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DAMPED WAVE EQUATIONS AND THE HEAT EQUATION*)

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1. INTRODUCTION

It has been pointed out, first by CATTANEO [1, 2] and then by VERNOTTE [7] that the heat equation

$$(1.1) \qquad u_t = u_{xx}$$

carries the physically unrealistic implication of instantaneous propagation of thermal disturbances, though the impact of the disturbance attenuates greatly with distance. Basing their arguments on thermodynamics and on statistical mechanics, they propose the inclusion of an inertial term with a small coefficient:

$$(1.2) \qquad \varepsilon^2 u_{tt} + u_t = u_{xx} .$$

This immediately raises questions concerning the manner and degree to which solutions of these two equations approximate each other. In particular the initial value problems for these two equations are of a rather different character, both requiring knowledge of

$$u(x, 0) = f(x) ,$$

and (1.2) in addition requiring

$$u_t(x, 0) = g(x) ,$$

so that the question of the behavior of the solution of the initial value problem for (1.2) as $\varepsilon \to 0$ falls in the category of singular perturbation problems.

One can argue physically that the solutions of these initial value problems should not differ greatly when ε is small. For (1.2) can be interpreted as describing the vibrations of a string of small linear density moving in a Newtonian fluid. It would thus seem reasonable that the effect of the initial velocity would be quickly overridden by the damping of the fluid, and that, in the limit as $\varepsilon \to 0$, this effect should vanish, except possibly for an initial "boundary layer" which, in fact, does not occur.

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In his Yale lectures HADAMARD [5] notes that the Riemann function for an equation similar to (1.2) tends in the limit to the fundamental solution of the heat equation. In a series of papers beginning in 1959, M. ZLÁMAL (see especially [11] where he surveys his results and refers to the previous papers) has studied this problem for equations with variable coefficients. He has treated, for various cases, both the Cauchy problem and the mixed initial-boundary value problem. He does however require rather heavy smoothness of the initial data.

More recently KOPÁČKOVÁ-SUCHÁ [6] has treated a mildly nonlinear case which, in its linearized form, is equivalent to the equation we discuss in the present paper. She gets theorems on the structure of the solution, again under heavy smoothness conditions on the data, and, for our purposes, an unnecessary positivity hypothesis on a coefficient.

We show in this paper that for the case of constant coefficients, a simple direct method based on Hadamard's observation serves to show the convergence under rather weak smoothness on f and g. In fact, we require little more than that the standard solution formulas make sense.

The most general second order hyperbolic equation in two dimensions with constant coefficients can be reduced to

$$\varepsilon^2 u_{tt} + u_t = u_{xx} + cu + F \tag{1.3}$$

where F may depend on x and t. This equation is equivalent to the linear case discussed by Kopáčková-Suchá, except that she requires $c > 0$. For our purposes this is the simplest form in which to consider the general case of this equation. The Cauchy problem asks for a solution of (1.3) satisfying the initial conditions

$$u(x, 0) = f(x), \quad u_t(x, 0) = g(x). \tag{1.4}$$

To write down the solution of this equation it is convenient to introduce the following notation:

$$B = \sqrt{(1 + 4\varepsilon^2 c)}, \quad R = R(x, t) = \sqrt{(1 - \varepsilon^2 x^2/t^2)},$$

and

$$I_\nu(z) = e^{-i\nu\pi/2} J_\nu(ze^{i\pi/2}) = \sum_{j=0}^{\infty} (\tfrac{1}{2}z)^{2j+\nu}/[m!\, \Gamma(m + \nu + 1)]$$

is the modified Bessel function of the first kind of order ν. The formal solution of our Cauchy problem (1.3), (1.4) is given by

$$\begin{aligned} u(x, t; \varepsilon) = {} & \frac{\partial}{\partial t}\left[\frac{\varepsilon}{2}\int_{x-t/\varepsilon}^{x+t/\varepsilon} f(\xi)\, e^{-t/2\varepsilon^2} I_0\left(BtR(x - \xi, t)/2\varepsilon^2\right) \mathrm{d}\xi\right] + \\ & + \frac{\varepsilon}{2}\int_{x-t/\varepsilon}^{x+t/\varepsilon} g(\xi)\, e^{-t/2\varepsilon^2} I_0(BtR(x - \xi, t)/2\varepsilon^2)\, \mathrm{d}\xi - \\ & - \frac{1}{2\varepsilon}\int_0^t \mathrm{d}\tau \int_{x-(t-\tau)/\varepsilon}^{x+(t-\tau)/\varepsilon} F(\xi, \tau)\, e^{-(t-\tau)/2\varepsilon^2} I_0(B(t - \tau)\, R(x - \xi, t - \tau)/2\varepsilon^2)\, \mathrm{d}\xi \end{aligned}$$

