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ON SOME COVERING PROPERTIES OF METRIC SPACES

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1. Introduction. The well known theorem of LINDELÖF reads (cf. [2], p. 49): there is a countable subcover of each open cover (a subset) of a separable space, i.e. of a space whose topology has a countable base. A. LELEK raised the question whether *metric* separable spaces possess a stronger property: is there a countable subcover of each open base consisting of sets of diameters tending to 0? JAN MYCIELSKI has pointed out that the answer is "no" (cf. (4.5) below) and the present paper contains some results originated while discussing the Lelek's question.

Notions and notations not defined here come from [2] and [3].

2. Preliminaries. Throughout the paper all spaces are assumed to be metric (not necessarily separable) unless the contrary is explicitly stated. By a *zero sequence* of a metric space X of diameter $\delta(X) > 0$ we shall mean any sequence $\{A_n\}_{n=1,2,\dots}$ of its subsets such that $\delta(A_n) < \delta(X)$ for $n = 1, 2, \dots$ and $\lim_{n\to\infty} \delta(A_n) = 0$. For the sake of elegance it seemed natural to us to suppose also that a metric space consisting of one point only contains a zero sequence, too. The effects of this last supposition can be seen here and there (for instance, under this supposition Lelek's "strong Lindelöf property" is now completely equivalent to our Property III) but, as a rule, we shall not attract attention to it later on.

In what follows we shall be interested mainly in zero sequences consisting of open sets only. A zero sequence of open sets which is simultaneously a cover (resp., a base) will be called shortly a *zero cover* (resp., a *zero base*).

The paper is devoted to the first two of the following three (the third will be treated more completely by Lelek in [4] and [5]), more and more stronger properties possessed by some metric[1]) spaces:

[1]) Because a *uniform structure* in the sense of N. BOURBAKI [1] is similar to a metric structure of a topological space in that it allows to compare the size of any two subsets of the space, we could extend the definition of a zero sequence in a metric space to such a sequence in a uniform space and, consequently, to define Properties I—III for uniform spaces. However, this is not the case we are interested in.

I. *There exists a zero cover.*

II. *There exists a zero base.*

III. *Each base contains a zero base.*

Clearly, III $\Rightarrow$ II and II $\Rightarrow$ I. However, the converse implications are not true. Actually, in the space $\mathcal{N}$ of irrational numbers of the segment [0, 1], which obviously has Property II, Lelek has defined in [4] a base which contains no zero base. Thus $\mathcal{N}$ does not possess Property III, which shows that II $\nrightarrow$ III. Furthermore, any metric space which possesses Property II is necessarily separable. However, there exists a metric space enjoying Property I which is not separable (e.g., the space Z defined in IV of § 3 below). Hence I $\nrightarrow$ II either. (In fact, it is not difficult to construct, following the idea of constructing Z, also a metric separable space which enjoys Property I but not Property II.)

Note also that, in view of our additional supposition, a one point metric space satisfies all three Properties I–III.

As Jan Mycielski has pointed out, a 0-dimensional Baire space, which is known to be homeomorphic to $\mathcal{N}$ ([6], p. 177), does not possess Property I and thus does not possess Property II either. This means that Properties I and II are of *metric* but not of topological character. However, the similar question concerning the character of Property III remains open.

3. Some examples. Now we shall describe some examples and prove some of their properties related to our basic Properties I–III.

I. The space $\mathcal{N}$. It is the space of irrational numbers of the segment [0, 1] with the ordinary metric

$$\varrho(x, y) = |x - y| \quad \text{for all} \quad x, y \in \mathcal{N}\,.$$

The space $\mathcal{N}$ has Property II and has not Property III. In fact, if $\mathfrak{B}_k$ is a finite open cover of $\mathcal{N}$ consisting of sets with diameters equal to $1/2^k$, then $\bigcup_{k=2}^{\infty} \mathfrak{B}_k$ is a zero base. On the other hand, however, Lelek has defined in [4] a base β for $\mathcal{N}$ which contains no zero subbase.

II. 0-dimensional Baire space $\mathcal{B}$ is the set of all sequences of positive integers, where the distance $\varrho(\{a_k\}, \{b_k\})$ between two distinct sequences $\{a_k\}$ and $\{b_k\}$ is the inverse $1/k$ of the least index k for which $a_k \neq b_k$ ([6], p. 175).

As is well known, the metric space $\mathcal{B}$ is separable and complete.

It is also known that $\mathcal{B}$ is homeomorphic to $\mathcal{N}$ ([6], p. 177). However, we shall proceed to show (see (4.5)) that $\mathcal{B}$, in contrast with $\mathcal{N}$, has not Property I and so has not Property II either.

The space $\mathcal{B}$ together with some its subspaces forms a basic set of examples and counter-examples to many questions related to Properties I–III. One of its surprising features which seems to be largely responsible for it, is that each totally bounded subset of $\mathscr{B}$ is nowhere dense in $\mathscr{B}$. In fact, we shall prove this in a slightly more complicated case of its subspace $\mathscr{B}_0$ but the proof with obvious simplifications works to the same effect in $\mathscr{B}$ itself.

III. The subspace $\mathscr{B}_0$ of $\mathscr{B}$. It is homeomorph of $\mathscr{B}$: to any sequence $\{a_k\}$ of positive integers we assign a "rarefied" sequence

$$r\{a_1, a_2, \ldots\} = \{a_1, \ldots, a_1, a_2, \ldots, a_2, \ldots\}$$

in which each a_k is repeated a_k-times. $\mathscr{B}_0$ is the subspace of $\mathscr{B}$ consisting of all "rarefied" sequences.

We shall prove now some properties of $\mathscr{B}_0$.

(3.1) *Each totally bounded subset of* $\mathscr{B}_0$ *is nowhere dense in* $\mathscr{B}_0$.

Proof. First of all observe that if A is a totally bounded subset of $\mathscr{B}_0$ and $r\{a_1, a_2, \ldots\}$ is any point of A, then for each k there are infinitely many points $r\{a_1, a_2, \ldots, a_{k-1}, x_k, \ldots\}$ of $\mathscr{B}_0$ which do not belong to the closure $\bar{A}$ of A. In fact, the distance between any two of these points is equal to $1/n$, where n is an index of x_k in the "rarefied" sequence $r\{a_1, a_2, \ldots, a_{k-1}, x_k, \ldots\}$, and a totally bounded set (closure of a totally bounded set is totally bounded too) cannot contain infinitely many points such that the distance between any two of them is larger than some positive number.

To conclude the proof of (3.1) it suffices now to observe that for each point $a \in A$ and for a positive integer m there exists a point $b \in \mathscr{B}_0 - A$ such that $\varrho(a, b) \leqq 1/m$, and this is quite obvious in view of the above.

