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ASYMPTOTIC INTEGRATION OF A NONHOMOGENEOUS DIFFERENTIAL EQUATION WITH INTEGRABLE COEFFICIENTS

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1. INTRODUCTION

In this article the asymptotic behavior of the solutions of the nth order nonhomogeneous linear system of differential equations

$$x^{(n)} = A_0(t)\,x + h(t) \tag{1}$$

where $n \geqq 2$ will be determined by exhibiting the asymptotic expansions of the solutions. In equation (1), x denotes a d-dimensional vector; $A_0 = A_0(t)$ is a $d \times d$ complex-valued matrix which is continuous on the interval $J = [t_0, \infty)$; and $h = h(t)$ is a continuous complex-valued d-vector defined on J.

A basic assumption which will be made concerning A_0 is that the integral

$$\int^{\infty} A_0(s)\,\mathrm{d}s = \lim_{T \to \infty} \int^{T} A_0(s)\,\mathrm{d}s$$

converges. The fact that this integral can be conditionally convergent indicates the direction of the new results of the paper. Under this hypothesis the function

$$A_1(t) = \int_t^{\infty} A_0(s)\,\mathrm{d}s$$

is well defined; in general, assuming that the following integrals exist, we define A_j inductively as

$$A_j(t) = \int_t^{\infty} A_{j-1}(s)\,\mathrm{d}s\,, \quad j = 1, 2, \ldots, n\,.$$

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Section 2 develops the case where the asymptotic expansions of the solutions of (1) are polynomials of order less than n. To obtain this conclusion, hypotheses are imposed in order that the forcing function has a negligible effect upon the solutions of (1). More precisely, up to terms of $o(1)$ as $t \to \infty$, the behavior of (1) is completely determined by the solutions of the associated homogeneous system. In Section 3, the above situation is reversed with a particular solution of (1) now completely determining the composition of the asymptotic expansion of the solutions of (1).

For $n > 2$ these results appear to be new even for homogeneous scalar differential equations. A related result for second order homogeneous equations which is due to HARTMAN and WINTNER may be found in HARTMAN [6, p. 382]. The main techniques used to establish the results obtained here are extensions of those used in Theorem 1 of [4]. In fact, when Theorem 1 below is specialized to second order homogeneous equations then its hypothesis reduces to that of Theorem 1 of [4]. For linear differential equations, results which possess conclusions similar to those obtained here may be found in COPPEL [1, p. 92], HARTMAN [6, p. 380], and HILLE [7, p. 428]. Related results which are valid for nonlinear differential equations may be found in [2], [3], and [5]. An example will be given to illustrate that our hypotheses are different from those required in the above references.

2. ALMOST HOMOGENEOUS EQUATIONS

Our first result reveals an asymptotic polynomial development for the solutions of (1) whenever the forcing function $h = h(t)$ satisfies

$$\int^{\infty} t^{n-1} \|h(t)\| \, dt < \infty \; ; \tag{2}$$

here, and for the remainder of the article, $\|.\|$ designates some convenient norm.

Theorem 1. *Let condition* (2) *and*

$$\int^{\infty} t^{n+k-1} \|A_n(t)\, A_0(t)\| \, dt < \infty \; ; \quad \int^{\infty} t^{j+k-1} \|A_{n-j}(t)\| \, dt < \infty \, , \tag{3}$$

$$j = 1, 2, \ldots, n-1 \, ,$$

be satisfied for some k, $k = 0, 1, \ldots, n-1$. Then, corresponding to any set of $k+1$ constant d-vectors $d_0, d_1, \ldots, d_k$, there exists a unique solution of (1) *which possesses the asymptotic expansion*

$$x(t) \sim \sum_{i=0}^{k} d_j t^i \, , \quad t \to \infty \, . \tag{4}$$

Proof. The Banach space $\mathfrak{F}_k$ of all continuous t^k-bounded functions defined on J (with norm of $x \in \mathfrak{F}_k$ given by $\|x\|_k = \sup_{t \in J} \|t^{-k} x(t)\|$) will be used in our arguments. For $\varrho_k = 2(\sum_{i=0}^{k} \|d_i\| + 1)$, consider the subset $\mathfrak{F}_{k,\varrho_k}$ of $\mathfrak{F}_k$ which consists of all $x \in \mathfrak{F}_k$ such that $\|x\|_k \leqq \varrho_k$.

