi}_{k0}\,,$$

$$\Gamma^{h'i'}_{k'j'} = g^k_{k'} g^{h'}_h A^{i'}_i A^j_{j'} \Gamma^{hi}_{kj}\,.$$

Conversely, given a family $\{\Gamma_{\mathscr{U}}\}$ of local differentiable isomorphisms (3.2) satisfying the transition formulae (3.7) or (3.8), the equation (3.3) defines a unique pseudo-connection on E.

Note that each of the local isomorphisms (3.1) satisfies $\Pi_T I_{\mathscr{U}} = \Pi_S$ and $I_{\mathscr{U}}|_{\mathrm{Ker}\,\Pi_S} = I_0$. This may suggest the following

Definition 3.2. A *connection* H (*of first order*) on the vector bundle E is a bundle morphism $H : S^1(E) \to T^1(E)$ satisfying

$$\Pi_T H = \Pi_S \tag{3.9}$$

and

$$H|_{\mathrm{Ker}\,\Pi_S} = I_0 \Leftrightarrow H^{-1} j^{1*}_T = I_0^{-1}\, j^{1*}_T\,. \tag{3.10}$$

Proposition 3.1. *Each connection H on E is simultaneously a pseudo-connection on E.*

Proof. Since the dimensions of $S^1(E)$ and $T^1(E)$ are equal it is sufficient to show that H being a bundle morphism is an injection at each $a \in M$. Thus let $\dot{f} \in S^1(E)_a$ and $H\dot{f} = 0$. From (3.9) we get $\dot{f} \in \operatorname{Ker} \Pi_S$ and hence $H\dot{f} = I_0\dot{f}$. But I_0 is an isomorphism and consequently $\dot{f} = 0$.

If H is a fixed pseudo-connection on E then each other pseudo-connection H' on E has the form $H' = KH$, where $K : T^1(E) \to T^1(E)$ is any bundle isomorphism. Moreover we have the

Proposition 3.2. *If H is a connection on E, then $KH : S^1(E) \to T^1(E)$ is another connection on E if and only if K satisfies*

$$\Pi_T K = \Pi_T \tag{3.11}$$

and

$$K|_{\operatorname{Ker} \Pi_T} = 1_{\operatorname{Ker} \Pi_T}, \quad i.e. \quad \Pi_T^* K j_T^{1*} = 1_{E \otimes T(M)^*}. \tag{3.12}$$

Proof. In fact, (3.9), i.e. $\Pi_T = \Pi_S H^{-1}$ implies the equivalence of (3.11) and $\Pi_T KH = \Pi_S$. Further $\dot{f} \in \operatorname{Ker} \Pi_S$ iff $I_0\dot{f} \in \operatorname{Ker} \Pi_T$ and $KI_0\dot{f} = I_0\dot{f}$ is equivalent with $KH\dot{f} = I_0\dot{f}$ since $H\dot{f} = I_0\dot{f}$.

This proposition characterizes the class of all possible connections on a vector bundle.

Now we shall pass to the explicite formulae for the local isomorphisms (3.2) in the case of a connection.

Proposition 3.3. *Let $\{\Gamma_{\mathscr{U}}\}$ correspond to a pseudo-connection H on E. Then H is a connection if and only if the local isomorphisms (3.2) given by (3.5) satisfy for each $(\mathscr{U}, r_x, \varphi) \in \mathring{\mathfrak{A}}$ the relations*

$$\Gamma^{h0}_{k0} = \delta^h_k; \quad \Gamma^{hi}_{k0} = 0 \tag{3.13}$$

and

$$\Gamma^{hi}_{kj} = \delta^h_k \delta^i_j. \tag{3.14}$$

Proof. It is not difficult to see that in a given chart $(\mathscr{U}, r_x, \varphi) \in \mathring{\mathfrak{A}}$ the condition (3.9) has the form

$$\Gamma^{h0}_{k0} f^k_h + \Gamma^{hi}_{k0} f^k_i = \delta^h_k f^k_0$$

whatever be f^k_0, f^k_i and this implies (3.13). On the other hand (3.10) is to be written as

$$\Gamma^{hi}_{kj} f^k_i + 0 = \delta^h_k \delta^i_j f^k_i$$

which yields (3.14).

One could also directly verify that this special choice of the quantities $\Gamma^{h\alpha}_{k\beta}$ is compatible with the transition formulae (3.7) or (3.8). Denoting in this case by Γ^{h}_{ki} the only non-trivial components Γ^{h0}_{ki} we see that the transition formulae are reduced to the only equation

$$(3.15) \qquad \Gamma^{h'}_{k'j'} = g^{k}_{k'} g^{h'}_{h} A^{j}_{j'} \Gamma^{h}_{kj} + \left(\frac{\partial}{\partial x^{j'}} g^{h}_{k'}\right) g^{h'}_{h} .$$

Remark 4. Let $E = T(M)$ be the tangent bundle of M and $\mathfrak{A}$ the natuual (semi-complete) atlas on it. Then $G(\mathfrak{A}) = GL(R, n)$ and $g^{k}_{k'} = A^{k}_{k'}$ for any two charts with not disjoint domains. In this case (3.15) become the usual transformation formulae defining a connection on M in the classical meaning of the word. The map $f \to (x \to (\Pi^{*}_{T}H)\, j^{1}_{x}f)$ defined on local sections in $T(M)$, i.e. on local vector fields, is nothing but the usual covariant differential of f.