It is a matter of a simple exercise to check that $\mathscr{B}_0$ is a closed subset of $\mathscr{B}$. Hence and from the completeness of $\mathscr{B}$ we infer that $\mathscr{B}_0$ is a complete space too ([3], I, p. 315). Consequently, by virtue of the Baire category theorem ([3], I, p. 321),

(3.2) $\mathscr{B}_0$ *is not a union of countably many totally bounded sets.*

One more property of $\mathscr{B}_0$ will be of importance to us:

(3.3) *For each* $\eta > 0$ *there exists a zero cover* $\{A_n\}$ *of* $\mathscr{B}_0$ *such that* $\delta(A_n) < \eta$ *for* $n = 1, 2, \ldots$

Proof. Let C_k be the set of those "rarefied" sequences of positive integers (i.e., of those points of $\mathscr{B}_0$) the first number of which is k. Obviously, each C_k is open (even closed-open) in $\mathscr{B}_0$ and of diameter $\delta(C_k) = 1/(k+1)$. If n_0 is a positive integer such that $1/n_0 < \eta$, then the sets C_{n_0+m}, $m = 1, 2, \ldots$, form a zero cover of $\bigcup_{k=n_0}^{\infty} C_k$ consisting of sets with diameters less than η. It remains then to construct such a zero cover for the set $\bigcup_{k=1}^{n_0-1} C_k$, but since it is a finite union of open sets, it suffices to do it for each C_k, $k = 1, 2, \ldots, n_0 - 1$, separately.

For that purpose observe that there are only countably many sequences of positive integers consisting of n_0 elements and so, for a given k, there are only countably many "rarefied" sequences of natural numbers consisting of n_0 elements, the first of which is k. If

$$a = \{a_1, \ldots, a_1, \ldots, a_l, \ldots, a_l\}$$

is such a sequence (i.e. $a_1 = k$ and each a_j is repeated a_j-times with the possible exception of a_l which is repeated $n_0 - \sum_{j=1}^{l-1} a_j$ times), then the set C^a of all "rarefied" sequences of positive integers which have a as initial segment is open and of diameter less than η,

$$\delta(C^a) = \frac{1}{1 + a_1 + \ldots + a_l} < \eta \,. \tag{1}$$

And since each positive integer $N \geqq n_0$ can be decomposed into the sum

$$N = a_1 + a_2 + \ldots + a_l \tag{2}$$

of positive integers $(a_1 = k)$ in finitely many ways only, then including yet permutations of $a_2, \ldots, a_l$ in (2) we easily infer from (1) that for each $N \geqq n_0$ there are only finitely many sequences a with $\delta(C^a) = 1/(N + 1)$.

Ordering now all sequences a into a sequence $\{a^n\}$ we obtain a sequence $\{C^{a^n}\}$ of open sets covering C_k, and such that $\delta(C^{a^n}) < \eta$ and $\lim_{n\to\infty} \delta(C^{a^n}) = 0$.

The just proved property (3.3) implies (cf. (5.5) below) that the space $\mathscr{B}_0$ enjoys Property II. However, surprisingly enough,

(3.4) *The cartesian product $\mathscr{B}_0 \times \mathscr{B}_0$ metrized by the pythagorean formula*

$$\varrho_1[(x_1, y_1), (x_2, y_2)] = \sqrt{[\varrho(x_1, x_2)^2 + \varrho(y_1, y_2)^2]}\,,$$

where ϱ is the metric in $\mathscr{B}$, does not possess Property I.

Proof. We have to show that for each sequence $\{A_n\}$ of open sets such that $\delta(A_n) < \delta(\mathscr{B}_0 \times \mathscr{B}_0)$ for $n = 1, 2, \ldots$, and

$$\lim_{n\to\infty} \delta(A_n) = 0\,, \tag{1}$$

there exists a point $p \in \mathscr{B}_0 \times \mathscr{B}_0$ not belonging to any A_n.

Let $\mathfrak{A} = \{A_n\}$ be such a sequence. By $\mathfrak{B}^*$, where $\mathfrak{B} \subset \mathfrak{A}$, we shall denote the union of all elements of $\mathfrak{B}$.

To show that $(\mathscr{B}_0 \times \mathscr{B}_0) - \mathfrak{A}^* \neq 0$ we shall proceed by induction. To start with, denote by C_k the set of all "rarefied" sequences of $\mathscr{B}_0$, the first number of which is k. Each C_k is a closed-open subset of $\mathscr{B}_0$ with the diameter $\delta(C_k) = 1/(k + 1)$, and the distance between any two points $x \in C_k$ and $y \in C_{k'}$, where $k \neq k'$, is equal to 1.

This implies that for each $n = 1, 2, \ldots$ we have either $A_n \subset \mathscr{B}_0 \times C_l$ or $A_n \subset C_k \times \times \mathscr{B}_0$ (because $\delta(A_n) = \sqrt{2} = \delta(\mathscr{B}_0 \times \mathscr{B}_0)$ otherwise), and for sufficiently large n, say for $n \geqq N$, we have $A_n \subset C_k \times C_l$ for some $k = k(n)$ and $l = l(n)$ (because (1) wouldn't hold otherwise).

Since N sets $A_1, \ldots, A_N$ meet only finitely many sets $\mathscr{B}_0 \times C_l$ and $C_k \times \mathscr{B}_0$, then there exist k_1 and l_0 such that

(2) $$\bigcup_{n=1}^{N} A_n \cap (C_{k_1} \times \bigcup_{l=l_0}^{\infty} C_l) = 0 .$$

Denote by $\tilde{\mathfrak{A}}_1$ the subfamily of $\mathfrak{A}$ consisting of all A_n with the diameters $\delta(A_n) \geqq \geqq 1/(k_1 + 1)$. By virtue of (1) the subfamily $\tilde{\mathfrak{A}}_1$ is finite. Since each A_n meeting $C_{k_1} \times \bigcup_{l=l_0}^{\infty} C_l$ is contained in some $C_{k_1} \times C_l$, then by virtue of (2) there exists l_1 such that

$$(C_{k_1} \times C_{l_1}) \cap \tilde{\mathfrak{A}}_1^* = 0 .$$

Now let $C_{n_1,\ldots,n_m}$ denote the set of all "rarefied" sequences $r\{n_1, \ldots, n_m, x_{m+1}, x_{m+2}, \ldots\}$ of $\mathscr{B}_0$, where $n_1, \ldots, n_m$ are fixed. Each set $C_{n_1,\ldots,n_{m-1}}$ is the union of countably many closed-open in $\mathscr{B}_0$ sets $C_{n_1,\ldots,n_{m-1},n}$, where $n = 1, 2, \ldots$, and the distance between any two points $x \in C_{n_1,\ldots,n_{m-1},n}$ and $y \in C_{n_1,\ldots,n_{m-1},n'}$, where $n \neq n'$, is equal to $\varrho(x, y) = 1/(n_1 + \ldots + n_{m-1} + 1)$. The diameter of $C_{n_1,\ldots,n_m}$ is equal to $\delta(C_{n_1,\ldots,n_m}) = 1/(n_1 + \ldots + n_m + 1)$.