The operator $\mathscr{I}$ which is defined on $\mathfrak{F}_{k,\varrho_k}$ by

$$\text{(5)} \qquad \mathscr{I}x(t) = \sum_{i=0}^{k} d_i t^i - A_n(t)\, x(t) + \\ + \sum_{j=1}^{n-1} (-1)^{j+1} \binom{n}{j} \int_t^\infty \frac{(s-t)^{j-1}}{(j-1)!} A_{n-j}(s)\, x(s)\, ds + \\ + (-1)^n \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} \{A_n(s)\, A_0(s)\, x(s) + [I_d + A_n(s)]\, h(s)\}\, ds\,,$$

(where I_d denotes the $d \times d$ identity matrix) maps $\mathfrak{F}_{k,\varrho_k}$ into itself. To establish this fact, we note that the following inequality can be obtained from (5):

$$\text{(6)} \qquad \|t^{-k} \mathscr{I}x(t)\| \leqq \varrho_{k/2} - 1 + \varrho_k\, \gamma_k(t) + \beta t^{-k} \int_t^\infty \frac{s^{n-1}}{(n-1)!} \|h(s)\|\, ds\,;$$

where

$$\beta = \sup_{t \in J} \|I_d + A_n(t)\|\,;$$

and

$$\gamma_k(t) = \|A_n(t)\| + \sum_{j=1}^{n-1} \binom{n}{j} t^{-k} \int_t^\infty \frac{s^{j+k-1}}{(j-1)!} \|A_{n-j}(s)\|\, ds + \\ + t^{-k} \int_t^\infty \frac{s^{n+k-1}}{(n-1)!} \|A_n(s)\, A_0(s)\|\, ds\,.$$

By virtue of (3), we can assume that $\gamma_k(t) < \frac{1}{2}$ for $t \in J$. As a consequence of (2), we can also assume that t_0 is sufficiently large so that

$$\beta t^{-k} \int_t^\infty \frac{s^{n-1}}{(n-1)!} \|h(s)\|\, ds < 1\,, \quad t \in J\,.$$

The combination of these comments into (6) shows that $\|\mathscr{I}x\|_k \leqq \varrho_k$; hence, $\mathscr{I}\mathfrak{F}_{k,\varrho_k} \subset \mathfrak{F}_{k,\varrho_k}$.

Let x_1 and x_2 be in $\mathfrak{F}_{k,\varrho_k}$; then, from (5), it follows that

$$\|t^{-k}[\mathscr{I}x_1(t) - \mathscr{I}x_2(t)]\| \leqq \gamma_k(t)\, \|x_1 - x_2\|_k \leqq \tfrac{1}{2}\|x_1 - x_2\|_k\,, \quad t \in J\,.$$

Thus, $\mathscr{J}$ is a contraction map. By the Contraction Principle, there is an $x \in \mathfrak{F}_{k,\varrho_k}$ such that $\mathscr{J}x = x$; that is,

$$x(t) = \sum_{i=0}^{k} d_i t^i - A_n(t)\, x(t) + \tag{7}$$

$$+ \sum_{j=1}^{n-1} (-1)^{j+1} \binom{n}{j} \int_t^\infty \frac{(s-t)^{j-1}}{(j-1)!} A_{n-j}(s)\, x(s)\, ds +$$

$$+ (-1)^n \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} \{A_n(s)\, A_0(s)\, x(s) + [I_d + A_n(s)]\, h(s)\}\, ds\,.$$

It remains to establish that x, as defined by (7), is a solution of (1) which possesses the asymptotic expansion (4). The verification of (4) follows directly from the inequality

$$\left\| x(t) - \sum_{i=0}^{k} d_i t^i \right\| \leq \varrho_k t^k \gamma_k(t) + \beta \int_t^\infty s^{n-1} \|h(s)\|\, ds\,. \tag{8}$$

The function γ_k satisfies $t^k \gamma_k(t) = o(1)$ as $t \to \infty$ by virtue of (3). The condition (2) applied to the inequality (8) now implies that the expansion (4) is valid.

