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CONTRIBUTION TO THE FOUNDATIONS OF NETWORK THEORY USING THE DISTRIBUTION THEORY, II

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In [1], linear and continuous operators on the space of distributions have been studied. Let $\mathbf{D}^l$ be the set of all distributions f such that every f vanishes on some interval $(-\infty, a)$ which in general depends on f. In this paper we shall study analogous properties of linear and uniformly continuous operators on a space of distributions from $\mathbf{D}^l$.

1. INTRODUCTION

The terminology and notation will follow [1] and [2]. Let $\mathbf{D}$ be the set of all distributions on $\mathbf{K}$. Put $\mathbf{D}_0^l = \mathbf{D}_0 \cap \mathbf{D}^l$. Let us also remark that in this paper $x(t)$ for $x \in \mathbf{D}_0^l$ always means a continuous function on $(-\infty, +\infty)$ such that $x(t) = 0$ on some interval $(-\infty, a)$ which in general depends on x. Let $n \geqq 1$ be an integer, and let $\mathbf{D}_n^l$ be the set of all distributions having the following property: if $f \in \mathbf{D}_n^l$ then there is a distribution $z \in \mathbf{D}_0^l$ such that $f = z^{(n)}$. Evidently $\mathbf{D}_n^l \subset \mathbf{D}^l$. Finally, let $\mathbf{D}_*^l = \bigcup_{n=0}^{\infty} \mathbf{D}_n$. Clearly, $\mathbf{D}^l$, $\mathbf{D}_n^l$ $(n = 0, 1, \ldots)$ and $\mathbf{D}_*^l$ are linear time-invariant subspaces of $\mathbf{D}$.

From Lemma 3.1.2 [2] there follows

Lemma 1.1. *Let a, b $(a < b)$ be real numbers and let $f \in \mathbf{D}$. If f vanishes on $(-\infty, b)$, then f vanishes on $(-\infty, a)$.*

Lemma 1.2. *Let a, b, α, β be real numbers and let $f, g \in \mathbf{D}$. If f vanishes on $(-\infty, a)$ and g vanishes on $(-\infty, b)$, then $\alpha f + \beta g$ vanishes on $(-\infty, c)$, where* $c = \min(a, b)$.

Lemma 1.3. *Let a, b be real numbers and let $f \in \mathbf{D}$. If f vanishes on $(-\infty, a)$, then $g = P_b[f]$ vanishes on $(-\infty, a + b)$.*

Lemma 1.4. *Let $x, x_n \in \mathbf{D}$, $n = 1, 2, \ldots$ and let $x_n \to x$. If x_n vanishes on some interval $(-\infty, b)$ for $n = 1, 2, \ldots$, then x vanishes on $(-\infty, b)$.*

Proof. If $\varphi \in \mathbf{K}$ and $\varphi(t) = 0$ on $(c, +\infty)$ where $c < b$, then by Lemma 1.3 [1] we have $\langle x_n, \varphi \rangle = 0$ for every $n = 1, 2, \ldots$ Thus $\langle x_n, \varphi \rangle \to 0 = \langle x, \varphi \rangle$ and therefore from Lemma 1.3 [1] it follows that x vanishes on $(-\infty, b)$.

Definition 1.1. Let $x, x_n \in \mathbf{D}$, $n = 1, 2, \ldots$; the sequence x_n will be called *iniformly convergent to* x (this fact will be symbolized by $x_n \rightrightarrows x$), if $x_n \to x$ and x_n vanishes on some interval $(-\infty, b)$ for every $n = 1, 2, \ldots$

Lemma 1.5. *Let $x \in \mathbf{D}^l$; then there is a sequence $x_n \in \mathbf{D}^l_*$, $n = 1, 2, \ldots$ such that $x_n \rightrightarrows x$.*

Proof. The proof is analogous to that of Lemma 5.4.5 in [2].

Let $\mathbf{K}^p$ denote the set of all infinitely differentiable real functions $\varphi(t)$ such that every $\varphi(t)$ vanishes on some interval $(a, +\infty)$ (which in general depends on φ).

Definition 1.2. Let $\mathscr{F}^l$ be the set of all distributions $\{f_a\}$ from $\mathbf{D}^l$ depending on a parameter a (where a is an arbitrary real number) with the following properties:

1. If $\varphi \in \mathbf{K}$, then $\psi \in \mathbf{K}^p$, where $\psi(a) = \langle f_a, \varphi \rangle$ for every real number a.

2. If a_0 is an arbitrary real number, then there exists a real number b_0 such that f_a vanishes on $(-\infty, b_0)$ for every real number $a \geqq a_0$.

Note. The statement 1 of Definition 1.2 holds if and only if the partial derivative $\partial^n f_a / \partial a^n$ exists for every positive integer n and $\alpha_n f_{a_n} \to 0$ for every two sequences of real numbers α_n, a_n with $a_n \to +\infty$, $n = 1, 2, \ldots$

The proof is similar to the proof of Theorem 1.1 [1].

Theorem 1.1. *Let $\{f_a\}, \{g_a\} \in \mathscr{F}^l$.*

1. *If α, β are real numbers, then $\{\alpha f_a + \beta g_a\} \in \mathscr{F}^l$.*
2. *If n is a positive integer, then $\{\partial^n f_a / \partial a^n\} \in \mathscr{F}^l$.*
3. *If b is a real number, then $\{h_a\} \in \mathscr{F}^l$, where $h_a = P_b[f_a]$.*
4. *If b is a real number, then $\{h_a\} \in \mathscr{F}^l$, where $h_a = f_{a-b}$.*

Proof. Put $\psi(a) = \langle f_a, \varphi \rangle$ and $\chi(a) = \langle g_a, \varphi \rangle$ for every $\varphi \in \mathbf{K}$.