The set C_{k_1} is then the union of countably many closed-open sets $C_{k_1,k}$, $k = 1, 2, \ldots$, such that the distance between any two points $x \in C_{k_1,k}$ and $y \in C_{k_1,k'}$, where $k \neq k'$, is equal to $1/(k_1 + 1)$. In view of the definition of $\tilde{\mathfrak{A}}_1$, each $A_n \in \mathfrak{A} - \tilde{\mathfrak{A}}_1$ meeting $C_{k_1} \times C_{l_1}$ is contained in some $C_{k_1,k} \times C_{l_1}$. Denote by $\mathfrak{A}_1$ the following subfamily of $\mathfrak{A}$

(3) $$\mathfrak{A}_1 = \left\{A_n : \delta(A_n) \geqq \frac{1}{l_1 + 1}\right\}.$$

By virtue of (1) the family $\mathfrak{A}_1$ is finite and therefore a set C_{k_1,k_2} must exist such that

(4) $$(C_{k_1,k_2} \times C_{l_1}) \cap \mathfrak{A}_1^* = 0.$$

The first step of induction is completed. Its purpose is to define two sequences of positive integers $\{k_m\}$ and $\{l_m\}$ such that for each $m = 2, 3, \ldots$

(5) $$(C_{k_1,\ldots,k_m} \times C_{l_1,\ldots,l_{m-1}}) \cap \mathfrak{A}_{m-1}^* = 0 ,$$

where

(6) $$\mathfrak{A}_{m-1} = \left\{A_n : \delta(A_n) \geqq \frac{1}{l_1 + l_2 + \ldots + l_{m-1} + 1}\right\}.$$

Since we have already defined k_1, k_2 and l_1 with (3) and (4) let us suppose that there are already known $k_1, k_2, \ldots, k_m$ and $l_1, l_2, \ldots, l_{m-1}$ with (5) and (6).

The set $C_{l_1,\dots,l_{m-1}}$ is the union of countably many closed-open sets $C_{l_1,\dots,l_{m-1},l}$ $l = 1, 2, \dots$, such that the distance between any two points $x \in C_{l_1,\dots,l_{m-1},l}$ and $y \in C_{l_1,\dots,l_{m-1},l'}$, where $l \neq l'$, is equal to $1/(l_1 + \dots + l_{m-1} + 1)$. By virtue of (6),

(7) each $A_n \in \mathfrak{A} - \mathfrak{A}_{m-1}$ meeting $C_{k_1,\dots,k_m} \times C_{l_1,\dots,l_{m-1}}$ is contained in some $C_{k_1,\dots,k_m} \times C_{l_1,\dots,l_{m-1},l}$.

Denote by $\tilde{\mathfrak{A}}_m$ the subfamily of $\mathfrak{A}$

$$\tilde{\mathfrak{A}}_m = \left\{A_n : \delta(A_n) \geqq \frac{1}{k_1 + \dots + k_m + 1}\right\}. \tag{8}$$

In view of (1) the subfamily $\tilde{\mathfrak{A}}_m$ is finite. We have either $\tilde{\mathfrak{A}}_m \subset \mathfrak{A}_{m-1}$ or $\tilde{\mathfrak{A}}_m - \mathfrak{A}_{m-1} \neq 0$. If $A_n \in \mathfrak{A}_{m-1}$, then by (5) it is disjoint with $C_{k_1,\dots,k_m} \times C_{l_1,\dots,l_{m-1}}$, and if $A_n \in \tilde{\mathfrak{A}}_m - \mathfrak{A}_{m-1}$ and meets $C_{k_1,\dots,k_m} \times C_{l_1,\dots,l_{m-1}}$, then by (7) it is contained in some $C_{k_1,\dots,k_m} \times C_{l_1,\dots,l_{m-1},l}$. Hence in both cases there must exist l_m such that

$$(C_{k_1,\dots,k_m} \times C_{l_1,\dots,l_m}) \cap \tilde{\mathfrak{A}}_m^* = 0\,. \tag{9}$$

Similarly, the set $C_{k_1,\dots,k_m}$ is the union of countably many closed-open sets $C_{k_1,\dots,k_m,k}$, $k = 1, 2, \dots$, such that the distance between any two points $x \in C_{k_1,\dots,k_m,k}$ and $y \in C_{k_1,\dots,k_m,k'}$, where $k \neq k'$, is equal to $1/(k_1 + \dots + k_m + 1)$. Hence by virtue of (8),

(10) each $A_n \in \mathfrak{A} - \tilde{\mathfrak{A}}_m$ meeting $C_{k_1,\dots,k_m} \times C_{l_1,\dots,l_m}$ is contained in some $C_{k_1,\dots,k_m,k} \times C_{l_1,\dots,l_m}$.

Denote by $\mathfrak{A}_m$ the subfamily of $\mathfrak{A}$

$$\mathfrak{A}_m = \left\{A_n : \delta(A_n) \geqq \frac{1}{l_1 + \dots + l_m + 1}\right\}.$$

In view of (1) the subfamily $\mathfrak{A}_m$ is finite and so in view of (9) and (10) there must exist k_{m+1} such that

$$(C_{k_1,\dots,k_m,k_{m+1}} \times C_{l_1,\dots,l_m}) \cap \mathfrak{A}_m^* = 0\,.$$

Hence the induction is completed.

Now to finish the proof observe first that the diameter of each set $D_m = C_{k_1,\dots,k_m} \times C_{l_1,\dots,l_{m-1}}$ is equal to

$$\delta(D_m) = \sqrt{[(\sum_{i=1}^{m} k_i + 1)^{-2} + (\sum_{i=1}^{m-1} l_i + 1)^{-2}]}\,, \quad m = 2, 3, \dots$$

and so we have

$$\lim_{m\to\infty} \delta(D_m) = 0\,.$$

The space $\mathscr{B}_0$ is complete and so is $\mathscr{B}_0 \times \mathscr{B}_0$ (cf. [3], I, p. 313). On the other hand, all the sets D_m are closed in $\mathscr{B}_0 \times \mathscr{B}_0$ and form a decreasing sequence of diameters tending to 0. Hence (cf. [3], I, p. 319) $\bigcap_{m=2}^{\infty} D_m \neq 0$. Let $p = \bigcap_{m=2}^{\infty} D_m$. Since $p \in D_m$ for each $m = 2, 3, \ldots$, then, by (5) $p \notin \mathfrak{A}_m^*$ for any $m = 1, 2, 3, \ldots$ However, $\bigcup_{m=1}^{\infty} \mathfrak{A}_m = \mathfrak{A}$ and so $\bigcup_{m=1}^{\infty} \mathfrak{A}_m^* = \mathfrak{A}^*$. Hence $p \notin \mathfrak{A}^*$.