The conditions (3.11) and (3.12) can also be expressed locally. In fact, let K be given in a fixed chart $(\mathscr{U}, r_x, \varphi)$ of $\mathring{\mathfrak{A}}$ by means of the (local) relation

$$(3.16) \qquad \hat{y}^{h}_{\beta} = K^{h\alpha}_{k\beta} y^{k}_{\alpha} .$$

The corresponding transition formulae are evident since K is a bundle isomorphism. In particular K^{h0}_{ki} are the components of a "tensor field on the whole", these components being taken with respect to the frame in $E^* \otimes E \otimes T(M)^*$ corresponding to the chart $(\mathscr{U}, r_x, \varphi)$. Analogously one can see that the conditions (3.11) and (3.12) give $K^{h0}_{k0} = \delta^{h}_{k}$; $K^{hi}_{k0} = 0$; $K^{hj}_{ki} = \delta^{h}_{k}\delta^{i}_{j}$. If $E = T(M)$, this is again a well known result. Namely if $\{\Gamma_{\mathscr{U}}\}$ corresponds to a connection H on E and K satisfies (3.11) and (3.12) we get in each chart $(\mathscr{U}, r_x, \varphi) \in \mathring{\mathfrak{A}}$ the relation

$$(K\Gamma)^{h0}_{kj} = \Gamma^{h0}_{kj} + K^{h0}_{kj}$$

stating that two connections on M differ by a tensor field.

Let again the family $\{\Gamma_{\mathscr{U}}\}$ of local isomorphisms (3.2) correspond to a pseudoconnection H. If $(\mathscr{U}, r_x, \varphi)$ is fixed, $\Gamma_{\mathscr{U}}$ can be represented as a section over $\mathscr{U}$ in $\mathscr{L}(E \oplus E \otimes T(M)^*, E \oplus E \otimes T(M)^*)$ or symbolically as a "matrix" of sections in

$$(3.17) \qquad \begin{pmatrix} \mathscr{L}(E, E); & \mathscr{L}(E, E \otimes T(M)^*) \\ \mathscr{L}(E \otimes T(M)^*, E); & \mathscr{L}(E \otimes T(M)^*, E \otimes T(M)^*) \end{pmatrix} .$$

The transition formulae (3.8) show that if we write accordingly symbolically

$$\Gamma_{\mathscr{U}} = \begin{pmatrix} \xi_{\mathscr{U}}; & \varpi_{\mathscr{U}} \\ \eta_{\mathscr{U}}; & \zeta_{\mathscr{U}} \end{pmatrix},$$

then $\eta_{\mathscr{U}}$ and $\zeta_{\mathscr{U}}$ respectively are nothing but a restriction to the domain $\mathscr{U}$ of a section on the whole in $\mathscr{L}(E \otimes T(M)^*, E)$ and $\mathscr{L}(E \otimes T(M)^*, E \otimes T(M)^*)$ respectively.

If now H is a connection, then these sections are the zero section and the identity section respectively. Moreover in this case also $\xi_{\mathscr{U}}$ is a restriction of the identity section in $\mathscr{L}(E, E)$ and only $\bar{\omega}_{\mathscr{U}}$ being a non-evident local section characterizes the connection. Suppose now $(\mathscr{U}', r'_x, \varphi') \in \mathring{\mathfrak{A}}$ is another chart and put, for the sake of simplicity, $\mathscr{U} = \mathscr{U}'$. Both $\bar{\omega}_{\mathscr{U}}$ and $\bar{\omega}'_{\mathscr{U}}$ are local sections over $\mathscr{U}$ in $\mathscr{L}(E, E \otimes T(M)^*)$ but they differ by an element that we are going to write down in a coordinate-free form.

Let $\omega_{\mathscr{U}}$ be the local differential form on $\mathscr{U}$ with values in $\mathscr{L}(E_0, E_0)$ associated with $\bar{\omega}_{\mathscr{U}}$. This means that, identifying $\mathscr{L}(E, E \otimes T(M)^*)$ with $\mathscr{L}(T(M), \mathscr{L}(E, E))$, we put for a fixed point $x \in \mathscr{U}$ and fixed vector of $T(M)_x$

$$y \in E_0 \Rightarrow \omega_{\mathscr{U}}(y) = r_x \bar{\omega}_{\mathscr{U}}(r_x^{-1} y) . \tag{3.18}$$

Denoting now $g = r'_x r_x^{-1}$ for each $x \in \mathscr{U}$, we have clearly $g \in \mathscr{L}(E_0, E_0)$ and $\mathrm{d}g$ is a local differential form over $\mathscr{U}$ with values in $\mathscr{L}(E_0, E_0)$. Note that, as a matter of fact, $\mathscr{L}(E_0, E_0)$ represents the Lie algebra of the "full" structure group $GL(R, E_0) \cong \cong GL(R, m)$.

Under these notations (3.15) can be written as

$$\omega'_{\mathscr{U}} = g \omega_{\mathscr{U}} g^{-1} + g \, \mathrm{d}g^{-1} . \tag{3.19}$$

Here the multiplications are simply compositions of (linear) mappings. Since $0 = = \mathrm{d}(1) = g \, \mathrm{d}g^{-1} + \mathrm{d}g \, g^{-1}$, the condition (3.19) is equivalent to

$$\omega'_{\mathscr{U}} = g \omega_{\mathscr{U}} g^{-1} - \mathrm{d}g \, g^{-1} . \tag{3.20}$$

These are the transformation formulae for differential forms defining a connection on a vector bundle as introduced in [2]. They represent a special case of the formulae dealt with in the general theory of connections working with principal fibre bundles (see e.g. [4]).

Now we obtain immediately the relation between $\bar{\omega}_{\mathscr{U}}$ and $\bar{\omega}'_{\mathscr{U}}$. Given again a fixed $x \in \mathscr{U}$ and a fixed vector of $T(M)_x$, we get for each $y \in E_x$ from (3.19) and (3.18)

$$\bar{\omega}'_{\mathscr{U}}(y) = (r'_x)^{-1} \omega'_{\mathscr{U}}(r'_x y) = (r'_x)^{-1} r'_x r_x^{-1} \omega_{\mathscr{U}}(r_x (r'_x)^{-1} r'_x y) + (r'_x)^{-1} r'_x r_x^{-1} \mathrm{d}(r_x (r'_x)^{-1}) \, r'_x y$$

and hence

$$\bar{\omega}'_{\mathscr{U}} = \bar{\omega}_{\mathscr{U}} + r_x^{-1} \mathrm{d}(r_x (r'_x)^{-1}) \, r'_x . \tag{3.21}$$

Note that the last term in (3.21) is a section over $\mathscr{U}$ in $\mathscr{L}(T(M), \mathscr{L}(E, E)) = = \mathscr{L}(E, E \otimes T(M)^*)$ and thus the summation is really defined.