or

$$(1.5)\quad u(x,t;\varepsilon) = \tfrac{1}{2}[f(x+t/\varepsilon) + f(x-t/\varepsilon)]\, e^{-t/2\varepsilon^2} + \\
+ \frac{1}{4\varepsilon}\int_{x-t/\varepsilon}^{x+t/\varepsilon} f(\xi)\, e^{-t/2\varepsilon^2} I_0(BtR(x-\xi,t)/2\varepsilon^2)\, d\xi + \\
+ \frac{B}{4\varepsilon}\int_{x-t/\varepsilon}^{x+t/\varepsilon} f(\xi)\, e^{-t/2\varepsilon^2} \frac{1}{R} I_1(BtR(x-\xi,t)/2\varepsilon^2)\, d\xi + \\
+ \frac{\varepsilon}{2}\int_{x-t/\varepsilon}^{x+t/\varepsilon} g(\xi)\, e^{-t/2\varepsilon^2} I_0(BtR(x-\xi,t)/2\varepsilon^2)\, d\xi + \\
+ \frac{1}{2\varepsilon}\int_0^t d\tau \int_{x-(t-\tau)/\varepsilon}^{x+(t-\tau)/\varepsilon} F(\xi,\tau)\, e^{-(t-\tau)/2\varepsilon^2} I_0(B(t-\tau)\, R(x-\xi,t-\tau)/2\varepsilon^2)\, d\xi$$

which can be derived by Riemann's method, and is a well known, standard formula.

We define k_0, k_1 and k by

$$k_0(x,t;\varepsilon) = \begin{cases} \dfrac{1}{2\varepsilon} e^{-t/2\varepsilon^2} I_0(BtR(x,t)/2\varepsilon^2), & \varepsilon^2 x^2 < t^2, \\ 0, & \varepsilon^2 x^2 \geqq t^2; \end{cases}$$

$$k_1(x,t;\varepsilon) = \begin{cases} [B/2\varepsilon R(x,t)]\, e^{-t/2\varepsilon^2} I_1(BtR(x,t)/2\varepsilon^2), & \varepsilon^2 x^2 < t^2, \\ 0, & \varepsilon^2 x^2 \geqq t^2; \end{cases}$$

and

$$k(x,t) = (4\pi t)^{-1/2}\, e^{ct - x^2/4t};$$

where we assume $t > 0$ in all three formulas. With these definitions (1.5) becomes

$$(1.6)\quad u(x,t;\varepsilon) = \tfrac{1}{2}[f(x+t/\varepsilon) + f(x-t/\varepsilon)]\, e^{-t/2\varepsilon^2} + \\
+ \frac{1}{2}\int_{-\infty}^{\infty} [k_0(x-\xi,t;\varepsilon) + k_1(x-\xi,t;\varepsilon)]\, f(\xi)\, d\xi + \\
+ \varepsilon^2 \int_{-\infty}^{\infty} k_0(x-\xi,t;\varepsilon)\, g(\xi)\, d\xi - \int_0^t d\tau \int_{-\infty}^{\infty} k_0(x-\xi,t-\tau;\varepsilon)\, F(\xi,\tau)\, d\xi\,.$$

With $\varepsilon = 0$ equation (1.3) reduces to

$$(1.7)\qquad u_t = u_{xx} + cu + F\,.$$

The initial value problem for this equation requires only the data

$$(1.8)\qquad u(x,0) = f(x)$$

for the unique determination of a solution, which is then formally given by

$$(1.9)\quad u(x, t) = \int_{-\infty}^{\infty} k(x - \xi, t) f(\xi)\, d\xi - \int_0^t d\tau \int_{-\infty}^{\infty} k(x - \xi, t - \tau)\, F(\xi, \tau)\, d\xi\,.$$

In the following we want to compare these formulas for $u(x, t; \varepsilon)$ and $u(x, t)$ for small ε.