IV. Non-separable metric spaces. Let W_n be any uncountable set with the metric

$$\varrho_n(x, y) = \begin{cases} \frac{1}{n} & \text{if} \quad x \neq y, \\ 0 & \text{if} \quad x = y, \end{cases}$$

and let Z be the set $\bigcup_{n=1}^{\infty} W_n$ with the metric

$$\varrho(x, y) = \begin{cases} \varrho_n(x, y) & \text{if} \quad x, y \in W_n \quad \text{for some} \quad n = 1, 2, \ldots, \\ \sqrt{2} & \text{otherwise}. \end{cases}$$

(3.5) *The metric space Z has Property I but not Property II.*

In fact, each set W_n is open in Z (even closed-open) and since $\delta(W_n) = 1/n < \sqrt{2}$, then $\{W_n\}$ is a zero cover of Z. However, the space Z is obviously non-separable and so it cannot possess Property II.

(3.6) *The metric space $Z \times Z$ metrized by the pythagorean formula does not possess Property I.*

Proof. To prove it we shall show that for any countable cover $\{A_n\}$ of $Z \times Z$ consisting of sets with diameters $\delta(A_n) < 2 = \delta(Z \times Z)$ the second condition of Property I, i.e., $\lim_{n\to\infty} \delta(A_n) = 0$, does not hold. For, if $\mathfrak{A} = \{A_n\}$ were such a sequence, then we must have $A_n \subset W_k \times Z$ or $A_n \subset Z \times W_k$ for each $n = 1, 2, \ldots$ and some $k = k(n)$, because $\delta(A_n) = 2$ otherwise. Moreover, for all n sufficiently large, say for $n \geqq N$, we must also have $A_n \subset W_k \times W_l$ for some $k = k(n)$ and $l = l(n)$, because otherwise we would not have $\lim_{n\to\infty} \delta(A_n) = 0$. Hence k_1 and l_0 must exist such that $\bigcup_{n=1}^{N} A_n \cap (W_{k_1} \times \bigcup_{l=l_0}^{\infty} W_l) = 0$. Now if A_n meets $W_{k_1} \times \bigcup_{l=l_0}^{\infty} W_l$, then it must be contained in some $W_{k_1} \times W_l$, where $l = l_0, l_0 + 1, \ldots$ Since $\lim_{n\to\infty} \delta(A_n) = 0$ by assumption, then there are only finitely many A_n with diameters $\delta(A_n) \geqq 1/k_1$ and so there exists l_1 such that if A_n meets $W_{k_1} \times \bigcup_{l=l_1}^{\infty} W_l$, then it is contained in some $W_{k_1} \times W_l$, $l = l_1, l_1 + 1, \ldots$, and has diameter $\delta(A_n) < 1/k_1$. However, since the distance between any two points $(a, b), (c, d)$ of $W_{k_1} \times W_{l_1}$ such that $a \neq c$ is not less than

$1/k_1$, then each A_n meeting $W_{k_1} \times W_{l_1}$ is contained in some $\{z_n\} \times W_{l_1}$. This, however, is a contradiction, because the cover $\mathfrak{A}$ is countable and there are uncountably many sets $\{z\} \times W_{l_1}$, $z \in W_{k_1}$.

4. Property I. This is the weakest property from the three ones and has the worst formal properties. We shall show, for instance, that it is neither hereditary (in a rather strong sense) nor productive, and is only finitely additive.

Property I is possessed by all metric spaces X which can be represented as finite unions of sets with diameters less than the diameter $\delta(X)$ of the space X itself (but not exclusively, as the example of the space Z defined in IV of § 3 shows). In particular, if X is any metric space with a finite diameter, then adding to X a new point p and introducing into the space $X_1 = X \cup \{p\}$ metric ϱ_1 by the formula

$$\varrho_1(x, y) = \begin{cases} \varrho(x, y) & \text{if} \quad x \in X \quad \text{and} \quad y \in X\,, \\ \delta(X) + 1 & \text{if either} \quad x = p \quad \text{or} \quad y = p \quad \text{but} \quad x \neq y\,, \end{cases}$$

we get a new metric space X_1 which contains X isometrically and has Property I *independently* of whether X itself has it or not. Hence, in contrast to Property II, *Property I is not hereditary.*

However,

(4.1) *Property I is hereditary with respect to dense subsets, i.e., if X is a metric space which has Property I and D is its dense subset, then D also has Property I.*

The proof is trivial, because $\delta(A) = \delta(A \cap D)$ for each open $A \subset X$, and thus if $\{A_n\}$ is a zero cover for X, then $D \cap A_n$ is a zero cover for D.

Let us recall that a property P of a metric space is *productive* if the product of two metric spaces (X, ϱ_1) and (Y, ϱ_2) enjoying it and metrized by the pythagorean formula

$$\varrho[(x_1, y_1), (x_2, y_2)] = \sqrt{[\varrho_1^2(x_1, x_2) + \varrho_2^2(y_1, y_2)]}$$

enjoys it too (cf. [2], p. 133). As the example of the cartesian square $Z \times Z$ of the metric space Z enjoying Property I shows (see (3.6)), *Property I is not productive.* The "not" is so decisive that even the cartesian square $X \times X$ of a metric separable space X enjoying the by far stronger Property II may not possess Property I (see (3.4))!

(4.2) *Property I is finitely additive, i.e. if X is a metric space, $X = \bigcup_{n=1}^{N} X_n$, and each X_n has Property I, then X has Property I too.*

Proof. Let, for each $n = 1, 2, \ldots, N$, $\{A_{n,k}\}_{k=1,2,\ldots}$ be a zero cover of X_n. Choosing now a number $\varepsilon_n > 0$ such that

$$\sup_k \delta(A_{n,k}) + 2\varepsilon_n < \delta(X_n)\,,$$

consider a generalized open ball $A^*_{n,k} = K(A_{n,k}, \varepsilon_n/k)$, i.e. the set of all those points $x \in X$ whose distance from $A_{n,k}$ is less than ε_n/k. Each $A^*_{n,k}$ is open in X and, in view of

$$\delta(A^*_{n,k}) \leqq \delta(A_{n,k}) + 2\varepsilon_n/k < \delta(X_n) \leqq \delta(X),$$

we have $\delta(A^*_{n,k}) < \delta(X)$. Moreover,

$$\lim_{k\to\infty} \delta(A^*_{n,k}) \leqq \lim_{k\to\infty} \delta(A_{n,k}) + \lim_{k\to\infty} 2\varepsilon_n/k = 0.$$

Ordering then N sequences $\{A^*_{n,k}\}_{k=1,2,\ldots}$ in a single sequence we get a zero cover of the entire X. Hence (4.2) is proved.