To complete the proof, we will show that differentiation in (7) leads to

$$[I_d + A_n(t)]\, x^{(n)}(t) = [I_d + A_n(t)]\, [A_0(t)\, x(t) + h(t)]\,. \tag{9}$$

The factor $[I_d + A_n(t)]$ is invertible for large t because the definition of A_n implies that $A_n(t) = o(1)$ as $t \to \infty$. This fact used in (9) will then establish that x is a solution of (1).

To derive (9), we note the following identity which may be obtained from Leibnitz's rule for derivative of a product:

$$\{[I_d + A_n(t)]\, x(t)\}^{(n)} = [I_d + A_n(t)]\, x^{(n)}(t) + \tag{10}$$

$$+ \sum_{j=1}^{n} (-1)^j \binom{n}{j} A_{n-j}(t)\, x^{(n-j)}(t)\,.$$

We will also need the identity

$$\left[\int_t^\infty \frac{(s-t)^{j-1}}{(j-1)!} A_{n-j}(s)\, x(s)\, ds\right]^{(n)} = (-1)^j \left[A_{n-j}(t)\, x(t)\right]^{(n-j)} = \tag{11}$$

$$= \sum_{k=0}^{n-j} (-1)^{k+j} \binom{n-j}{k} A_{n-k-j}(t)\, x^{(n-k-j)}(t)\,, \quad j = 1, 2, \ldots, n-1\,.$$

An n-fold differentiation in (7) and then applying the results in (10) and (11) yields the equation

$$(12)\qquad [I_d + A_n(t)]\, x^{(n)}(t) = \sum_{j=1}^{n} (-1)^{j+1} \binom{n}{j} A_{n-j}(t)\, x^{(n-j)}(t) + $$
$$+ \sum_{j=1}^{n-1} \sum_{k=0}^{n-j} (-1)^{k+1} \binom{n}{j} \binom{n-j}{k} A_{n-k-j}(t)\, x^{(n-k-j)}(t) + $$
$$+ A_n(t)\, A_0(t)\, x(t) + [I_d + A_n(t)]\, h(t)\,.$$

The second group of terms on the right of (12) can be written as

$$(13)\qquad \sum_{j=1}^{n} (-1)^j \binom{n}{j} \sum_{k=1}^{j} (-1)^{k+1} \binom{j}{k} A_{n-j}(t)\, x^{(n-j)}(t) + A_0(t)\, x(t) = $$
$$= \sum_{j=1}^{n} (-1)^j \binom{n}{j} A_{n-j}(t)\, x^{(n-j)}(t) + A_0(t)\, x(t)\,.$$

In the last step in (13) we have used the fact that the binomial expansion for $[1 + (-1)]^j \equiv 0$ is

$$\sum_{k=0}^{j} (-1)^{k+1} \binom{j}{k}, \quad j = 1, 2, \ldots, n\,.$$

The insertion of (13) into (12) yields (9) which, as previously noted, verifies that x is a solution of (1) which satisfies (4). This completes the proof of the theorem.

The above theorem is consistent with known results which are formulated with other sets of hypotheses; in particular, stronger integrability conditions are needed to obtain a higher order asymptotic expansion of a solution than are necessary to obtain an asymptotic representation of the solution (see [6, p. 429]).

In the known results for linear equations, the hypotheses which imply that an asymptotic representation is valid for a solution of (1) also implies that there exists a representation for the remaining solutions. It does not appear that the integration by parts technique used here gives such a representation for the nonconvergent solutions of (1).

The techniques used here are readily extendable to include (in (1)) a nonlinear perturbation term of the form $f(t, x)$ which satisfies a Lipschitz condition in x; see [4] for a second order analogue.