2. ESTIMATES ON THE KERNELS

We establish the following

Theorem. *Let $T > 0$ be given. If $c > -1/4\varepsilon^2$, then there is a constant C_0, depending only on the lower bound of $B = \sqrt{(1 + 4c\varepsilon^2)}$, and a constant C_1, depending only on the upper bound of B for which*

$$(2.1)\qquad 0 \leqq k_0(x, t; \varepsilon) \leqq C_0 e^{|c|T}\, k(x, t)\,, \quad 0 < t \leqq T\,;$$

$$(2.2)\qquad 0 \leqq k_1(x, t; \varepsilon) \leqq C_1 e^{|c|T}\, k(x, t)\,, \quad 0 < t \leqq T\,;$$

and if $c \geqq 0$, then

$$(2.1')\qquad 0 \leqq k_0(x, t; \varepsilon) \leqq 4k(x, t)\,, \qquad 0 < t\,;$$

$$(2.2')\qquad 0 \leqq k_1(x, t; \varepsilon) \leqq C_1 k(x, t)\,, \qquad 0 < t\,.$$

Proof. We remark that for $z \geqq 0$ we have

$$(2.3)\qquad 0 \leqq I_0(z) \leqq e^z\,, \quad 0 \leqq I_0(z) \leqq e^z/\sqrt{z}\,,$$

the first being of interest for small z, and the second for large. That $I_0(z)$ can be estimated by some constant times e^z, and by some constant times $e^z/\sqrt{z}$, follow from continuity and the standard asymptotic formula for $I_0(z)$, and that is all that is really needed here, except for the verification of the constant $C_0 = 4$ on the right hand side of (2.1′), and the number 4 plays no significant role in the following. The estimates (2.3) as stated, however, can easily be calculated, for example, from formula (7) p. 81 of [4].

Choose r between 1 and 2 and consider the case $|x| < t\,(2r - 1)^{1/2}/\varepsilon r$ so that $R(x, t) = R > (r - 1)/r$. Then we have by (2.3),

$$k_0(x, t; \varepsilon) \leqq e^{-t/2\varepsilon^2}\, e^{BtR/2\varepsilon^2}/2\varepsilon(BtR/2\varepsilon^2)^{1/2} \leqq e^{-c\varepsilon^2 x^2/2t} \left(\frac{2\pi r}{B(r - 1)}\right)^{1/2} k(x, t)\,.$$

If $c \geqq 0$ we estimate the exponential by unity. If $c < 0$, the exponent is positive and

$x^2/t < t/\varepsilon^2$ so

$$k_0(x, t; \varepsilon) \leqq e^{-ct/2}\left(\frac{2\pi r}{B(r-1)}\right)^{1/2} k(x, t) \leqq e^{|c|T}\left(\frac{2\pi r}{B(r-1)}\right)^{1/2} k(x, t) .$$

With the same choice of r we consider the case $t(2r-1)^{1/2}/\varepsilon r \leqq |x| \leqq t/\varepsilon$, so that $R \leqq (r-1)/r$. Then

$$k_0(x, t; \varepsilon) \leqq e^{-t/2\varepsilon^2} e^{BtR/2\varepsilon^2}/2\varepsilon \leqq e^{-ct/r} e^{ct} e^{-t/2\varepsilon^2 r}/2\varepsilon .$$

The first factor $e^{-ct/r}$ is estimated by unity for $c \geqq 0$, and by $e^{|c|T}$ for $c \leqq 0$. Thus

$$k_0(x, t; \varepsilon) \leqq e^{|c|T}(4\pi)^{1/2}\left(\frac{t}{4\varepsilon^2}\right)^{1/2} e^{-(t/4\varepsilon^2)(2-r)/r}\left[\frac{e^{ct}e^{-t/4\varepsilon^2}}{(4\pi t)^{1/2}}\right].$$

Again $t/\varepsilon^2 > x^2$, so the bracketed factor is bounded by $k(x, t)$. And by observing that $ze^{-az^2} \leqq 1/(2ea)^{1/2}$ for $z \geqq 0$, we get

$$k_0(x, t; \varepsilon) \leqq e^{|c|T}\left(\frac{2\pi r}{e(2-r)}\right)^{1/2} k(x, t) .$$