However, *Property I*, in contrast to Property II, *is not countably additive.* In fact, take a union of countably many copies of the space Z defined in § 3. As was shown in (3.5), Z has Property I. However, if we introduce into this union a metric ϱ' by the formula

$$\varrho'(x, y) = \begin{cases} \varrho(x, y) & \text{if } x \text{ and } y \text{ belong to the same copy},\\ 2 & \text{otherwise}, \end{cases}$$

then this new metric space does not possess Property I, because each countable cover of it consisting of sets (not necessarily open) with diameters less than 2 contains a countable subfamily composed of sets with diameters larger or equal to 1, and hence it cannot be a zero cover.

Now let X be a metric space and let $\lambda(X)$ denote the greatest lower bound of those real numbers η for which X can be represented as a countable union of its (not necessarily open) subsets A_n such that

(1) $$\delta(A_n) < \eta \quad \text{for} \quad n = 1, 2, \ldots \quad \text{and} \lim_{n\to\infty} \delta(A_n) = 0.$$

(4.3) *A metric space X has Property I if and only if $\lambda(X) < \delta(X)$.*

Proof. First of all observe that the value of $\lambda(X)$ will not change if we restrict ourselves to open sets A_n only. In fact, if $X = \bigcup_{n=1}^{\infty} A_n$, where A_n are arbitrary subsets of X satisfying (1), then for each $\varepsilon > 0$ the generalized open balls $A^*_n = K(A_n, \varepsilon/n)$ are open in X and $\delta(A^*_n) \leqq \delta(A_n) + 2\varepsilon/n$.

Hence $X = \bigcup_{n=1}^{\infty} A^*_n$, $\delta(A^*_n) < \eta + \varepsilon$ for each $n = 1, 2, \ldots$, and $\lim_{n\to\infty} \delta(A^*_n) = 0$. This means that the g.l.b. for open sets is not larger than that for arbitrary sets, and the converse inequality is quite trivial.

Applying this we can now say that if $\lambda(X) > 0$, then no sequence $\{A_n\}$ of open sets such that

$$\delta(A_n) < \lambda(X) \quad \text{for} \quad n = 1, 2, \ldots \quad \text{and} \quad \lim_{n\to\infty} \delta(A_n) = 0$$

is a cover of X. This means that if $\lambda(X) = \delta(X)$, then the space X does not possess Property I.

On the other hand, if $\lambda(X) < \delta(X)$, then for each η, $\lambda(X) < \eta < \delta(X)$, there exists an open cover $\{A_n\}$ of X which satisfies (1). Since $\eta < \delta(X)$, then it is a zero cover and thus the space X has Property I. The proof of (4.3) is completed.

(4.4) *For a 0-dimensional Baire space $\mathscr{B}$ we have $\lambda(\mathscr{B}) = \delta(\mathscr{B}) = 1$.*

Proof. It suffices to show that if $\mathfrak{A} = \{A_i\}$ is a sequence of open subsets of $\mathscr{B}$ such that

$$(1) \qquad \delta(A_i) < 1 \quad \text{for} \quad i = 1, 2, \ldots \quad \text{and} \quad \lim_{i \to \infty} \delta(A_i) = 0\,,$$

then $\mathfrak{A}$ is not a cover of $\mathscr{B}$. For that purpose let $\mathfrak{A}_n$ denote the set of those A_i, for which $\delta(A_i) \geqq 1/n$, and let $\mathfrak{A}^*$ (resp. $\mathfrak{A}_n^*$) be the union of all elements of $\mathfrak{A}$ (resp. $\mathfrak{A}_n$). By virtue of (1), each $\mathfrak{A}_n$ is finite and $\mathfrak{A}_1 = 0$.

Let n_0 be the least positive integer for which $\mathfrak{A}_{n_0+1} \neq 0$. The diameter of each element $A_i \in \mathfrak{A}_{n_0+1}$ is then $\delta(A_i) = 1/(n_0 + 1)$ and this means that to each element $A_i \in \mathfrak{A}_{n_0+1}$ there corresponds a sequence $\{c_1, \ldots, c_{n_0}\}$ of n_0 positive integers such that A_i is contained in the set of all sequences of the form $\{c_1, \ldots, c_{n_0}, x_{n_0+1}, x_{n_0+2}, \ldots\}$, where x_{n_0+j} are arbitrary positive integers.

Since $\mathfrak{A}_{n_0+1}$ is finite, then there must exist a sequence of n_0 positive integers $\{a_1, \ldots, a_{n_0}\}$ such that no sequence $\{a_1, \ldots, a_{n_0}, x_{n_0+1}, x_{n_0+2}, \ldots\}$ belongs to $\mathfrak{A}^*_{n_0+1}$.

Assume now that for some $k \geqq n_0$ we have already defined a sequence consisting of k positive integers $\{a_1, \ldots, a_k\}$ such that no sequence $\{a_1, \ldots, a_k, x_{k+1}, \ldots\}$ belongs to $\mathfrak{A}^*_{k+1}$. The family $\mathfrak{A}_{k+2}$ differs from the family $\mathfrak{A}_{k+1}$ for finitely many sets each of which has diameter equal to $1/(k + 2)$ and thus is contained in the set of all sequences of positive integers $\{c_1, \ldots, c_{k+1}, x_{k+2}, \ldots\}$, where $c_1, \ldots, c_{k+1}$ are fixed. Hence a positive integer a_{k+1} must exist such that no sequence $\{a_1, a_2, \ldots, a_k, a_{k+1}, x_{k+2}, \ldots\}$ belongs to $\mathfrak{A}^*_{k+2}$.

In this inductive way we have constructed a sequence $\{a_1, a_2, \ldots\}$ which is not covered by $\mathfrak{A}$, because it does not belong to any $\mathfrak{A}_k^*$ and obviously $\bigcup_{k=1}^{\infty} \mathfrak{A}_k^* = \mathfrak{A}^*$.

It is a simple corollary of (4.3) and (4.4) that

(4.5) *0-dimensional Baire space $\mathscr{B}$ does not possess Property I.*

Let us recall (see [3], I, p. 318) that $\alpha(X)$ denotes the greatest lower bound of those real numbers η for which metric space X can be represented as a union of finitely many (but not necessarily open) sets with diameters $<\eta$.