1. If α, β are real numbers, then $\omega(a) = \langle \alpha f_a + \beta g_a, \varphi \rangle = \alpha\, \psi(a) + \beta\, \chi(a)$. Thus $\omega = \alpha\psi + \beta\chi \in \mathbf{K}^p$. If a_0 is an arbitrary real number, then there exist real numbers b_1, b_2 such that f_a vanishes on $(-\infty, b_1)$ and g_a vanishes on $(-\infty, b_2)$ for every real number $a \geqq a_0$. From Lemma 1.2 it follows that $\alpha f_a + \beta g_a$ vanishes on $(-\infty, b_0)$

(where $b_0 = \min(b_1, b_2)$) for every real number $a \geqq a_0$. Hence $\{\alpha f_a + \beta g_a\} \in \mathscr{F}^l$.

2. If n is a positive integer, then from Lemma 1.6 [1] it follows that $\omega(a) = \langle \partial^n f_a / \partial a^n, \varphi \rangle = \psi^{(n)}(a)$. Thus $\omega = \psi^{(n)} \in \mathbf{K}^p$. If a_0 is an arbitrary real number, then there exists a real number b_0 such that f_a vanishes on $(-\infty, b_0)$ for every real number $a \geqq a_0$. Let c be a real number, $c < b_0$, and let $\varphi(t) = 0$ on the interval $(c, +\infty)$. Then by Lemma 1.3 [1] we have $\psi(a) = \langle f_a, \varphi \rangle = 0$ for every real number $a \geqq a_0$. From this it follows that $\langle \partial^n f_a / \partial a^n, \varphi \rangle = \psi^{(n)}(a) = 0$ for every real number $a \geqq a_0$. Then by Lemma 1.3 [1] $\partial^n f_a / \partial a^n$ vanishes on $(-\infty, b_0)$ for every real number $a \geqq a_0$. Hence $\{\partial^n f_a / \partial a^n\} \in \mathscr{F}^l$.

3. If b is a real number, then $\omega(a) = \langle h_a, \varphi \rangle = \langle P_b[f_a], \varphi \rangle = \langle f_a, \varphi(t + b) \rangle$. Evidently $\varphi(t + b) \in \mathbf{K}$ and thus we have $\omega \in \mathbf{K}^p$. If a_0 is an arbitrary real number, then there exists a real number b_0 such that f_a vanishes on $(-\infty, b_0)$ for every real number $a \geqq a_0$. From Lemma 1.3 it follows that $h_a = P_b[f_a]$ vanishes on $(-\infty, b_0 + b)$ for every real number $a \geqq a_0$. Hence $\{h_a\} \in \mathscr{F}^l$.

4. If b is a real number, then $\omega(a) = \langle h_a, \varphi \rangle = \langle f_{a-b}, \varphi \rangle = \psi(a - b)$. Thus $\omega \in \mathbf{K}^p$. Let a_0 be an arbitrary real number. Since $a_0 - b$ is a real number, there exists a real number b_0 such that f_a vanishes on $(-\infty, b_0)$ for every real number $a \geqq a_0 - b$. From this it follows that $h_a = f_{a-b}$ vanishes on $(-\infty, b_0)$ for every real number $a \geqq a_0$. Consequently $\{h_a\} \in \mathscr{F}^l$.

Example 1. Let a be a real number and let H_a be the shifted Heaviside's distribution, i.e.

$$\langle H_a, \varphi \rangle = \int_a^{+\infty} \varphi(t)\, \mathrm{d}t$$

for every $\varphi \in \mathbf{K}$. Evidently, H_a vanishes on $(-\infty, a)$ for every real number a. If $\varphi \in \mathbf{K}$, then $\psi \in \mathbf{K}^p$, where $\psi(a) = \langle H_a, \varphi \rangle$. From this it follows that $\{H_a\} \in \mathscr{F}^l$.

Example 2. From Theorem 1.1 it follows that $\{\delta_a\} \in \mathscr{F}^l$. Also, we clearly have $\partial H_a / \partial a = -\delta_a$.

Definition 1.3. Let b be a real number and let $\mathscr{N}_b$ be the set of all real infinitely differentiable functions $\mu(t)$ on $(-\infty, +\infty)$ such that

$$\mu(t) = \begin{cases} 0 & \text{on } (-\infty, c), \\ 1 & \text{on } (d, +\infty), \end{cases}$$

where $c < d < b$.

Lemma 1.6. *If* $\{f_a\} \in \mathscr{F}^l$ *and* $\mu \in \mathscr{N}_b$, *then* $\{\mu(a) f_a\} \in \mathscr{F}$.

Proof. Let $\varphi \in \mathbf{K}$. Put $\psi(a) = \langle f_a, \varphi \rangle$ for every real number a. Evidently, $\psi \in \mathbf{K}^p$. We have $\langle \mu(a) f_a, \varphi \rangle = \mu(a)\, \psi(a)$ and thus $\mu\psi \in \mathbf{K}$. Hence $\{\mu(a) f_a\} \in \mathscr{F}$.