In what now follows we are going to show how connections H_E and H_F on vector bundles E and F over M respectively define canonical connections on vector bundles $E \oplus F$, $E \otimes F$ and E^*. We shall obtain formulae independent of local coordinate expressions which correspond to classical results if $E = T(M)$ and $F = T(M)$ or $T(M)^*$ (see also [2]).

Let the direct sum $E \oplus F$ of vector bundles E and F be represented by the diagram

$$(3.22) \qquad E \underset{j}{\overset{\Pi}{\leftrightarrows}} E \oplus F \underset{j^*}{\overset{\Pi^*}{\rightleftarrows}} F\,.$$

There are natural isomorphisms $S^1(E \oplus F) = S^1(E) \oplus S^1(F)$ and $T^1(E \oplus F) = T^1(E) \oplus T^1(F)$ and we get the diagrams

$$S^1(E) \underset{S^1(j)}{\overset{S^1(\Pi)}{\leftrightarrows}} S^1(E \oplus F) \underset{S^1(j^*)}{\overset{S^1(\Pi^*)}{\rightleftarrows}} S^1(F)$$

$$T^1(E) \underset{T^1(j)}{\overset{T^1(\Pi)}{\leftrightarrows}} T^1(E \oplus F) \underset{T^1(j^*)}{\overset{T^1(\Pi^*)}{\rightleftarrows}} T^1(F)\,.$$

If now H_E and H_F are pseudo-connections on E and F respectively, then

$$(3.23) \qquad H_E(\oplus)\, H_F = T^1(j)\, H_E S^1(\Pi) + T^1(j^*)\, H_F S^1(\Pi^*)$$

is a pseudo-connection on $E \oplus F$ and it is not difficult to see that it is a connection if H_E and H_F are connections.

The problem is however not nearly so simple in the case of the tensor product $E \otimes F$. Thus let $\mathfrak{A}^E$ and $\mathfrak{A}^F$ be atlases of E and F respectively. They induce in a natural manner an atlas $\mathfrak{A}^{E\otimes F}$ of $E \otimes F$ and thus the local isomorphisms $\{I^E_{\mathcal{U}}\}$ and $\{I^F_{\mathcal{U}}\}$ of first order jet and tensor prolongations, connected with $\mathfrak{A}^E$ and $\mathfrak{A}^F$ respectively, induce the corresponding family $\{I^{E\otimes F}_{\mathcal{U}}\}$ of local isomorphisms. If $j^1_a(f_1 \otimes f_2) \in \in \overset{-1}{p}_{S^1(E\otimes F)}(\mathcal{U})$, we can write with slight inaccuracy

$$(3.24) \qquad I^{E\otimes F}_{\mathcal{U}}\, j^1_a(f_1 \otimes f_2) = I^E_{\mathcal{U}}\, j^1_a f_1 \otimes f_2(a) + f_1(a) \otimes I^F_{\mathcal{U}} j^1_a f_2 - f_1(a) f_2(a)\,,$$

where f_1 and f_2 are local sections in E and F respectively. Even the local formula (3.24) suggests the definition of connection $H_E(\otimes)\, H_F$ on $E \otimes F$ that we are going to give below.

Denote by $\gamma^E = \gamma^E_{E\otimes F} : T^1(E) \otimes F \to T^1(E \otimes F)$ and $\gamma^F = \gamma^F_{E\otimes F} : E \otimes T^1(F) \to \to T^1(E \otimes F)$ the natural bundle isomorphisms. They satisfy the following evident

Lemma 3.1. *Let E_1, E_2 and F_1, F_2 be vector bundles over M and $\Phi : E_1 \to E_2$, $\Psi : F_1 \to F_2$ bundle morphisms. Then the diagram*

$$(3.25) \qquad \begin{array}{ccccc} E_1 \otimes T^1(F_1) & \xrightarrow{\gamma^{F_1}} & T^1(E_1 \otimes F_1) & \xleftarrow{\gamma^{E_1}} & T^1(E_1) \otimes F_1 \\ \Big\downarrow {\scriptstyle \Phi \otimes T^1(\Psi)} & & \Big\downarrow {\scriptstyle T^1(\Phi \otimes \Psi)} & & \Big\downarrow {\scriptstyle T^1(\Phi) \otimes \Psi} \\ E_2 \otimes T^1(F_2) & \xrightarrow{\gamma^{F_2}} & T^1(E_2 \otimes F_2) & \xleftarrow{\gamma^{E_2}} & T^1(E_2) \otimes F_2 \end{array}$$

is commutative, i.e.

$$T^1(\Phi \otimes \Psi)\,\gamma^{E_1}_{E_1\otimes F_1} = \gamma^{E_2}_{E_2\otimes F_2}[T^1(\Phi) \otimes \Psi] \tag{3.26}$$

and

$$T^1(\Phi \otimes \Psi)\,\gamma^{F_1}_{E_1\otimes F_1} = \gamma^{F_2}_{E_2\otimes F_2}[\Phi \otimes T^1(\Psi)]\,. \tag{3.27}$$

If now $H_E : S^1(E) \to T^1(E)$ and $H_F : S^1(F) \to T^1(F)$ are bundle morphisms, define first the "product"

$$[H_E; H_F] : S^1(E) \otimes S^1(F) \to T^1(E \otimes F) \tag{3.28}$$

by

$$[H_E; H_F] = \gamma^E(H_E \otimes \Pi_S) + \gamma^F(\Pi_S \otimes H_F) - j^1_T(\Pi_S \otimes \Pi_S)\,. \tag{3.29}$$

It is clearly a bundle morphism. Define further the bundle projection

$$\sigma \equiv \sigma_{E,F} : S^1(E) \otimes S^1(F) \to S^1(E \otimes F)$$

by

$$\sigma(j^1_a f_1 \otimes j^1_a f_2) = j^1_a(f_1 \otimes f_2)\,.$$

It is not difficult to see that σ is well defined and that it is a projection.