Obviously,

(4.6) *For any metric space X, $\lambda(X) \leqq \alpha(X) \leqq \delta(X)$.*

Hence and from (4.3) we infer that

(4.7) *If X is a metric space such that $\alpha(X) < \delta(X)$, then X has Property I.*

However, there exist metric spaces which have Property I and for which $\alpha(X) = \delta(X)$. Such is, for instance, the metric space Z of § 3 for which $\lambda(Z) = 1$ and $\alpha(Z) = \delta(Z) = 2$.

We shall complete this section with two formal properties of the function λ.

(4.8) *λ is monotone, i.e., if $Y \subset X$ then $\lambda(Y) \leqq \lambda(X)$.*

The proof is trivial, because if $X = \bigcup_{n=1}^{\infty} A_n$, where $\delta(A)_n < \eta$ for $n = 1, 2, \ldots$ and $\lim_{n\to\infty} \delta(A_n) = 0$, then $Y = \bigcup_{n=1}^{\infty} Y \cap A_n$ and $\delta(Y \cap A_n) < \eta$ for $n = 1, 2, \ldots$ as well as $\lim_{n\to\infty} \delta(Y \cap A_n) = 0$.

(4.9) *λ is countably subadditive, i.e., $\lambda(\bigcup_{i=1}^{\infty} X_i) \leqq \sum_{i=1}^{\infty} \lambda(X_i)$.*

Suppose the contrary, i.e., that there exists a metric space X such that $X = \bigcup_{i=1}^{\infty} X_i$ and $\lambda(\bigcup_{i=1}^{\infty} X_i) > \sum_{i=1}^{\infty} \lambda(X_i)$. This means that for each $i = 1, 2, \ldots$ there is a number $\eta_i > 0$ such that $\lambda(X_i) < \eta_i$ and

$$\lambda(\bigcup_{i=1}^{\infty} X_i) > \sum_{i=1}^{\infty} \eta_i . \tag{1}$$

By virtue of $\lambda(X_i) < \eta_i$ each set X_i can be decomposed into a union $X_i = \bigcup_{n=1}^{\infty} A_n^i$ such that $\delta(A_n^i) < \eta_i$ for $n = 1, 2, \ldots$ and $\lim_{n\to\infty} \delta(A_n^i) = 0$. Ordering all sets A_n^i into a single sequence $\{B_k\}_{k=1,2,\ldots}$ we infer that $X = \bigcup_{k=1}^{\infty} B_k$ and that

$$\delta(B_k) \leqq \sup_i \eta_i \leqq \sum_{i=1}^{\infty} \eta_i < \lambda(\bigcup_{i=1}^{\infty} X_i) ,$$

and since by (1) the series $\sum_{i=1}^{\infty} \eta_i$ is convergent and therefore $\lim_{i\to\infty} \eta_i = 0$, also $\lim_{k\to\infty} \delta(B_k) = 0$. But this is a contradiction with the value of $\lambda(\bigcup_{i=1}^{\infty} X_i)$.

5. Property II. This Property appears to be the most regular one. For instance, it is hereditary while Property I is not and Property III only partially, and it is also countably additive.

(5.1) *Property II is hereditary, i.e., if X is a metric space which possesses Property II and $Z \subset X$, then Z also has Property II.*

Actually, if Z is a single point, then it enjoys Property II by the definition. Thus, let $\delta(z) > 0$. If $\{A_n\}$ is a zero base for X, then $\{Z \cap A_n\}$ is a base for Z consisting of sets with diameters tending to 0. And since from any base in a metric space we can remove all sets with large diameter without changing the property "to be a base" (cf. [3], p. 133), then in order to receive a zero base for Z it suffices to remove from the base $\{Z \cap A_n\}$ all sets $Z \cap A_n$ with $\delta(Z \cap A_n) = \delta(Z)$.

(5.2) *Property II is countably additive, i.e., if X is a metric space, $X = \bigcup_{n=1}^{\infty} X_n$, and each member X_n has Property II, then the space X itself also has Property II.*

Proof. Let $\{A_{n,k}\}_{k=1,2,\dots}$ be a zero base for X_n, $n = 1, 2, \dots$ Choosing arbitrarily $\varepsilon > 0$ consider the generalized open balls

$$A^*_{n,k} = K(A_{n,k}, \varepsilon/nk)\,, \quad n, k = 1, 2, \dots$$

Obviously, $\delta(A^*_{n,k}) \leqq \delta(A_{n,k}) + 2\varepsilon/nk$ and thus, for a fixed $n = 1, 2, \dots$,

$$\lim_{k\to\infty} \delta(A^*_{n,k}) = 0\,. \tag{1}$$

This means, in particular, that the sets $A^*_{n,k}$, where n is fixed and $k = 1, 2, \dots$, form a base for X_n. Hence we may assume that (after, perhaps, removing some of the sets $A^*_{n,k}$ with large diameter)

$$\delta(A^*_{n,k}) < \frac{1}{n} \min\left[1, \delta(X)\right] \quad \text{for each} \quad k = 1, 2, \dots \quad \text{and} \quad n = 1, 2, \dots \tag{2}$$

Ordering now the double sequence $\{\{A^*_{n,k}\}_{k=1,2,\dots}\}_{n=1,2,\dots}$ into a single one $\{B_l\}_{l=1,2,\dots}$ we shall show that it is a zero base for X.

Actually, it is a base for X, because it is a base for each X_n and $X = \bigcup_{n=1}^{\infty} X_n$. Moreover, it is a zero base, because, by virtue of (1) and (2), for each $\eta > 0$ there are only finitely many sets B_l with $\delta(B_l) > \eta$, and this implies $\lim_{l\to\infty} \delta(B_l) = 0$. Also, by (2), $\delta(B_l) < \delta(X)$ for $l = 1, 2, \dots$

To show that Property II is not productive we need some characterization of it. Before turning to this question let us insert here, for the sake of completeness, a simple observation

(5.3) *Let $f : X \to Y$ be an open mapping of a metric space (X, ϱ_1) onto a metric space (Y, ϱ_2), and such that*

$$\varrho_1(x, y) \geqq \varrho_2[f(x), f(y)] \quad \text{for all} \quad x, y \in X\,.$$

If X enjoys Property II, then Y does it too.

Proof. If $\{A_n\}$ is a zero base for X, then the sequence $\{f(A_n)\}$ of sets open in Y is a base for Y such that

$$\lim_{n\to\infty} f(A_n) = 0\,.$$

Removing now, if necessary, sets $f(A_n)$ with diameter $\delta[f(A_n)] = \delta(Y)$ we obtain a zero base for Y.

Let us turn now to some characterizations of Property II.