Let us recall the fact that if $\{f_a\} \in \mathscr{F}$ and $x \in \mathbf{D}$, then

$$\left\langle \int_{-\infty}^{+\infty} (x, f_a)\, \mathrm{d}a, \varphi \right\rangle = \langle x, \psi \rangle \tag{2.1}$$

for every $\varphi \in \mathbf{K}$, where $\psi(a) = \langle f_a, \varphi \rangle$ for every real number a. (See Definition 2.1 [1].)

Lemma 2.1. *Let b, c be real numbers $(c < b)$. Let $\{f_a\} \in \mathscr{F}$ and $f_a = 0$ for every real number $a > c$. If $x \in \mathbf{D}^l$ and x vanishes on $(-\infty, b)$, then*

$$\int_{-\infty}^{+\infty} (x, f_a)\, \mathrm{d}a = 0 .$$

Proof. Put $y = \int_{-\infty}^{+\infty} (x, f_a)\, \mathrm{d}a$. If $\varphi \in \mathbf{K}$ then by (2.1) it follows that $\langle y, \varphi \rangle = \langle x, \psi \rangle$, where $\psi(a) = \langle f_a, \varphi \rangle$ for every real number a. Since $\psi(t) = 0$ on the interval $(c, +\infty)$, it follows from Lemma 1.3 [1] that $\langle x, \psi \rangle = 0$. Thus $y = 0$, q.e.d.

Lemma 2.2. *Let b, c be real numbers $(c < b)$. Let $\{f_a\}, \{g_a\} \in \mathscr{F}$ and let $f_a = g_a$ for every real number $a > c$. If $x \in \mathbf{D}^l$ and x vanishes on $(-\infty, b)$, then*

$$\int_{-\infty}^{+\infty} (x, f_a)\, \mathrm{d}a = \int_{-\infty}^{+\infty} (x, g_a)\, \mathrm{d}a .$$

Proof follows from Lemma 2.1 and Theorem 2.3 [1].

Lemma 2.3. *Let $\{f_a\} \in \mathscr{F}$ and let $x \in \mathbf{D}^l$ vanish on the interval $(-\infty, b)$. If $\mu \in \mathscr{N}_b$, then $\{\mu(a) f_a\} \in \mathscr{F}$ and*

$$\int_{-\infty}^{+\infty} (x, f_a)\, \mathrm{d}a = \int_{-\infty}^{+\infty} (x, \mu(a) f_a)\, \mathrm{d}a .$$

Proof. Put $\psi(a) = \langle f_a, \varphi \rangle$ for every $\varphi \in \mathbf{K}$. Then $\omega(a) = \langle \mu(a) f_a, \varphi \rangle = \mu(a)\, \psi(a)$. Thus $\omega \in \mathbf{K}$. Hence $\{\mu(a) f_a\} \in \mathscr{F}$. The rest of the proof follows from Lemma 2.2 and from Definition 1.3.

Definition 2.1. Let $\{f_a\} \in \mathscr{F}^l$, $x \in \mathbf{D}^l$ and let the integral $\int_{-\infty}^{+\infty} (x, f_a)\, \mathrm{d}a$ be defined by

$$\int_{-\infty}^{+\infty} (x, f_a)\, \mathrm{d}a = \int_{-\infty}^{+\infty} (x, \mu(a) f_a)\, \mathrm{d}a \tag{2.2}$$

where $\mu \in \mathscr{N}_b$ and x vanishes on the interval $(-\infty, b)$.

Note. The integral $\int_{-\infty}^{+\infty}(x, f_a)\,da$ is defined by (2.2) uniquely, i.e. it does not depend on $\mu \in \mathcal{N}_b$. Indeed, if $\mu_1 \in \mathcal{N}_{b_1}$, $\mu_2 \in \mathcal{N}_{b_2}$, then $\mu_1(a) f_a = \mu_2(a) f_a$ for every real number $a > c$ where c is some real number ($c < \max(b_1, b_2)$). From Lemma 2.2 it follows that

$$\int_{-\infty}^{+\infty} (x, \mu_1(a) f_a)\,da = \int_{-\infty}^{+\infty} (x, \mu_2(a) f_a)\,da\,.$$

Theorem 2.1. *Let $\{f_a\} \in \mathcal{F}^l$ and $x \in \mathbf{D}^l$. Then $\int_{-\infty}^{+\infty}(x, f_a)\,da \in \mathbf{D}^l$.*

Proof. Let x vanish on $(-\infty, b)$. From Definition 2.1 it follows that $y = \int_{-\infty}^{+\infty}(x, f_a)\,da = \int_{-\infty}^{+\infty}(x, \mu(a) f_a)\,da$, where $\mu \in \mathcal{N}_b$. Let $\mu(t) = 0$ on $(-\infty, a_0)$, where a_0 is some real number ($a_0 < b$). According to Definition 1.2 there exists a real number b_0 such that f_a vanishes on $(-\infty, b_0)$ for every real number $a \geqq a_0$. Let $\varphi \in \mathbf{K}$ and let $\varphi(t) = 0$ on $(c, +\infty)$, where c is some real number ($c < b_0$). Then by (2.1) $\langle y, \varphi\rangle = \langle x, \psi\rangle$, where $\psi(a) = \langle \mu(a) f_a, \varphi\rangle = \mu(a)\langle f_a, \varphi\rangle$. If $a < a_0$, then $\mu(a) = 0$. Thus $\psi(a) = 0$. If $a \geqq a_0$, then f_a vanishes on $(-\infty, b_0)$. By Lemma 1.3 [1] it follows that $\langle f_a, \varphi\rangle = 0$ and thus $\psi(a) = 0$. Hence $\langle x, \psi\rangle = 0$. Lemma 1.3 [1] implies that y vanishes on $(-\infty, b_0)$. Consequently, $y \in \mathbf{D}^l$, q.e.d.