Lemma 3.2. *Let H_E and H_F be connections on E and F respectively. Then $\sigma(X) = 0$ implies $[H_E; H_F]\, X = 0$ for any $X \in S^1(E) \otimes S^1(F)$.*

Proof. Let $a \in \mathcal{U}$, $(\mathcal{U}, r_x, \varphi) \in \mathring{\mathfrak{A}}^{E\otimes F}$ and let $\{\underset{(E)}{s^\alpha_k}\}$, $\{\underset{(F)}{s^\beta_r}\}$, $\{s^\gamma_{ut}\}$ and $\{t^\omega_{ut}\}$ be the corresponding frames in $S^1(E)$, $S^1(F)$, $S^1(E \otimes F)$ and $T^1(E \otimes F)$ respectively. Let

$$X = X^{kr}_{\alpha\beta}\,\underset{(E)}{s^\alpha_k(a)} \otimes \underset{(F)}{s^\beta_r(a)}\,. \tag{3.30}$$

We can write

$$\underset{(E)}{s^\alpha_k(a)} = j^1_a(\xi^\alpha_k)\,, \quad \text{where} \quad \frac{\partial}{\partial x^\gamma}(\xi^\alpha_k)^u = \delta^\alpha_\gamma \delta^u_k$$

and

$$\underset{(F)}{s^\beta_r(a)} = j^1_a(\eta^\beta_r)\,, \quad \text{where} \quad \frac{\partial}{\partial x^\gamma}(\eta^\beta_r)^t = \delta^\beta_\gamma \delta^t_r\,.$$

But since

$$\sigma(\underset{(E)}{s^\alpha_k(a)} \otimes \underset{(F)}{s^\beta_r(a)}) = \frac{\partial}{\partial x^\gamma}[(\xi^\alpha_k)^u (\eta^\beta_r)^t]\, s^\gamma_{ut}(a) = \delta^\alpha_0\delta^\beta_0\delta^u_k\delta^t_r s^0_{ut}(a) + (\delta^\alpha_i\delta^\beta_0\delta^u_k\delta^t_r +$$
$$+ \delta^\alpha_0\delta^\beta_i\delta^u_k\delta^t_r)\, s^i_{ut}(a)$$

we have

$$(3.31)\qquad \sigma(X) = X^{ut}_{00} s^{0}_{ut}(a) + (X^{ut}_{i0} + X^{ut}_{0i})\, s^{i}_{ut}(a)$$

and thus $\sigma(X) = 0$ implies

$$(3.32)\qquad X^{ut}_{00} = X^{ut}_{i0} + X^{ut}_{0i} = 0\,.$$

On the other hand we get analogously

$$[H_E; H_F]\,(s^{\alpha}_{k}(a) \otimes s^{\beta}_{r}(a)) = \{\underset{(E)}{\Gamma^{u\alpha}_{k\omega}}\delta^{t}_{r}\delta^{\beta}_{0} + \delta^{u}_{k}\delta^{\alpha}_{0}\underset{(F)}{\Gamma^{t\beta}_{r\omega}} - \delta^{u}_{k}\delta^{t}_{r}\delta^{\alpha}_{0}\delta^{\beta}_{0}\delta^{\omega}_{0}\}\, t^{\omega}_{ut}(a)\,,$$

where $\underset{(E)}{\Gamma^{u\alpha}_{k\omega}}$ and $\underset{(F)}{\Gamma^{t\beta}_{r\omega}}$ denote the components of H_E and H_F respectively in the chart in view according to (3.5). Thus

$$(3.33)\qquad [H_E; H_F]\, X = (X^{kt}_{\alpha 0}\underset{(E)}{\Gamma^{u\alpha}_{k\omega}} + X^{ur}_{0\beta}\underset{(F)}{\Gamma^{t\beta}_{r\omega}} - X^{ut}_{00}\delta^{\omega}_{0})\, t^{\omega}_{ut}(a)\,.$$

According to Proposition 3.3 the components of the connections H_E and H_F satisfy

$$(3.34)\qquad \underset{(E)}{\Gamma^{u0}_{k0}} = \delta^{u}_{k}\,;\quad \underset{(F)}{\Gamma^{t0}_{r0}} = \delta^{t}_{r}$$

$$(3.35)\qquad \underset{(E)}{\Gamma^{ui}_{k0}} = 0\,;\quad \underset{(F)}{\Gamma^{ti}_{r0}} = 0$$

$$(3.36)\qquad \underset{(E)}{\Gamma^{ui}_{kj}} = \delta^{u}_{k}\delta^{i}_{j}\,;\quad \underset{(F)}{\Gamma^{ti}_{rj}} = \delta^{t}_{r}\delta^{i}_{j}\,.$$

Using these relations we can write (3.33) in the form

$$(3.37)\qquad \begin{aligned}[H_E; H_F]\, X &= (X^{ut}_{00} + X^{ut}_{00} - X^{ut}_{00})\, t^{0}_{ut}(a) + \\ &+ (X^{ut}_{j0} + X^{ut}_{0j} + X^{kt}_{00}\underset{(E)}{\Gamma^{u0}_{kj}} + X^{ur}_{00}\underset{(F)}{\Gamma^{t0}_{rj}})\, t^{j}_{ut}(a)\end{aligned}$$

and a comparison with (3.32) yields the required implication.

This lemma justifies the definition of the bundle morphism

$$H_E(\otimes)\, H_F : S^1(E \otimes F) \to T^1(E \otimes F)$$

according to

$$(H_E\,(\otimes)\, H_F)\, \sigma_{E,F} = [H_E; H_F]$$

or, with less accuracy

$$H_E\,(\otimes)\, H_F = [H_E; H_F]\, \sigma^{-1}_{E,F}$$

(c.f. the "switchback rule" in [5]).