(5.4) *A metric space X has Property II if and only if there exists an increasing sequence $\{N_k\}$ of its finite subsets N_k such that each point of X belongs to $K(N_k, 1/k)$ for infinitely many $k = 1, 2, \ldots$*

Proof. The condition is sufficient. Actually, the countable family of open balls

$$(1) \qquad K(p, 1/k), \quad \text{where} \quad p \in N_k \quad \text{and} \quad k = 1, 2, \ldots,$$

is a base for X, because each point of X is contained in arbitrarily small balls of the form (1). Since each N_k is finite, then ordering balls (1) into a sequence $\{K_n\}$ we have also $\lim_{n\to\infty} \delta(K_n) = 0$. Removing now, if necessary, from the base $\{K_n\}$ all sets K_n with diameters $\delta(K_n) = \delta(X)$ we get a zero base for X.

The condition is necessary. Indeed, if $\mathfrak{A}$ is a zero base for X, then replacing each $U \in \mathfrak{A}$ with $\delta(U) < 1$ by an open ball $K(p, 1/k)$, where k is the largest integer such that $\delta(U) < 1/k$ and $p \in U$, we obtain another zero base for X. This is so, because each U is contained in the ball $K(p, 1/k)$ by which we have replaced it and therefore the family of these balls forms the base for X. Furthermore, if for some k_0 there would be infinitely many balls with radius $1/k_0$, then there would be also infinitely many U's from the original base $\mathfrak{A}$ with diameter $\delta(U) \geqq 1/(k_0 + 1)$ which is impossible as $\mathfrak{A}$ is a zero base by hypothesis. It remains then, if necessary, to remove all balls K with diameter $\delta(K) = \delta(X)$.

Now, since the family of balls $K(p, 1/k)$ forms a zero base for X, then denoting by N'_l the set of centers of balls with radius $1/l$, we obtain a sequence of finite sets $\{N'_l\}_{l=1,2,\ldots}$ with the property that each point of X belongs to $K(N'_l, 1/l)$ for infinitely many $l = 1, 2, \ldots$ Putting then $N_k = \bigcup_{l=1}^{k} N'_l$ we get a non-decreasing sequence of finite sets with that property.

The coefficient $\lambda(X)$, which has helped us to give a characterization of metric spaces that have Property I (cf. (4.3)), enables us also to prove the following simple characterization of metric spaces that have Property II.

(5.5) *A metric space X has Property II if and only if $\lambda(X) = 0$.*

Proof. If X has Property II, then there exists a base $\{A_n\}$ for X such that

$$\delta(A_n) < \delta(X) \quad \text{for} \quad n = 1, 2, \ldots \quad \text{and} \quad \lim_{n\to\infty} \delta(A_n) = 0\,.$$

Given any $\eta > 0$, we remove from the base $\{A_n\}$ all sets A_n with $\delta(A_n) \geqq \eta$ and thus remain with a base $\{B^\eta_m\}$ such that

$$(1) \qquad \delta(B^\eta_m) < \eta \quad \text{for} \quad m = 1, 2, \ldots \quad \text{and} \quad \lim_{m\to\infty} \delta(B^\eta_m) = 0\,,$$

which proves that $\lambda(X) = 0$.

Conversely, if $\lambda(X) = 0$, i.e. if for each $\eta > 0$ there exists an open cover $\{B_m^\eta\}_{m=1,2,\dots}$ of X for which (1) holds, then considering the sequence of covers $\{B_m^{1/k}\}_{m=1,2,\dots}$ for $k = k_0, k_0 + 1, \dots$, where k_0 is a positive integer such that $1/k_0 < \delta(X)$, and ordering this sequence of sequences of open sets into a single one, we can easily verify that it is a zero base for X.

Now we may show that *Property II is not productive.* Indeed, as we have already proved in (3.3), the metric space $\mathscr{B}_0$ satisfies $\lambda(\mathscr{B}_0) = 0$ and so, by (5.5), it enjoys Property II. However, by virtue of (3.4), the cartesian square $\mathscr{B}_0 \times \mathscr{B}_0$ possesses even not so much as Property I!

Further on we shall establish a sufficient condition under which the cartesian product of two metric spaces enjoying Property II does it too (cf. (6.7) below).

It is a trivial conclusion from (5.1) that if a metric space X has Property II, then it has it *locally*, i.e., for each point $p \in X$ there exists an open (even arbitrarily small) neighbourhood U which has Property II. In what follows we shall need a lemma which concerns the converse of this remark for metric separable spaces.

(5.6) *If a metric separable space X has Property II locally at each point $p \in X - A$ and $\lambda(A) = 0$, then X has Property II globally.*

Indeed, if for each point $p \in X - A$ there exists an open neighbourhood U which has Property II, then by Lindelöf Theorem ([2], p. 49) there is a countable subcover $\{U_k\}_{k=1,2,\dots}$ of a subset $X - A$, and thus $X = A \cup \bigcup_{k=1}^{\infty} U_k$. Since each member on the right-hand side of the last equation has Property II, then by (5.2) we infer that X itself also has Property II.

Now we are about to draw some conclusions from what we have proved so far. First of all observe that for a totally bounded metric space X we have $\alpha(X) = 0$ and so, by (4.6) and (5.5),

(5.7) *Each totally bounded metric space has Property II.*

By a *countably totally bounded metric space* we mean any metric space which is a union of countably many totally bounded sets. From (5.2) and (5.7) we infer that

(5.8) *Each countably totally bounded metric space has Property II.*

It is well known that compact metric spaces are totally bounded (cf. [3], II, p. 2). Hence and from (5.1) and (5.7) it follows that

(5.9) *Each subspace* [2]) *of a compact metric space has Property II.*

An *F_σ-absolute space* is a union of countably many compact spaces. Lemmas (5.1) and (5.8) imply the following generalization of (5.9):

(5.10) *Each subspace of an F_σ-absolute metric space has Property II.*

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[2]) By a subspace of a metric space (X, ϱ) we mean any metric space (Y, ϱ') such that $Y \subset X$ and $\varrho' = \varrho \mid Y \times Y$.

In particular, if X is a complete metric and F_σ-absolute space, then X has Property II. However, there exists a complete metric space which enjoys Property II but is not F_σ-absolute (e.g., the space $\mathscr{B}_0$ of § 3).

A locally compact metric separable space is by Lindelöf Theorem an F_σ-absolute metric space. Hence we infer from (5.10) that

(5.11) *Each subspace of a locally compact metric separable space has Property II.*

In particular,

(5.12) *Each subspace of a Euclidean space has Property II.*

Although conclusions (5.7) to (5.12) may lead to the conjecture that the family of metric spaces which possess Property II is enormously vast, it is good to remember that equally many metric spaces have not that Property. For instance, 0-dimensional Baire space $\mathscr{B}$ and any metric space containing $\mathscr{B}$ isometrically do not enjoy Property II.