Example 3. Let b be a real number and $\{f_a\} \in \mathcal{F}^l$. If $\mu \in \mathcal{N}_b$, then $\mu(b) = 1$. From Definition 2.1 and from Example to Definition 2.1 [1] it follows that $\int_{-\infty}^{+\infty}(\delta_b, f_a)\,da = \int_{-\infty}^{+\infty}(\delta_b, \mu(a) f_a)\,da = \mu(b) f_b = f_b$. Thus

$$f_b = \int_{-\infty}^{+\infty} (\delta_b, f_a)\,da\,.$$

Theorem 2.2. *If α, β are real numbers, $\{f_a\} \in \mathcal{F}^l$ and $x, y \in \mathbf{D}^l$, then*

$$\int_{-\infty}^{+\infty} (\alpha x + \beta y, f_a)\,da = \alpha \int_{-\infty}^{+\infty} (x, f_a)\,da + \beta \int_{-\infty}^{+\infty} (y, f_a)\,da\,.$$

Proof. Let x vanish on $(-\infty, b_1)$ and let y vanish on $(-\infty, b_2)$. By Lemma 1.1 x and y vanish on $(-\infty, b)$, where $b = \min(b_1, b_2)$. If $\mu \in \mathcal{N}_b$, then from Definition 2.1 and Theorem 2.2 [1] it follows that $\int_{-\infty}^{+\infty}(\alpha x + \beta y, f_a)\,da = \int_{-\infty}^{+\infty}(\alpha x + \beta y, \mu(a) f_a)\,da = \alpha \int_{-\infty}^{+\infty}(x, \mu(a) f_a)\,da + \beta \int_{-\infty}^{+\infty}(y, \mu(a) f_a)\,da = \alpha \int_{-\infty}^{+\infty}(x, f_a)\,da + \beta \int_{-\infty}^{+\infty}(y, f_a)\,da$, q.e.d.

Theorem 2.3. *If α, β are real numbers, $\{f_a\}, \{g_a\} \in \mathcal{F}^l$ and $x \in \mathbf{D}^l$, then*

$$\int_{-\infty}^{+\infty} (x, \alpha f_a + \beta g_a)\,da = \alpha \int_{-\infty}^{+\infty} (x, f_a)\,da + \beta \int_{-\infty}^{+\infty} (x, g_a)\,da\,.$$

Proof. According to Theorem 1.1 $\{\alpha f_a + \beta g_a\} \in \mathcal{F}^l$. Let x vanish on $(-\infty, b)$. If $\mu \in \mathcal{N}_b$, then from Definition 2.1 and Theorem 2.3 [1] it follows that $\int_{-\infty}^{+\infty}(x, \alpha f_a +$

$+ \beta g_a)\,da = \int_{-\infty}^{+\infty} (x, \alpha\mu(a) f_a + \beta\,\mu(a)\,g_a)\,da = \alpha \int_{-\infty}^{+\infty} (x, \mu(a) f_a)\,da + \beta \int_{-\infty}^{+\infty} (x, \mu(a)\,g_a)\,da = \alpha \int_{-\infty}^{+\infty} (x, f_a)\,da + \beta \int_{-\infty}^{+\infty} (x, g_a)\,da$, q.e.d.

Theorem 2.4. *If* $x, x_n \in \mathbf{D}^l$, $n = 1, 2, \ldots$, $x_n \rightrightarrows x$ *and* $\{f_a\} \in \mathscr{F}^l$, *then*

$$\int_{-\infty}^{+\infty} (x_n, f_a)\,da \rightrightarrows \int_{-\infty}^{+\infty} (x, f_a)\,da\ .$$

Proof. Let x_n, $n = 1, 2, \ldots$, vanish on $(-\infty, b)$. By Lemma 1.4 it follows that x vanishes on $(-\infty, b)$. If $\mu \in \mathscr{N}_b$, then Definition 2.1 and Theorem 2.4 [1] yield $y_n = \int_{-\infty}^{+\infty} (x_n, f_a)\,da = \int_{-\infty}^{+\infty} (x_n, \mu(a) f_a)\,da \to \int_{-\infty}^{+\infty} (x, \mu(a) f_a)\,da = \int_{-\infty}^{+\infty} (x, f_a)\,da = y$. By the proof of Theorem 2.1 there exists a real number b_0 such that y_n, $n = 1, 2, \ldots$, vanish on $(-\infty, b_0)$. Hence $y_n \rightrightarrows y$, which completes the proof.