Proposition 3.4. *If H_E and H_F are connections on E and F respectively then $H_E\,(\otimes)\, H_F$ is a connection on $E \otimes F$.*

Proof. Since $H_E\,(\otimes)\,H_F$ is really a bundle morphism, it suffices to verify the relations (3.9) and (3.10) of Porposition 3.3. Thus let, the notations being as in the preceding lemma, $Y = \sigma(X) \in S^1(E \otimes F)_a$. A comparison of (3.31) and (3.37) gives

$$(H_E\,(\otimes)\,H_F)\,(Y^{kr\alpha}_{\alpha} s^{\alpha}_{kr}(a)) = Y^{ut}_0 t^0_{ut}(a) + \\ + (Y^{ut}_j + Y^{kt}_0 \underset{(E)}{\Gamma^u_{kj}} + Y^{ur}_0 \underset{(F)}{\Gamma^t_{rj}})\, t^j_{ut}(a)$$

and thus the components of $H_E\,(\otimes)\,H_F$ are

$$\underset{(E\otimes F)}{\Gamma^{ut0}_{kr0}} = \delta^u_k \delta^t_r\,; \quad \underset{(E\otimes F)}{\Gamma^{uti}_{kr0}} = 0\,; \quad \underset{(E\otimes F)}{\Gamma^{uti}_{krj}} = \delta^u_k \delta^t_r \delta^i_j$$

and

$$\underset{(E\otimes F)}{\Gamma^{ut0}_{krj}} = \delta^t_r \underset{(E)}{\Gamma^u_{kj}} + \delta^u_k \underset{(F)}{\Gamma^t_{rj}} \tag{3.38}$$

and this proves the proposition.

It can be seen immediately from (3.38) that the product $(\otimes)$ is associative. Furthermore one sees from (3.38) that this definition of the "tensor product" of connections coincides, in the case $E = F = T(M)$, with the classical one (c.f. also the example below).

Note that (3.24) can now be written also in the form

$$I^{E\otimes F}_{\mathscr{U}} = I^E_{\mathscr{U}}\,(\otimes)\,I^F_{\mathscr{U}}\,.$$

Next we give two lemmas which will be of use in the following paragraph.

Lemma 3.3. *Let $\Phi : E \to F$ be a bundle morphism and let A be a third vector bundle over M. Further let H_E, H_F and H_A be connections on the vector bundles E, F and A respectively and let*

$$H_F\, S^1(\Phi) = T^1(\Phi)\, H_E\,. \tag{3.39}$$

Then

$$T^1(\Phi \otimes 1)\,(H_E\,(\otimes)\,H_A) = (H_F\,(\otimes)\,H_A)\, S^1(\Phi \otimes 1)\,.$$

Proof. First note that

$$S^1(\Phi \otimes 1)\,\sigma_{E,A} = \sigma_{F,A}[S^1(\Phi) \otimes 1]\,. \tag{3.40}$$

Since $\sigma_{E,A}$ is a projection, it suffices to show

$$T^1(\Phi \otimes 1)\,(H_E\,(\otimes)\,H_A)\,\sigma_{E,A} = (H_F\,(\otimes)\,H_A)\, S^1(\Phi \otimes 1)\,\sigma_{E,A}\,. \tag{3.41}$$

From Lemma 3.1 we get the relations

$$T^1(\Phi \otimes 1)\,\gamma^E_{E\otimes A} = \gamma^F_{F\otimes A}[T^1(\Phi) \otimes 1]\,,$$
$$T^1(\Phi \otimes 1)\,\gamma^A_{E\otimes A} = \gamma^A_{F\otimes A}[\Phi \otimes 1]$$

and thus the left hand side in (3.41) can be transformed into

$$T^1(\Phi \otimes 1)[H_E; H_A] = T^1(\Phi \otimes 1)\{\gamma^E_{E\otimes A}(H_E \otimes \Pi_S) + \\ + \gamma^A_{E\otimes A}(\Pi_S \otimes H_A) - j^1_T(\Pi_S \otimes \Pi_S) = \gamma^F_{F\otimes A}(T^1(\Phi) H_E \otimes \Pi_S) + \\ + \gamma^A_{F\otimes A}(\Phi\Pi_S \otimes H_A) - T^1(\Phi \otimes 1) j^1_T(\Pi_S \otimes \Pi_S).$$

By virtue of (3.39) and (2.15), (2.13) this transforms into

$$\gamma^F_{F\otimes A}(H_F \otimes \Pi_S)(S^1(\Phi) \otimes 1) + \gamma^A_{F\otimes A}(\Pi_S \otimes H_A)(S^1(\Phi) \otimes 1) - \\ - j^1_T(\Pi_S \otimes \Pi_S)(S^1(\Phi) \otimes 1) = [H_F; H_A)(S^1(\Phi) \otimes 1).$$

Finally using (3.40) we see that this is equal to

$$(H_F(\otimes) H_A)\,\sigma_{F,A}(S^1(\Phi) \otimes 1) = (H_F(\otimes) H_A)\, S^1(\Phi \otimes 1)\,\sigma_{E,A}$$

and this completes the proof.

Lemma 3.4. *Let the direct sum $E \oplus F$ of vector bundles E, F be represented by the diagram* (3.22) *and let A be a third vector bundle over M. Let H_E, H_F and H_A be connections on E, F and A respectively. Then*

$$(H_E(\oplus) H_F)(\otimes) H_A = \\ = T^1(j \otimes 1)(H_E(\otimes) H_A)\, S^1(\Pi \otimes 1) + T^1(j^* \otimes 1)(H_F(\otimes) H_A)\, S^1(\Pi^* \otimes 1).$$

Proof. Note that (3.23) implies

$$(H_E(\oplus) H_F)\, S^1(j) = T^1(j)\, H_E$$

and

$$(H_E(\oplus) H_F)\, S^1(j^*) = T^1(j^*)\, H_F.$$

Thus applying the preceding lemma to $\Phi = j : E \to E \oplus F$ and $\Phi = j^* : F \to E \oplus F$ we get simultaneously

$$T^1(j \otimes 1)(H_E(\otimes) H_A) = \{[H_E(\oplus) H_F](\otimes) H_A\}\, S^1(j \otimes 1)$$

$$T^1(j^* \otimes 1)(H_F(\otimes) H_A) = \{[H_E(\oplus) H_F](\otimes) H_A\}\, S^1(j^* \otimes 1).$$

"Multiplying" these relations by $S^1(\Pi \otimes 1)$ and $S^1(\Pi^* \otimes 1)$ respectively and adding the obtained equations we get immediately the required result.