Next, an example is given to illustrate that Theorem 1 may be applicable in certain cases where known results do not apply. A result of Ghizzetti (see [1, p. 92]) shows that if $\int^\infty t^{n-1} \|A_0(t)\|\, dt < \infty$ then the homogeneous equation (2) has a solution $x = x(t)$ with $\lim_{t\to\infty} x(t) = 1$. If $A_0(t) = e^{it^p} t^{1-n}$ with $p > n - 1$ then the result of Ghizzetti is not applicable since $\int^\infty t^{n-1} \|A_0(t)\|\, dt = \infty$. However, for this equation, the hypotheses of Theorem 1 are satisfied for this function A_0; hence, there exists a solution $x = x(t)$ of the homogeneous equation (1) such that $\lim_{t\to\infty} x(t) = 1$. A more detailed discussion on the comparison of known results for second order equations and the second order homogeneous analogue of Theorem 1 may be found in [4].

3. DOMINANT PARTICULAR SOLUTIONS

The case where a particular solution of (1) is dominant at infinity will now be discussed. Because the forcing function h is a vector it is convenient to have a positive continuous scalar valued function $\psi = \psi(t)$ which is an exact measurement for large t of the dominant part of certain integrals of h. That is, there exists a nonzero constant vector h_0 with the property that

$$(14) \qquad \int_{t_0}^{t} \frac{(t-s)^{n-1}}{(n-1)!}\, h(s)\, ds \sim h_0\, \psi(t)\,, \quad t \to \infty\,.$$

In order that a particular solution of (1) be dominant at infinity, it will be required that

$$(15) \qquad \psi^{-1}(t)\, t^{n-1} = o(1)\,, \quad t \to \infty\,.$$

In this section stronger integrability conditions on the coefficient in (1) are required than those used in the previous theorem.

Theorem 2. *Let the scalar function $\psi = \psi(t)$ be positive and continuous on J and satisfy conditions* (14) *and* (15). *If*

$$(16) \qquad \int^{\infty} t^{j-1} \|A_{n-j}(t)\|\, \psi(t)\, dt < \infty\,, \quad j = 1, 2, \ldots, n-1\,;$$

$$\int^{\infty} t^{n-1} \|A_n(t)\, A_0(t)\|\, \psi(t)\, dt < \infty\,;$$

holds then every solution $x = x(t)$ of (1) *has the asymptotic representation*

$$(17) \qquad x(t) \sim h_0\, \psi(t)\,, \quad t \to \infty\,.$$

Proof. The proof again uses the contraction principle to establish the existence of a solution of an appropriate integral equation. Let d_i, $i = 0, 1, \ldots, n-1$, be n given constant vectors; then set

$$\beta_1 = \sum_{i=0}^{n-1} \|d_i\|\,, \quad \beta = \sup_{t\in J} \|I_d + A_n(t)\|\,.$$

Define $\varrho = 2[\beta_1 + \beta\|h_0\| + \beta]$ and consider the set $\mathfrak{F}_{\psi,\varrho}$ of all continuous ψ-bounded functions, $x = x(t)$, defined on J, with $\|x\|_\psi = \sup_{t\in J} \|\psi^{-1}(t)\, x(t)\| \leqq \varrho$.

Let $\mathscr{J}$ be defined on $\mathfrak{F}_{\psi,\varrho}$ by the equation

$$\mathscr{J}x(t) = \sum_{i=0}^{n-1} d_i t^i - A_n(x)\, x(t) +$$
$$+ \sum_{j=1}^{n-1} (-1)^{j+1} \binom{n}{j} \int_t^\infty \frac{(s-t)^{j-1}}{(j-1)!} A_{n-j}(s)\, x(s)\, ds +$$
$$+ (-1)^n \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} A_n(s)\, A_0(s)\, x(s)\, ds +$$
$$+ \int_{t_0}^t \frac{(t-s)^{n-1}}{(n-1)!} [I_d + A_n(s)]\, h(s)\, ds\,.$$