Theorem 2.5. *If* $x \in \mathbf{D}^l$ *and* $\{f_a\} \in \mathscr{F}^l$, *then*

$$\int_{-\infty}^{+\infty} (x^{(n)}, f_a)\,da = (-1)^n \int_{-\infty}^{+\infty} \left(x, \frac{\partial^n f_a}{\partial a^n}\right) da\ .$$

Proof. By Theorem 1.1 we have $\{\partial^n f_a/\partial a^n\} \in \mathscr{F}^l$. Let x vanish on $(-\infty, b)$. If $\mu \in \mathscr{N}_b$, then there exists a real number c $(c < b)$ such that $\mu(a) = 1$ for every real number $a > c$. If $a > c$, then by Note to Lemma 1.6 [1] we have $\langle(\partial^n \mu(a) f_a)/\partial a^n, \varphi\rangle = [\mu(a)\,\psi(a)]^{(n)} = \mu(a)\,\psi^{(n)}(a) = \langle \mu(a)\,\partial^n f_a/\partial a^n, \varphi\rangle$ for every $\varphi \in \mathbf{K}$ where $\psi(a) = \langle f_a, \varphi\rangle$. Thus $(\partial^n \mu(a) f_a)/\partial a^n = \mu(a)\,\partial^n f_a/\partial a^n$ for all real numbers $a > c$. Using Definition 2.1, Theorem 2.5 [1] and Lemma 2.2 it follows that

$$\int_{-\infty}^{+\infty} (x^{(n)}, f_a)\,da = \int_{-\infty}^{+\infty} (x^{(n)}, \mu(a) f_a)\,da = (-1)^n \int_{-\infty}^{+\infty} \left(x, \frac{\partial^n \mu(a) f_a}{\partial a^n}\right) da =$$
$$= (-1)^n \int_{-\infty}^{+\infty} \left(x, \mu(a) \frac{\partial^n f_a}{\partial a^n}\right) da = (-1)^n \int_{-\infty}^{+\infty} \left(\frac{x, \partial^n f_a}{\partial a^n}\right) da\ ,$$

q.e.d.

Theorem 2.6. *If b is a real number,* $\{f_a\} \in \mathscr{F}^l$ *and* $x \in \mathbf{D}^l$, *then*

$$P_b[y] = \int_{-\infty}^{+\infty} (x, g_a)\,da\ ,$$

where $y = \int_{-\infty}^{+\infty} (x, f_a)\,da$ *and* $g_a = P_b[f_a]$.

Proof. By Theorem 1.1 we have $\{g_a\} \in \mathscr{F}^l$. Let x vanish on $(-\infty, c)$. If $\mu \in \mathscr{N}_c$, then $y = \int_{-\infty}^{+\infty} (x, \mu(a) f_a)\,da$ and $P_b[\mu(a) f_a] = \mu(a)\,P_b[f_a] = \mu(a)\,g_a$. From Definition 2.1 and Theorem 2.6 [1] it follows that $P_b[y] = \int_{-\infty}^{+\infty} (x, \mu(a)\,g_a)\,da = \int_{-\infty}^{+\infty} (x, g_a)\,da$.

Theorem 2.7. *If b is a real number, $\{f_a\} \in \mathscr{F}^l$ and $x \in \mathbf{D}^l$, then*

$$\int_{-\infty}^{+\infty} (x, f_a)\, \mathrm{d}a = \int_{-\infty}^{+\infty} (P_b[x], f_{a-b})\, \mathrm{d}a \cdot$$

Proof. According to Theorem 1.1 $\{g_a\} \in \mathscr{F}^l$ where $g_a = f_{a-b}$. Let x vanish on $(-\infty, c)$. If $\mu \in \mathscr{N}_c$, then from Definition 2.1 and Theorem 2.7 [1] it follows that $\int_{-\infty}^{+\infty} (x, f_a)\, \mathrm{d}a = \int_{-\infty}^{+\infty} (x, \mu(a) f_a)\, \mathrm{d}a = \int_{-\infty}^{+\infty} (P_b[x], \mu(a-b) f_{a-b})\, \mathrm{d}a = \int_{-\infty}^{+\infty} (P_b[x]\, \nu(a) f_{a-b})\, \mathrm{d}a = \int_{-\infty}^{+\infty} (P_b[x], f_{a-b})\, \mathrm{d}a$, where $\nu(t) = \mu(t-b)$ on $(-\infty, +\infty)$ and $\nu \in \mathscr{N}_{c+b}$.

Theorem 2.8. *Let $\{f_a\} \in \mathscr{F}^l$ and let f_a vanish on $(-\infty, a)$ for every real number a. If b is a real number, $x \in \mathbf{D}^l$ and x vanishes on $(-\infty, b)$, then $\int_{-\infty}^{+\infty} (x, f_a)\, \mathrm{d}a$ vanishes on $(-\infty, b)$.*

Proof. If $\mu \in \mathscr{N}_b$, then $\mu(a) f_a$ vanishes on $(-\infty, a)$ for every real number a (see Lemma 1.2). From Definition 2.1 and Theorem 2.8 [1] it follows that $\int_{-\infty}^{+\infty} (x, f_a)\, \mathrm{d}a = \int_{-\infty}^{+\infty} (x, \mu(a) f_a)\, \mathrm{d}a$ vanishes on $(-\infty, b)$, q.e.d.

Example 4. From Example 2 and from Example to Definition 2.1 [1] it follows that

$$x = \int_{-\infty}^{+\infty} (x, \delta_a)\, \mathrm{d}a$$

for every $x \in \mathbf{D}^l$.

3. LINEAR UNIFORMLY CONTINUOUS OPERATORS

Let $\mathbf{P}$ be a non-empty subset of the set $\mathbf{D}^l$. A mapping T of $\mathbf{P}$ into $\mathbf{D}^l$ will be called the *operator* on $\mathbf{P}$. The set $\mathbf{P}$ will be termed the *domain* of the operator T.

Definition 3.1. Let $\mathbf{P}$ be a non-empty subset of $\mathbf{D}^l$. An operator T on $\mathbf{P}$ will be called *uniformly continuous* if the following implication holds:
If $x, x_n \in \mathbf{P}$, $n = 1, 2, \ldots$, $x_n \rightrightarrows x$, then $T[x_n] \rightrightarrows T[x]$.