Let now E^* be the dual of the vector bundle E, $R = R(M)$ denotes as usually the trivial bundle of real valued functions on M. Denote by $c : E \otimes E^* \to R$ the "contraction" of elements, i.e. the natural bundle morphism assigning to $x \otimes y \in \in (E \otimes E^*)_a$ the element $\langle x, y\rangle = y(x)$.

If $X \in T^1(E)_a = \mathscr{L}(E_a^*, T^1(R)_a)$ and $y \in E_a^*$, denote by $\langle X, y\rangle \in T^1(R)_a$ the image of y under X. We see easily that

$$(3.42) \qquad \langle X, y\rangle = T^1(c)\,\gamma^E_{E\otimes E^*}(X \otimes y)\,.$$

Analogously if $Y \in T^1(E^*)_a = \mathscr{L}(E_a, T^1(R)_a)$ and $x \in E_a$, write

$$(3.43) \qquad \langle x, Y\rangle = T^1(c)\,\gamma^{E^*}_{E\otimes E^*}(x \otimes Y) \in T^1(R)\,.$$

The atlas $\mathfrak{A}^E$ of E induces in a natural way an atlas $\mathfrak{A}^{E^*}$ and consequently a family $\{I^{E^*}_{\mathscr{U}}\}$ of local isomorphisms of the jet and tensor prolongations of E^*. If f and g are local sections in E and E^* respectively and $(\mathscr{U}, r_x, \varphi) \in \mathring{\mathfrak{A}}^E$ an arbitrary chart, one verifies easily the relation

$$(3.44) \qquad \langle I^E_{\mathscr{U}} j^1_a f, g(a)\rangle + \langle f(a), I^{E^*}_{\mathscr{U}} j^1_a g\rangle = j^1_a\langle f, g\rangle + \langle f(a), g(a)\rangle$$

for each $a \in \mathscr{U}$. This relation in fact suggests again the definition of the dual connection given below. Note that if $f(a) = 0$ then (3.44) reduces to

$$(3.45) \qquad \langle I^E_0 j^1_a f, g(a)\rangle = j^1_a\langle f, g\rangle\,.$$

Lemma 3.5. *Let $H_E : S^1(E) \to T^1(E)$ be a bundle morphism satisfying* (3.10). *Then there exists exactly one bundle morphism $H_{E^*} : S^1(E^*) \to T^1(E^*)$ such that*

$$(3.46) \qquad T^1(c)\,[H_E; H_{E^*}] = S^1(c)\,\sigma_{E,E^*}\,.$$

Proof. The condition (3.46) can be written in the form

$$(3.47) \qquad T^1(c)\,\gamma^{E^*}_{E\otimes E^*}(\Pi_S \otimes H_{E^*}) = -T^1(c)\,\gamma^{E^*}_{E\otimes E^*}(H_E \otimes \Pi_S) + \\ + j^1_T c(\Pi_S \otimes \Pi_S) + S^1(c)\,\sigma_{E,E^*}\,.$$

Applying this relation to $j^1_a f \otimes j^1_a g \in S^1(E)_a \otimes S^1(E^*)_a$ we get

$$(3.48) \qquad \langle f(a), H_{E^*} j^1_a g\rangle = -\langle H_E j^1_a f, g(a)\rangle + \langle f(a), g(a)\rangle + j^1_a\langle f, g\rangle\,.$$

Since $H_E|_{\mathrm{Ker}\,\Pi_S} = I^E_0$, we see from (3.45) that the right hand side of (3.48) vanishes if $f(a) = 0$. But that means that (3.48) defines uniquely $H_{E^*} \in \mathscr{L}(S^1(E^*), \mathscr{L}(E, T^1(R)))$ and this proves the lemma.

Proposition 3.5. *If H_E is a connection then H_{E^*} defined in the preceding lemma is also a connection.*

Proof. In fact, applying Π_T to (3.48) we get

$$\langle f(a), (\Pi_T H_{E^*})\, j^1_a g\rangle = -\langle(\Pi_T H_E)\, j^1_a f, g(a)\rangle + \\ + \langle f(a), g(a)\rangle + \langle f(a), g(a)\rangle = \langle f(a), g(a)\rangle\,,$$

i.e.

$$\langle f(a), (\Pi_T H_{E*} - \Pi_S) j_a^1 g \rangle = 0$$

for each $f(a) \in E_a$, $g(a) \in E_a^*$ and thus $\Pi_T H_{E*} = \Pi_S$. On the other hand we have similarly as in (3.45)

$$\langle f(a), I_0^{E*} j_a^1 g \rangle = j_a^1 \langle f, g \rangle$$

for each $j_a^1 g \in \operatorname{Ker} \Pi_S$ and hence for such $j_a^1 g$ and any $f(a)$ we get from (3.48)

$$\langle f(a), H_{E*} j_a^1 g \rangle = \langle f(a), I_0^{E*} j_a^1 g \rangle$$

and this completes the proof.