From this definition it follows that

$$\|\psi^{-1}(t)\, \mathscr{J}x(t)\| \leqq \beta_1 t^{n-1} \psi^{-1}(t) + \varrho\, \gamma_\psi(t) + \tag{18}$$
$$+ \beta \left\| \psi^{-1}(t) \int_{t_0}^t \frac{(t-s)^{n-1}}{(n-1)!} h(s)\, ds \right\|$$

where

$$\gamma_\psi(t) = \|A_n(t)\| + \psi^{-1}(t) \sum_{j=1}^{n-1} \binom{n}{j} \int_t^\infty \frac{s^{j-1}}{(j-1)!} \|A_{n-j}(s)\|\, \psi(s)\, ds +$$
$$+ \psi^{-1}(t) \int_t^\infty \frac{s^{n-1}}{(n-1)!} \|A_n(s)\, A_0(s)\|\, \psi(s)\, ds\,.$$

In view of (14), the last expression on the right in (18) satisfies the inequality

$$\left\| \psi^{-1}(t) \int_{t_0}^t \frac{(t-s)^{n-1}}{(n-1)!} h(s)\, ds \right\| \leqq \|h_0\| + 1 \tag{19}$$

provided t_0 is sufficiently large. It follows from (15) that $\psi^{-1}(t) = o(1)$ as $t \to \infty$. And, by definition $A_n(t) = o(1)$ as $t \to \infty$; hence, from (16), we can assume that t_0 is sufficiently large so that

$$\|\psi^{-1}(t)\, \mathscr{J}x(t)\| \leqq \beta_1 + \varrho/2 + \beta\|h_0\| + \beta \leqq \varrho\,, \quad t \in J\,.$$

This inequality establishes the inclusion $\mathscr{J}\mathfrak{F}_{\psi,\varrho} \subset \mathfrak{F}_{\psi,\varrho}$.

Next, it will be shown that $\mathscr{J}$ is a contraction map on $\mathfrak{F}_{\psi,\varrho}$. If x_1 and x_2 are in $\mathfrak{F}_{\psi,\varrho}$ then

$$\|\psi^{-1}(t)\, [\mathscr{J}\, x_1(t) - \mathscr{J}\, x_2(t)]\| \leqq \gamma_\psi(t)\, \|x_1 - x_2\|_\psi \leqq \tfrac{1}{2}\|x_1 - x_2\|_\psi\,, \quad t \in J\,.$$

The Contraction Principle implies that there exists a unique $x \in \mathfrak{F}_{\psi,\varrho}$ with the property that $\mathscr{J}x = x$.

That is,

$$(20)\qquad x(t) = \sum_{i=0}^{n-1} d_{n-i-1} t^{n-i-1} - A_n(t)\, x(t) +$$

$$+ \sum_{j=1}^{n-1} (-1)^{j+1} \binom{n}{j} \int_t^\infty \frac{(s-t)^{j-1}}{(j-1)!} A_{n-j}(s)\, x(s)\, ds +$$

$$+ (-1)^n \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} A_n(s)\, A_0(s)\, x(s)\, ds +$$

$$+ \int_{t_0}^t \frac{(t-s)^{n-1}}{(n-1)!} [I_d + A_n(s)]\, h(s)\, ds\,.$$

The function x defined by (20) can be shown to be a solution of (1) by using the procedure indicated in the proof of Theorem 1. In fact, the only difference occurs in the contribution of the forcing term where the observation

$$\left\{ \int_{t_0}^t \frac{(t-s)^{n-1}}{(n-1)!} [I_d + A_n(s)]\, h(s)\, ds \right\}^{(n)} = [I_d + A_n(t)]\, h(t)$$

is evident.

The remainder of the proof will be to verify that $x = x(t)$ as defined by (20) has the desired behavior (17). By using (15) and (16) in (20) we obtain

$$(21)\qquad \|\psi^{-1}(t)[x(t) - h_0\, \psi(t)]\| \leqq \psi^{-1}(t) \left\| \int_{t_0}^t \frac{(t-s)^{n-1}}{(n-1)!} h(s)\, ds - h_0\, \psi(t) \right\| +$$

$$+ \psi^{-1}(t) \int_{t_0}^t \frac{(t-s)^{n-1}}{(n-1)!} \|A_n(s)\, h(s)\|\, ds + o(1)\,, \quad t \to \infty\,.$$

The representation (14) implies that the first term on the right side of (21) is $o(1)$ as $t - \infty$. Since $A_n(t) = o(1)$ as $t \to \infty$ it follows that the second term on the right in (21) is also $o(1)$. This demonstrates that x has the asymptotic representation (17) and completes the proof of the theorem.