Example 5. Let $H = H_0$ be the Heaviside's distribution. If we put

$$T[x] = x^{(-1)} = H * x\,, \quad (x \in \mathbf{D}^l)$$

then the operator T is not continuous. Actually, we have $\delta_{-n} \to 0$ and, on the other hand, $T[\delta_{-n}] = H_{-n} \to 1 \neq 0 = T[0]$. However, the operator T is uniformly continuous (see p. 137 [3]).

Theorem 3.1. *Let T be a linear uniformly continuous operator on a linear subspace $\mathbf{P} \subset \mathbf{D}^l$. Let $\{f_a\} \in \mathscr{F}^l$ and $\partial^n f_a / \partial a^n \in \mathbf{P}$ for every $n = 0, 1, 2, \ldots$ and for every real number a. Then $\{g_a\} \in \mathscr{F}^l$ and $\partial^n g_a / \partial a^n = T[\partial^n f_a / \partial a^n]$ where $g_a = T[f_a]$.*

Proof. 1. Let $a_n \to a\,(a_n \neq a)$, $n = 1, 2, \ldots$ be a convergent sequence of real numbers. Then there exists a real number a_0 such that $a_0 \leqq a_n$ for every $n = 1, 2, \ldots$ By Definition 1.2 there exists a real number b_0 such that f_{a_n} vanishes on $(-\infty, b_0)$ for every $n = 1, 2, \ldots$. According to Lemma 1.5 [1] it follows that $(a_n - a)^{-1} \cdot (f_{a_n} - f_a) \rightrightarrows \partial f_a/\partial a$. Using the linearity and uniform continuity of the operator T, we obtain $(a_n - a)^{-1}(g_{a_n} - g_a) \rightrightarrows T\,[\partial f_a/\partial a]$ where $g_a = T[f_a]$. By Lemma 1.5 [1] there exists $\partial g_a/\partial a$ and $\partial g_a/\partial a = T[\partial f_a/\partial a]$. Similarly we obtain that there exists $\partial^n g_a/\partial a^n$ and $\partial^n g_a/\partial a^n = T[\partial^n f_a/\partial a^n]$ for every $n = 2, 3, \ldots$

2. Let a_n, α_n, $n = 1, 2, \ldots$ be two sequences of real numbers and let $a_n \to +\infty$. Then there exists a real number a_0 such that $a_0 \leqq a_n$ for every $n = 1, 2, \ldots$ By Definition 1.2 there exists a real number b_0 such that f_{a_n} vanishes on $(-\infty, b_0)$ for every $n = 1, 2, \ldots$ By Note to Definition 1.2, $\alpha_n f_{a_n} \to 0$ and thus it follows by Lemma 1.2 that $\alpha_n f_{a_n} \rightrightarrows 0$. Using the linearity and uniform continuity of the operator T we conclude that $\alpha_n g_{a_n} \rightrightarrows 0$. From Note to Definition 1.2 it follows that $\psi \in \mathbf{K}^p$ for every $\varphi \in \mathbf{K}$, where $\psi(a) = \langle g_a, \varphi \rangle$.

3. If $\{g_a\} \notin \mathscr{F}^l$, then by Definition 1.2 there exists a sequence a_n of real numbers such that $k \leqq a_n$ for every $n = 1, 2, \ldots$ and the statement "g_{a_n} vanishes on $(-\infty, -n)$" is not true for every $n = 1, 2, \ldots$

On the other hand $\{f_a\} \in \mathscr{F}^l$ and thus there exists a real number b such that f_{a_n} vanishes on $(-\infty, b)$ for every $n = 1, 2, \ldots$ If $\varphi \in \mathbf{K}$, then $\psi \in \mathbf{K}^p$ where $\psi(a) = \langle f_a, \varphi \rangle$.

Next, there clearly exists a subsequence b_n of a_n such that either $b_n \to a_0$ or $b_n \to +\infty$. If $\varphi \in \mathbf{K}$, then either $\langle f_{b_0}, \varphi \rangle = \psi(b_n) \to \psi(a_0) = \langle f_{a_0}, \varphi \rangle$ or $\langle f_{b_0}, \varphi \rangle = \psi(b_n) \to 0$. Thus $f_{b_n} \to f_{a0}$ or $f_{b_n} \to 0$. Since $f_{b_n} \rightrightarrows f_{a0}$ or $f_{b_n} \rightrightarrows 0$, by Definition 3.1 we have $g_{b_n} \rightrightarrows g_{a_0}$ or $g_{b_n} \rightrightarrows 0$. Hence g_{b_n} vanishes on some interval $(-\infty, c)$ for every $n = 1, 2, \ldots$ which is a contradiction. Consequently $\{g_a\} \in \mathscr{F}^l$, and the theorem is proved.