The connection H_{E*} on E^* associated with a connection H_E on E in the sense of the preceding proposition is called the dual of the connection H_E. We see from (3.46) that it satisfies

$$T^1(c)\,(H_E\,(\otimes)\,H_{E*}) = S^1(c)\,. \tag{3.49}$$

Before passing to the explicite formulae connecting the components of H_{E*} with those of H_E in a given chart, let us introduce the following convention which is a modification of the Einstein summation convention. In expressions and formulae given below we sum not only over repeated indices appearing "above and below" but also over repeated indices, both appearing as subscripts or superscripts, if they are denoted by the same letter and one of them provided with an asterisk. The "stared indices" will appear in expressions connected with the "dual" E^*. Therefore the analogue of the formula (3.5) defining the "components of H_E" in a fixed chart is to be written as

$$\hat{y}_\beta^{h*} = \Gamma_{k*\beta}^{h*\alpha} y_\alpha^{k*}\,. \tag{3.50}$$

Note that $g_{k'*}^{k*} = g_k^{k'}$ are the components of an element in the group $G(\mathfrak{A}^{E^*})$ which is the contragradient to $G(\mathfrak{A}^E)$.

In a fixed chart $(\mathscr{U}, r_x, \varphi) \in \mathring{\mathfrak{A}}^E$ the relation (3.48) gives rise to the system of equations

$$\Gamma_{h*\beta}^{h*\alpha} y_\alpha^{k*} y_0^h + \Gamma_{\beta k}^{h\alpha} y_\alpha^k y_0^{h*} = y_\beta^{k*} y_0^k + y_0^{k*} y_\beta^k\,.$$

Since H_E is a connection, we get from these equations the expected expressions for the components of $\Gamma_{\mathscr{U}}^*$ according to Proposition 3.3 and, in addition, the only "interesting" relation

$$\Gamma_{k*j}^{h*0} + \Gamma_{hj}^{k0} = 0\,, \tag{3.51}$$

which shows again that, in the case $E = T(M)$, the definition of the dual of a connection given here corresponds to that in the classical theory.

One can also easily introduce the notions of covariant differential and covariant derivatives with respect to a pseudo-connection H on E of local sections in E. Let f be a local section in E, $a \in M$. The covariant differential of f at a with respect to H is given by the element

$$\Pi_T^* H j_a^1 f \in E \otimes T(M)^* . \tag{3.52}$$

If $(\mathscr{U}, r_x, \varphi) \in \mathring{\mathfrak{A}}$ and if $\{dx^i \otimes i_k(a)\}$ is the frame in $E_a \otimes T(M)_a^*$ corresponding to this chart, then the covariant derivatives of f at a are the components of (3.52) with respect to this frame. Denoting the covariant derivatives by $D_i f^k$, we have

$$\Pi_T^* H j f_a^1 = (D_i f^k)_a \, dx^i \otimes i_k(a) . \tag{3.53}$$

In general if X is any tangent vector to M at a, i.e. $X \in T(M)_a$, then the element

$$D_X f = \langle X, \Pi_T^* H j_a^1 f \rangle$$

is called the covariant derivative of f in the direction of X. Note that if $X = (\partial/\partial x^i)(a)$, then

$$D_X f = \sum_{k=1}^{m} (D_i f^k)_a \, i_k(a) .$$

In the given chart we have $j_a^1 f = (\partial_\beta f^k)_a \, s_k^\beta(a)$ and

$$H j_a^1 f = \Gamma_{\mathscr{U}} I_{\mathscr{U}} j_a^1 f = \Gamma_{h\alpha}^{k\beta} (\partial_\beta f^h)_a \, dx^\alpha \otimes i_k(a) .$$

Comparing this with (3.53) we get the covariant derivatives in the explicite form

$$(D_i f^k)_a = \Gamma_{hi}^{k\beta} (\partial_\beta f^h)_a = \Gamma_{hi}^{kj} (\partial_j f^h)_a + \Gamma_{hi}^{k0} f^h(a) .$$

In particular, if H is a connection we get by virtue of Proposition 3.3 the "expected" relation

$$(D_i f^k)_a = (\partial_i f^k)_a + \Gamma_{hi}^k f^h(a) . \tag{3.54}$$

Example. In the form of an illustration we shall show that the connection on e.g. $T(M) \otimes T(M) \otimes T(M)^*$ induced by a connection on $T(M)$, in the manner described above, really corresponds to the classical connection on M. Just let $(\mathscr{U}, \varphi)$ be a chart of M and T a local section in $T(M) \otimes T(M) \otimes T(M)^*$ over $\mathscr{U}$, i.e. a twice contravariant and once covariant tensor field on M. Write

$$T = T_u^{rs} \, \partial_r \otimes \partial_s \otimes dx^u .$$

Then

$$(D_i T_u^{rs})_a = (\partial_i T_u^{rs})_a + \Gamma_{pqv*i}^{rsu*} T_v^{pq}(a) , \tag{3.55}$$

where Γ_{pqv*i}^{rsu*} are the components of the connection on $T(M) \otimes T(M) \otimes T(M)^*$. If this connection is generated by a connection on $T(M)$ with components Γ_{hi}^k we conclude from (3.38) and (3.51) that

$$\Gamma_{pqv*i}^{rsu*} = \delta_{v*}^{u*} (\delta_q^s \Gamma_{pi}^r + \delta_p^r \Gamma_{qi}^s) + \delta_p^r \delta_q^s (-\Gamma_{ui}^v)$$

or, substituing into (3.55)

$$(D_i T^{rs}_u)_a = (\partial_i T^{rs}_u)_a + \Gamma^r_{pi} T^{ps}_u(a) + \Gamma^s_{pi} T^{rp}_u(a) - \Gamma^p_{ui} T^{rs}_p(a),$$

which is the well known classical formula.

Definition 3.3. Let $\Phi : E \to F$ be a bundle morphism. The bundle morphism $H_\Phi : S^1(E) \to T^1(F)$ is called a *relative connection* (or briefly *R-connection*) with respect to Φ if it satisfies

$$\Pi_T H_\Phi = \Phi \Pi_S \tag{3.56}$$

and

$$H_\Phi\big|_{\mathrm{Ker}\,\Pi_S} = T^1(\Phi)\, I_0 \,. \tag{3.57}$$

Lemma 3.6. *If $\Phi : E \to F$ is an injection then each* R*-connection with respect to Φ is an injection.*

Proof. Let $\dot{f} \in S^1(E)_a$, $a \in M$ and $H_\Phi \dot{f} = 0$. From (3.56) we get $\Pi_T H_\Phi \dot{f} = \Phi \Pi_S \dot{f} = 0$ and since Φ is an injection we have $\dot{f} \in \mathrm{Ker}\,\Pi_S$. But (3.57) yields $0 = H_\Phi \dot{f} = T^1(\Phi)\, I_0 \dot{f}$ and since $T^1(\Phi)\, I_0$ is also an injection, we conclude that $\dot{f} = 0$.