If additional information is known about both the coefficient function A_0 in (1) and the asymptotic expansion of the forcing function then additional information may often be obtained about the solutions of (1). We will restrict our considerations to scalar equations ($d = 1$) and develop the expansions of the solutions for a case where the forcing function has an asymptotic power series expansions.

Theorem 3. *Let the hypotheses of Theorem 2 be satisfied with* $\psi(t) = t^{m+n}$; *in addition, suppose that*

$$(22)\qquad t^k A_n(t) = o(1)\,, \quad t \to \infty\,,$$

for some positive integer k. If h(t) has an asymptotic expansion given by

$$(23) \qquad t^{-m}\, h(t) = \sum_{i=0}^{N} b_i t^{-i} + R_N(t)$$

where $R_N(t) = o(t^{-N})$ *as* $t \to \infty$, $m > k - 1$, $N \geqq k$, *then any solution* $x = x(t)$ *of* (1) *possesses the expansion*

$$(24) \qquad x(t) \sim \sum_{i=0}^{k} a_i t^{m+n-i}, \quad t \to \infty,$$

with

$$(25) \qquad a_i = \frac{b_i}{(m + 1 - i)(m + 2 - i)\dots(m + n - i)}, \quad i = 0, 1, \dots, k.$$

Proof. The proof is inductive in nature. Theorem 2 implies that any solution x of (1) has the representation

$$(26) \qquad x(t) \sim \int_{t_0}^{t} \frac{(t - s)^{n-1}}{(n - 1)!} h(s)\, ds, \quad t \to \infty.$$

The use of the power series expansion (23) in (26) leads to

$$x(t) \sim \frac{b_0 t^{m+n}}{(m + 1)(m + 2)\dots(m + n)} = a_0 t^{m+n}, \quad t \to \infty.$$

This demonstrates that (24) is valid to one term; assume that (24) is valid to ϱ terms where $1 \leqq \varrho \leqq k$. From (20), we have

$$(26) \qquad \left[x(t) - \sum_{i=0}^{\varrho-1} a_i t^{m+n-i}\right] / t^{m+n-\varrho} = \sum_{i=0}^{n-1} d_{n-i-1} t^{\varrho-i-m-1} - A_n(t)\, x(t)\, t^{\varrho-m-n} +$$
$$+ \sum_{j=1}^{n-1} (-1)^{j+1} \binom{n}{j} t^{\varrho-m-n} \int_t^\infty \frac{(s - t)^{j-1}}{(j - 1)!} A_{n-j}(s)\, x(s)\, ds +$$
$$+ (-1)^n\, t^{\varrho-m-n} \int_t^\infty \frac{(s - t)^{n-1}}{(n - 1)!} A_n(s)\, A_0(s)\, x(s)\, ds +$$
$$+ t^{\varrho-m-n} \int_{t_0}^{t} \frac{(t - s)^{n-1}}{(n - 1)!} [1 + A_n(s)] \left[\sum_{i=0}^{N} b_i s^{m-i} + R_N(s)\right] ds - \sum_{i=0}^{\varrho-1} a_i t^{\varrho-i}.$$

We will now examine each group of terms on the right side of (26) individually to determine its asymptotic behavior. First, $m > k - 1 \geqq \varrho - 1$ implies $t^{\varrho-i-m-1} = o(1)$, $t \to \infty$, $i = 0, 1, \dots, n - 1$. There exists a constant $\eta > 0$ such that

$$\|A_n(t)\, x(t)\, t^{\varrho-m-n}\| \leqq \eta \|A_n(t)\|\, t^\varrho \leqq \eta \|A_n(t)\|\, t^k = o(1), \quad t \to \infty.$$