Theorem 3.2. *Let T be a linear uniformly continuous operator on a linear subspace $\mathbf{P} \subset \mathbf{D}^l$. Let $\{f_a\} \in \mathscr{F}^l$ and $\partial^n f_a/\partial a^n \in \mathbf{P}$ for every $n = 0, 1, 2, \ldots$ and for every real number a. If $x \in \mathbf{D}_0^l$ and $y \in \mathbf{P}$, where $y = \int_{-\infty}^{+\infty} x(a) f_a\,\mathrm{d}a$, then $T[y] = \int_{-\infty}^{+\infty} x(a)\, g_a\,\mathrm{d}a$, where $g_a = T[f_a]$.*

Proof. Let x vanish on some interval $(-\infty, b)$. From Theorem 3.1 it follows that $\{g_a\} \in \mathscr{F}^l$. By Lemma 1.7 [1] and Definition 1.2 there exist y and $u = \int_{-\infty}^{+\infty} x(a)\, g_a\,\mathrm{d}a$. Let $\mathscr{D}$ be an arbitrary subdivision of the interval $(-\infty, +\infty)$. Using the notation introduced in Definition 1.4 [1], we obtain for the integral sums, $\mathrm{s}_1(\mathscr{D}) = \sum_{i=1}^{m} (a_i - a_{i-1})\, x(\xi_i) f_{\xi_i}$, $\mathrm{s}_2(\mathscr{D}) = \sum_{i=1}^{m} (a_i - a_{i-1})\, x(\xi_i)\, g_{\xi_i}$. From the linearity of the operator T it follows that $T[\mathrm{s}_1(\mathscr{D})] = \mathrm{s}_2(\mathscr{D})$. By Definition 1.2 there exists a real number c

such that f_a vanishes on $(-\infty, c)$ for every $a \geqq b$. If $\xi_i \geqq b$, then f_{ξ_i} vanishes on $(-\infty, c)$. If $\xi_i < b$, then $x(\xi_i) = 0$. Hence, by Lemma 1.2, $s_1(\mathscr{D})$ vanishes on $(-\infty, c)$.

If $\mathscr{D}_n$ is an arbitrary zero sequence of subdivisions of the interval $(-\infty, +\infty)$, then $s_1(\mathscr{D}_n) \to y$ and $s_2(\mathscr{D}_n) \to u$. Since $s_1(\mathscr{D}_n) \rightrightarrows y$, we have by Definition 3.1 $s_2(\mathscr{D}_n) = T[s_1(\mathscr{D}_n)] \rightrightarrows T[y]$. Thus $u = T[y]$, q.e.d.

Theorem 3.3. *Let T be a linear uniformly continuous operator on a linear subspace $\mathbf{P} \subset \mathbf{D}^l$. Let $\{f_a\} \in \mathscr{F}^l$ and $\partial^n f_a / \partial a^n \in \mathbf{P}$ for every $n = 0, 1, 2, \ldots$ and for every real number a. If $x \in \mathbf{D}^l_*$ and $y \in \mathbf{P}$, where $y = \int_{-\infty}^{+\infty} (x, f_a)\, da$, then $T[y] = \int_{-\infty}^{+\infty} (x, g_a)\, da$, where $g_a = T[f_a]$.*

Proof. According to Theorem 3.1, $\{g_a\} \in \mathscr{F}^l$, and consequently, $\int_{-\infty}^{+\infty} (x, g_a)\, da$ exists. On the other hand, there is evidently a $z \in \mathbf{D}^l_0$ such that $z^{(n)} = x$. From this it follows that z vanishes on some interval $(-\infty, b)$. If $\mu \in \mathscr{N}_b$, then Theorem 2.5, Definition 2.1 and the proof of Theorem 2.1 [1] yield $y = \int_{-\infty}^{+\infty} (x, f_a)\, da = (-1)^n \int_{-\infty}^{+\infty} (z, (\partial^n f_a / \partial a^n))\, da = (-1)^n \int_{-\infty}^{+\infty} (z, \mu(a)(\partial^n f_a / \partial a^n))\, da = (-1)^n \int_{-\infty}^{+\infty} z(a) \,.\, \mu(a)(\partial^n f_a / \partial a^n)\, da$. However, according to Theorem 3.1, Theorem 3.2, proof of Theorem 2.1 [1], Definition 2.1 and Theorem 2.5 we have $T[y] = (-1)^n \int_{-\infty}^{+\infty} z(a) \,.\, \mu(a)(\partial^n g_a / \partial a^n))\, da = (-1)^n \int_{-\infty}^{+\infty} (z, \mu(a)(\partial^n g_a / \partial a^n))\, da = (-1)^n \int_{-\infty}^{+\infty} (z, (\partial^n g_a / \partial a^n)) \,.\, da = \int_{-\infty}^{+\infty} (x, g_a)\, da$, which completes the proof.

Note. Let $\{f_a\} \in \mathscr{F}^l$ and let $\delta_a \in \mathbf{P} \subset \mathbf{D}^l$ for every real number a. If the operator T on $\mathbf{P}$ has the form

$$T[x] = \int_{-\infty}^{+\infty} (x, f_a)\, da\,, \quad (x \in \mathbf{P})$$

then from Example 3 it follows that $f_a = T[\delta_a]$.

Theorem 3.4. *Let $\mathbf{P} \subset \mathbf{D}^l_*$ be a linear subspace and let $\delta^{(n)}_a \in \mathbf{P}$ for every $n = 0, 1, 2, \ldots$ and for every real number a. Then the operator T on $\mathbf{P}$ is linear and uniformly continuous if and only if it has the form*

$$T[x] = \int_{-\infty}^{+\infty} (x, f_a)\, da\,, \quad (x \in \mathbf{P})$$

and $f_a = T[\delta_a]$.

Proof follows from Theorem 2.2, Theorem 2.4, Theorem 3.3 and Example 4.