Equating the coefficients in these estimates we choose $r = (2e + B)/(e + B)$ leading to the common estimate

$$k_0(x, t; \varepsilon) \leqq e^{|c|T}\left(2\pi\left(\frac{2}{B} + \frac{1}{e}\right)\right)^{1/2} k(x, t) ,$$

with $e^{|c|T}$ replaced by unity for $c \geqq 0$. It only remains to get the estimate 4 for the radical if $c \geqq 0$. But then $B \geqq 1$, $e > 2$ and $\pi < 3.2$, whence $k_0(x, t; \varepsilon) \leqq 4k(x, t)$. For $|x| \geqq t/\varepsilon$ the theorem is trivial. Similar estimates on k_1, using $I_1(z) \leqq ze^z$ and $I_1(z) \leqq e^z/\sqrt{z}$, establish

$$k_1(x, t; \varepsilon) \leqq (2\pi B)^{1/2}\,[2 + 3B/e]^{3/2}\, e^{|c|T} k(x, t)$$

with $e^{|c|T}$ replaced by unity for $c \geqq 0$.

We close this section by observating that

$$(2.4)\quad \frac{1}{2}\int_{-\infty}^{\infty} k_0(x, t; \varepsilon)\,dx = \frac{1}{2}\int_{-t/\varepsilon}^{t/\varepsilon} k_0(x, t; \varepsilon)\,dx = B^{-1}e^{-t/2\varepsilon^2}\sinh(Bt/2\varepsilon^2)$$

and

$$(2.5)\quad \frac{1}{2}\int_{-\infty}^{\infty} k_1(x, t; \varepsilon)\,dx = \frac{1}{2}\int_{-t/\varepsilon}^{t/\varepsilon} k_1(x, t; \varepsilon)\,dx = e^{-t/2\varepsilon^2}[\cosh(Bt/2\varepsilon^2) - 1] .$$

These formulas follow by expanding I_0 and I_1 by their power series and integrating termwise.

3. AS $\varepsilon \to 0$.

On the basis of the asymptotic formula

$$I_n(z) \sim e^z/\sqrt{(2\pi z)} \quad \text{as} \quad z \to \infty$$

one easily establishes

$$\left.\begin{aligned} k_0(x, t; \varepsilon) \to k(x, t) \\ k_1(x, t; \varepsilon) \to k(x, t) \end{aligned}\right\} \quad \text{as} \quad \varepsilon \to 0\,,$$

and even that the convergence is uniform on compact subsets of the half plane $t \geqq \geqq t_0 > 0$. From this observation, and from growth conditions on f and g of the form

$$|f(x)| \leqq Ae^{x^2/4a}\,, \quad |g(x)| \leqq Ae^{x^2/4a}$$

it follows easily that $u(x, t; \varepsilon)$ converges to $u(x, t)$ for $0 < t < a$, and in fact that the convergence is uniform on compact subsets of the strip $\{0 < t < a, -\infty < x < \infty\}$. This still leaves open the possibility of a boundary layer effect for small t. It is, for example, clear that such an effect must occur for u_t: suppose $f \in C^2$, then $u_t(x, 0) = = f''(x)$ while $u_t(x, 0, \varepsilon) = g(x)$ for $\varepsilon > 0$, and the convergence of $u_t(x, t; \varepsilon)$ to $u_t(x, t)$ is, by the same sort of arguments outlined above, uniform on compact subsets of the strip. That no such effect occurs in the case of u itself is of some interest. It is perhaps physically surprising that the difficulty in establishing this arises not from the contribution of $u_t(x, 0; \varepsilon) = g(x)$ but rather from $u(x, 0; \varepsilon) = f(x)$, though a glance at the formulas for $u(x, t; \varepsilon)$ shows that f enters the solution in a much more complicated way than g, and that the integral involving g has an ε^2 factor compared to a similar integral involving f.

Theorem. *Let A, a, M, b be positive constants with $b < a$. Let g be locally integrable on the reals and F on the strip $\{0 < t < a, -\infty < x < \infty\}$. In addition let f be continuous,*

$$|f(x)| \leqq Ae^{x^2/4a}\,, \quad |g(x)| \leqq Ae^{x^2/4a} \tag{3.1}$$

and

$$|F(x, t)| \leqq Ae^{x^2/4a}\,, \quad 0 < t < a\,. \tag{3.2}$$

Then given $\eta > 0$, there is an $\varepsilon_0 > 0$ such that

$$|u(x, t; \varepsilon) - u(x, t)| < \eta \quad \textit{for} \quad 0 < \varepsilon < \varepsilon_0 \tag{3.3}$$

uniformly for $|x| \leqq M$, $0 < t \leqq b$.