From this it follows, by a comparison with (5.12), that no metric space containing $\mathscr{B}$ isometrically (in particular, $\mathscr{B}$ itself) can be embedded isometrically into an Euclidean space. However, it is well known (cf. [6], p. 177) that $\mathscr{B}$ can be embedded topologically into 1-dimensional Euclidean space.

6. Nucleus $\hat{X}$. We shall say that a metric space X is *locally, at a point p, countably totally bounded* if it contains an open neighbourhood U of p such that $U = \bigcup_{k=1}^{\infty} P_k$, where each P_k is totally bounded. The set of all points q at which X is not locally countably totally bounded will be called a *nucleus* and denoted by $\hat{X}$.

(6.1) *If X is a metric space, then the nucleus $\hat{X}$ is closed in X.*

Indeed, if $p \in X - \hat{X}$, then there exists an open neighbourhood U of p which is a union of countably many totally bounded sets. Hence each point q of U also belongs to $X - \hat{X}$ and so $U \subset X - \hat{X}$. This means that $X - \hat{X}$ is open or, in other words, that $\hat{X}$ is closed.

(6.2) *If X is a metric separable space, then $\hat{\hat{X}} = \hat{X}$, i.e., $\hat{X}$ is at no point of it locally countably totally bounded.*

Proof. To prove that $\hat{\hat{X}} = \hat{X}$ suppose the contrary. For some point $p \in \hat{X}$ there exists then its open (in $\hat{X}$) neighbourhood V which is countably totally bounded

(1) $$V = \bigcup_{k=1}^{\infty} W_k \quad \text{and } W_k \text{ is totally bounded for each } k = 1, 2, \ldots$$

Let U be an open subset of X such that $U \cap \hat{X} = V$ (cf. [3], I, p. 25). Hence the set

(2) $$U = (U - \hat{X}) \cup V \quad \text{is an open (in } X\text{) neighbourhood of } p\,.$$

Since $\hat{X}$ is closed in X, then $U - \hat{X}$ is open in X. For each $x \in U - \hat{X}$ there exists then an open in X neighbourhood $U_x \subset U - \hat{X}$ which is countably totally bounded. By virtue of Lindelöf Theorem there is a countable subcover $\{U_i\}$ of $U - \hat{X}$. Let

(3) $\quad U_i = \bigcup_{j=1}^{\infty} P_{ij}$, where P_{ij} is totally bounded for each $i, j = 1, 2, \ldots$

In view of (1), (2) and (3) we have then

$$U = \bigcup_{i=1}^{\infty} \bigcup_{j=1}^{\infty} P_{ij} \cup \bigcup_{k=1}^{\infty} W_k ,$$

whence we infer that $p \in X - \hat{X}$. A contradiction.

(6.3) *Let X be a metric separable space. Then $\hat{X} = 0$ if and only if X is countably totally bounded.*

Actually, if X is countably totally bounded, then obviously it is also locally totally bounded at each of its points, which means that $\hat{X} = 0$.

Conversely, if $\hat{X} = 0$, then for each point $x \in X$ choose an open neighbourhood U_x which is countably totally bounded. By virtue of Lindelöf Theorem there exists then a countable subcover $\{U_{x_n}\}$ of X and this means that X is countably totally bounded.

(6.4) *If X is a separable metric space, then $\lambda(X) = \lambda(\hat{X})$.*

Indeed, by (4.8) and (4.9) we infer that

$$\lambda(\hat{X}) \leqq \lambda(X) \leqq \lambda(\hat{X}) + \lambda(X - \hat{X}) .$$

And since as trivially follows from the definition of nucleus, the nucleus of $X - \hat{X}$ is an empty set, then by (6.3) the set $X - \hat{X}$ is countably totally bounded and therefore, by (5.8), enjoys Property II. In view of (5.5) we have then $\lambda(X - \hat{X}) = 0$ and so, finally, $\lambda(\hat{X}) = \lambda(X)$.

It is an easy consequence of (5.5) and (6.4) that

(6.5) *Let X be a separable metric space. The space X has Property II if and only if $\lambda(\hat{X}) = 0$.*

However, as the example of the space $\mathscr{B}_0$ defined in § 3 shows, the spaces which have Property II (cf. (3.3) and (5.5)) but whose nucleus $\hat{X}$ is not empty (cf. (3.2) and (6.3)) do exist.

(6.6) *If $X \times Y$ is the cartesian product of two metric spaces X and Y, then $\widehat{X \times Y} = \hat{X} \times Y \cup X \times \hat{Y}$.*

Proof. If $(p, q) \in X \times Y - \widehat{X \times Y}$, then by (6.1) there exists its open neighbourhood U disjoint with $\widehat{X \times Y}$ which is a union of countably many totally bounded sets

$U = \bigcup_{n=1}^{\infty} U_n$. Since the projections $r_X : X \times Y \to X$ and $r_Y : X \times Y \to Y$ are open and preserve total boundedness, then $r_X(U) = \bigcup_{n=1}^{\infty} r_X(U_n)$ is an open neighbourhood of p which is a union of countably many totally bounded sets and therefore $p \in X - \hat{X}$. Similarly, $q \in Y - \hat{Y}$. Hence

$$X \times Y - \widehat{X \times Y} \subset (X - \hat{X}) \times (Y - \hat{Y}).$$

However, since (cf. [3], I, p. 12)

$$(X - \hat{X}) \times (Y - \hat{Y}) = (X \times Y) - (\hat{X} \times Y \cup X \times \hat{Y}),$$

then $\hat{X} \times Y \cup X \times \hat{Y} \subset \widehat{X \times Y}$.

Conversely, if $(p, q) \in X \times Y - (\hat{X} \times Y \cup X \times \hat{Y})$, then $p \in X - \hat{X}$ and $q \in Y - \hat{Y}$. This means that there exists an open in X neighbourhood U of p which is a union of countably many totally bounded sets U_n, $U = \bigcup_{n=1}^{\infty} U_n$, and similarly, there exists an open in Y neighbourhood V of q which is a union of countably many totally bounded sets V_n, $V = \bigcup_{n=1}^{\infty} V_n$. Since each product $U_k \times V_l$ is totally bounded (cf. [3], I, p. 115), then an open in $X \times Y$ neighbourhood $U \times V$ of the point (p, q) is a union of countably many totally bounded sets, $U \times V = \bigcup_{k,l=1}^{\infty} U_k \times V_l$, and this means that $(p, q) \in X \times Y - \widehat{X \times Y}$. Hence $\widehat{X \times Y} \subset \hat{X} \times Y \cup X \times \hat{Y}$ and the proof is completed.

As a simple corollary of (6.6) and (6.3) we obtain

(6.7) *Let X and Y be two metric separable spaces both enjoying Property II. If both nuclei $\hat{X}$ and $\hat{Y}$ are empty, then the product $X \times Y$ metrized by the pythagorean formula enjoys Property II.*

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