Similarly, from (16) with $\psi(t) = t^{m+n}$, it follows that

$$t^{\varrho-m-n} \int_t^\infty (s - t)^{j-1} A_{n-j}(s)\, x(s)\, ds = o(1), \quad t \to \infty;$$

and

$$t^{\varrho-m-n}\int_t^\infty (s-t)^{n-1} A_n(s)\, A_0(s)\, x(s)\, ds = o(1)\,, \quad t \to \infty\,.$$

The behavior of the terms in (26) which are generated by the forcing function h has been essentially determined previously in [2]. Using Lemmas 1 and 2 of [2], we can write

$$t^{\varrho-m-n}\int_{t_0}^t \frac{(t-s)^{n-1}}{(n-1)!}\, h(s)\, ds = \tag{27}$$

$$= \sum_{i=0}^{N}\left[\sum_{r=0}^{n-1}\binom{n-1}{r}\frac{(-1)^r}{(n-1)!\,(r+m+1-i)}\right] b_i t^{\varrho-i} +$$

$$+ \sum_{r=0}^{n-1}\binom{n-1}{r}\frac{(-1)^r}{(n-1)!}\left\{C_r(t_0)\, t^{\varrho-m-r-1} + t^{\varrho-m-r-1}\int_{t_0}^t s^{r+m} R_N(s)\, ds\right\}$$

where $C_r(t_0)$ designates a constant function (with respect to t), $r = 0, 1, \ldots, n-1$. An application of L'Hospital's rule verifies the relationship

$$t^{\varrho-m-r-1}\int_{t_0}^t s^{r+m} R_N(s)\, ds = o(1)\,, \quad t \to \infty\,.$$

Noting that $\varrho \leqq k$ and $m > k - 1$, we use Lemma 2 of [2] to see that (27) can be written as

$$t^{\varrho-m-n}\int_{t_0}^t \frac{(t-s)^{n-1}}{(n-1)!}\, h(s)\, ds = \sum_{i=0}^{\varrho} a_i t^{\varrho-i} + o(1)\,, \quad t \to \infty\,,$$

where the a_i are given by (25). The remaining term in (26),

$$t^{\varrho-m-n}\int_{t_0}^t (t-s)^{n-1} A_n(s)\, h(s)\, ds$$

can be shown to be $o(1)$ as $t \to \infty$ by using L'Hospital's rule and (22).

The above asymptotic relations when used in (26) establish that the expansion (24) is valid to $\varrho + 1$ terms. This completes the induction and proves the theorem.

4. CONCLUDING REMARKS

A deeper analysis of the terms in the integral equation (20) leads to additional results which allow for an interplay between a particular solution and the complementary solutions of (1). The type of conclusions which may be obtained are similar to those in [3].

The problem of securing analogous results to those obtained here for differential equations which contain terms involving intermediate order derivatives can be solved in certain instances by using the integration by parts technique developed in this article. As an illustration of this remark, consider the second order homogeneous differential equation

$$x'' = B_0(t)\, x' + A_0(t)\, x\,, \quad t \in J\,.$$

If we assume that B_0 is differentiable and use the notation

$$B_1(t) = \int_t^\infty B_0(s)\, \mathrm{d}s\,,$$

then an analogue of Theorem 1 may be obtained. The integral equation which is used to find a convergent solution is

$$(28) \qquad x(t) = d_0 - A_2(t)\, x(t) + 2\int_t^\infty A_1(s)\, x(s)\, \mathrm{d}s - \int_t^\infty B_0(s)\, x(s)\, \mathrm{d}s -$$
$$- \int_t^\infty A_2(s)\, B_0(s)\, x(s)\, \mathrm{d}s - \int_t^\infty (s - t)\, B_0'(s)\, x(s)\, \mathrm{d}s + \int_t^\infty (s - t)\, A_2(s)\, A_0(s)\, x(s)\, \mathrm{d}s +$$
$$+ \int_t^\infty (s - t)\left[A_1(s)\, B_0(s) - A_2(s)\, B_0'(s)\right] x(s)\, \mathrm{d}s\,.$$

It appears that the condition $B_0 \in L^1(J)$ is required (among other integrability requirements) in order to prove the existence of a solution of (28). It would be nice to know if there are results available where both coefficients have appropriate conditionally convergent integrals.

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