Theorem 3.5. *Let $\mathbf{P}$ $(\mathbf{D}^l_* \subset \mathbf{P} \subset \mathbf{D}^l)$ be a linear space. Then the operator T on $\mathbf{P}$ is linear and uniformly continuous if and only if it has the form*

$$T[x] = \int_{-\infty}^{+\infty} (x, f_a)\, da\,, \quad (x \in \mathbf{P})$$

and $f_a = T[\delta_a]$.

Proof. The proof is analogous to that of Theorem 3.5 [1], and follows from Theorem 2.2, Theorem 2.4, Theorem 3.4 and Lemma 1.5.

Corollary. *If T_1, T_2 are two linear uniformly continuous operators on $\mathbf{D}^l$ and $T_1[\delta_a] = T_2[\delta_a]$ for every real number a, then $T_1[x] = T_2[x]$ for every $x \in \mathbf{D}^l$.*

Note. If $f \in \mathbf{D}^l$, then $\{f_a\} \in \mathscr{F}^l$ where $f_a = P_a[f]$ for every real number a. Actually, the operator

$$T[x] = f * x\,, \quad (x \in \mathbf{D}^l)$$

is linear and uniformly continuous (see p. 137 [3]). From Theorem 3.5 it follows that

$$T[x] = \int_{-\infty}^{+\infty} (x, f_a)\, \mathrm{d}a\,, \quad (x \in \mathbf{D}^l)$$

where $\{f_a\} \in \mathscr{F}^l$. However $f_a = T[\delta_a] = f * \delta_a = P_a[f]$.

Theorem 3.6. *The operator T on $\mathbf{D}^l$ is linear, uniformly continuous and time-invariant if and only if it has the form*

$$T[x] = \int_{-\infty}^{+\infty} (x, f_a)\, \mathrm{d}a\,, \quad (x \in \mathbf{D}^l)$$

where $f_a = P_a[f]$ and $f = T[\delta]$.

Proof. Let T be a linear, uniformly continuous and time-invariant operator. From Theorem 3.5 it follows that $T[x] = \int_{-\infty}^{+\infty} (x, f_a)\, \mathrm{d}a\ (x \in \mathbf{D}^l)$ where $\{f_a\} \in \mathscr{F}^l$. Also, $f_a = T[\delta_a] = T[P_a[\delta]] = P_a[T[\delta]] = P_a[f]$.

If the operator T has the form $T[x] = \int_{-\infty}^{+\infty} (x, f_a)\, \mathrm{d}a\ (x \in \mathbf{D}^l)$, where $f_a = P_a[f]$, then by Note to Theorem 3.5, $\{f_a\} \in \mathscr{F}^l$. According to Theorem 3.5 T is linear and uniformly continuous. From Theorem 2.6 and Theorem 2.7 it follows that $T[P_b[x]] = \int_{-\infty}^{+\infty} (P_b[x], f_a)\, \mathrm{d}a = \int_{-\infty}^{+\infty} (x, f_{a+b})\, \mathrm{d}a = \int_{-\infty}^{+\infty} (x, P_b[f_a])\, \mathrm{d}a = P_b[T[x]]$ for every $x \in \mathbf{D}^l$ and for every real number b. Thus the operator T is time-invariant.

Note. If $f_a = P_a[f]$ for some $f \in \mathbf{D}^l$, then from Note to Theorem 3.5 it follows that

$$\int_{-\infty}^{+\infty} (x, f_a)\, \mathrm{d}a = f * x\,, \quad (x \in \mathbf{D}^l)\,.$$

Theorem 3.7. *The operator T on $\mathbf{D}^l$ is linear, uniformly continuous and causal if and only if it has the form*

$$T[x] = \int_{-\infty}^{+\infty} (x, f_a)\, \mathrm{d}a\,, \quad (x \in \mathbf{D}^l)$$

where $\{f_a\} \in \mathscr{F}^l$ and f_a vanishes on $(-\infty, a)$ for every real number a.

Proof. Let T be a linear, uniformly continuous and causal operator. From Theorem 3.5 it follows that $T[x] = \int_{-\infty}^{+\infty} (x, f_a) \, da$ $(x \in \mathbf{D}^l)$, where $\{f_a\} \in \mathscr{F}^l$ and $f_a = T[\delta_a]$. Since δ_a vanishes on $(-\infty, a)$, f_a vanishes on $(-\infty, a)$ for every real number a.

Let the operator T have the form $T[x] = \int_{-\infty}^{+\infty} (x, f_a) \, da$ $(x \in \mathbf{D}^l)$ where $\{f_a\} \in \mathscr{F}^l$ and f_a vanishes on $(-\infty, a)$ for every real number a. From Theorem 3.5 it follows that T is linear and uniformly continuous. Finally, from Theorem 2.8 and Lemma 3.1 [1] it follows that the operator T is causal.

Theorem 3.8. *The operator T on $\mathbf{D}^l$ is linear, uniformly continuous, time-invariant and causal if and only if it has the form*

$$T[x] = \int_{-\infty}^{+\infty} (x, f_a) \, da \,, \quad (x \in \mathbf{D}^l)$$

where $f_a = P_a[f]$ and $f = T[\delta]$ vanishes on the interval $(-\infty, 0)$.

The proof follows from Theorem 3.6 and Theorem 3.7.

References

[1] *Ponděliček B.:* Contribution to the Foundations of Network Theory Using the Distribution Theory, Czech. Math. J., *19* (*94*) 1969, 697—710.

[2] *Doležal V.:* Dynamics of linear systems, Prague 1964.

[3] *Гельфанд И.И. и Шилов Г. Е.:* Обобщенные функции и действия над ними, Москва 1959.

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