Corollary. *If Φ is an isomorphism then H_Φ is also an isomorphism.*

Note that a connection on E is an R-connection with respect to the identity.

Proposition 3.6. *Let $\Phi : E \to F$ and let H_E and H_F be connections on E and F respectively. Then both $T^1(\Phi)\, H_E$ and $H_F\, S^1(\Phi)$ are* R*-connections with respect to Φ.*

Proof. We get immediately from (3.9) and (2.13) or (2.14) $\Pi_T\, T^1(\Phi)\, H_E = \Phi \Pi_T H_E = \Phi \Pi_S$; $\Pi_T H_F\, S^1(\Phi) = \Pi_S\, S^1(\Phi) = \Phi \Pi_S$. On the other hand let $\dot{f} \in \mathrm{Ker}\,\Pi_S \in S^1(E)$. Then clearly $T^1(\Phi)\, H_E \dot{f} = T^1(\Phi)\, I_0 \dot{f}$ and applying (2.65) we have also $H_F\, S^1(\Phi) \dot{f} = H_F\, S^1(\Phi)\, I_0^{-1} I_0 \dot{f} = H_F I_0^{-1} T^1(\Phi)\, I_0 \dot{f} = T^1(\Phi)\, I_0 \dot{f}$. This completes the proof.

Proposition 3.7. *If $\Phi : E \to F$ is a bundle isomorphism and H_Φ is an* R*-connection with respect to Φ then $T^1(\Phi)^{-1} H_\Phi$ and $H_\Phi\, S^1(\Phi)^{-1}$ are connections one E and F respectively.*

Proof. We have again using (2.13), (2.14) and (3.56): $\Pi_T\, T^1(\Phi)^{-1} H_\Phi = \Phi^{-1} \Pi_T H_\Phi = \Phi^{-1} \Phi \Pi_S$ and $\Pi_T H_\Phi\, S^1(\Phi)^{-1} = \Phi \Pi_S\, S^1(\Phi)^{-1} = \Phi \Phi^{-1} \Pi_S$. If $\dot{f} \in \mathrm{Ker}\,\Pi_S \subset S^1(E)$, $\dot{g} \in \mathrm{Ker}\,\Pi_S \in S^1(F)$, we get from (2.65) and (3.57) $T^1(\Phi)^{-1} H_\Phi \dot{f} = T^1(\Phi)^{-1}\, T^1(\Phi)\, I_0 \dot{f}$, or $H_\Phi\, S^1(\Phi)^{-1} \dot{g} = H_\Phi I_0^{-1} I_0\, S^1(\Phi)^{-1} \dot{g} = H_\Phi I_0^{-1} T^1(\Phi)^{-1} \,.\, I_0 \dot{g} = T^1(\Phi)\, T^1(\Phi)^{-1} I_0 \dot{g}$ which completes the proof.

Note that the condition (3.39) in Lemma 3.3 means that H_E and H_F induce the same R-connection with respect to Φ. Analogously (3.49) states that $H_E\,(\otimes)\,H_{E*}$ and the identity induce the same R-connection with respect to the "contraction" c.

Given an arbitrary bundle morphism $\Phi : E \to F$ one could also introduce the notion of right and left pseudo-connections with respect to Φ. A bundle morphism $H_\Phi : S^1(E) \to T^1(F)$ would be a right (or accordingly a left) R-pseudo-connection with respect to Φ if H_Φ admited the decomposition $H_\Phi = H_F\, S^1(\Phi)$ (accordingly $H_\Phi = T^1(\Phi)\, H_E$) where H_F is some pseudo-connection on F (or H_E is some pseudo-connection on E). However we shall not need this complicated terminology and say only that the pseudo-connections H_E and H_F on E and F respectively *induce the same* R-*pseudo-connection with respect to* Φ *if* $H_F\, S^1(\Phi) = T^1(\Phi)\, H_E$ *holds.*

On the other hand if $\Phi : E \to F$ is a bundle isomorphism then each bundle isomorphism $H_\Phi : S^1(E) \to T^1(F)$ is simultaneously a right and a left R-pseudo-connection with respect to Φ. In this case both the corresponding H_E and H_F are uniquely determined by H_Φ, and conversely any of these pseudo-connections determines uniquely H_Φ and the second one. This can be briefly expressed by saying that there is a one-to-one-to-one correspondence between pseudo-connections on E, pseudo-connections on F and bundle isomorphisms of $S^1(E)$ onto $T^1(F)$. Moreover we have seen that if (H_E, H_Φ, H_F) is such a triple in correspondence and any one of the components is a connection, then the same is true about the other two.

Remark 5. In [6] a definition of a connection D relative to a bundle morphism $\Phi : E \to F$ due to BOTT is given as follows. D is a first order linear differential operator assigning to local differentiable sections in E local diflerentiable sections in $F \otimes T(M)^*$ with the property that if f is such a local section in E, $a \in M$ and α is a differentiable function defined in a neighbourhood of a, then

$$D(\alpha f)(a) = \alpha(a)(Df)(a) + (\Phi f)(a) \otimes \mathrm{d}\alpha\,.$$

It can be shown by direct calculations in local coordinates that this definition is equivalent to the definition of an R-connection H_Φ with respect to Φ given here, this correspondence being given by

$$(Df)(a) = \Pi_T^* H_\Phi j_a^1 f\,.$$

In particular if Φ = identity, then $(Df)(a)$ is nothing but the covariant differential of f at a in our terminology (c.f. (3.52)).

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