Before we begin the proof we comment that for $u(x, t; \varepsilon)$ and $u(x, t)$ to be classical solutions of the problems posed in Section 1, additional smoothness is required, and the growth conditions (3.1) and (3.2) are not needed for $u(x, t; \varepsilon)$. For example, it is sufficient that $f \in C^2$, $g \in C^1$, $F \in C^1$ for $u(x, t; \varepsilon)$ to be a classical solution of its Cauchy problem, and it is sufficient that f be continuous, $F \in C^1$ and f and F satisfy (3.1) and (3.2) for $u(x, t)$ to be a classical solution of its problem. Under the assumptions of the however, both are generalized solutions.

We assume $0 < \varepsilon \leqq 1$, and is so small that $\frac{1}{2} \leqq B \leqq \frac{3}{2}$ so that C_0 and C_1 in the theorem of Section 2 are absolute constants. We will use K as a generic symbol for absolute constants, and H as a generic symbol for constants depending on at most A, a, M, b, and the coefficient c. The Landau symbols "O" and "o" may also depend on A, a, M, b, and c. In particular, all constants and Landau symbols will be independent of both ε and the symbol D which we will shortly introduce.

Initially we suppose $F \equiv 0$ and return to the case $F \not\equiv 0$ later.

First we dispose of the contribution arising from $u_t(x, 0; \varepsilon) = g(x)$. We have

$$\left|\varepsilon^2 \int_{-\infty}^{\infty} k_0(x - \xi, t; \varepsilon)\, g(\xi)\, d\xi\right| \leqq \varepsilon^2 A C_0 e^{|c|a} \int_{-\infty}^{\infty} k(x - \xi, t)\, e^{\xi^2/4a}\, d\xi =$$

$$= \varepsilon^2 A C_0 e^{|c|a} e^{ct} \left(\frac{a}{a - t}\right)^{1/2} e^{x^2/4(a-t)} \leqq \varepsilon^2 A C_0 e^{|c|(a+b)} \left(\frac{a}{a - b}\right)^{1/2} e^{M^2/4(a-b)} = \varepsilon^2 H .$$

We make some preliminary estimates on f. If $\alpha \neq 0$ then

$$2My \leqq \frac{M^2}{\alpha^2} + \alpha^2 y^2 ,$$

which follows from $(\alpha y - M/\alpha)^2 \geqq 0$. Thus from (3.1) we get, for $|x| \leqq M$

$$|f(x + y)| \leqq A e^{(x+y)^2/4a} \leqq A e^{(M+|y|)^2/4a} \leqq A e^{M^2(1+1/\alpha^2)/4a}\, e^{y^2(1+\alpha^2)/4a} .$$

Choosing $\alpha = ((a - b)/(a + b))^{1/2}$ gives

$$|f(x + y)| \leqq H e^{y^2/2(a+b)} , \tag{3.4}$$

for $|x| \leqq M$.

And for $0 < t \leqq b$ we have

$$\frac{1}{2(a + b)} - \frac{1}{4t} = \frac{2t - a - b}{4(a + b)\, t} \leqq - \frac{a - b}{4(a + b)\, t}$$

so that

$$e^{y^2/2(a+b) - y^2/4t} \leqq e^{-y^2(a-b)/4(a+b)t} = e^{-y^2\alpha^2/4t} \tag{3.5}$$

for $0 < t \leqq b$.

Since f is assumed continuous it is uniformly continuous on any compact interval, and there is, for each such interval a uniform modulus of continuity δ such that

$$|f(x) - f(y)| \leqq \delta(|x - y|)$$

for any x and any y in the interval. We take the interval to be of the form

$$|x| \leqq M + 2D$$

where D will be specified later.

We examine $[f(x \pm t/\varepsilon) - f(x)]\, e^{-t/2\varepsilon^2}$ for $|x| \leqq M$, $0 < t \leqq b$. Then for any D with $t/2\varepsilon^2 < D$ we have

$$|f(x \pm t/\varepsilon) - f(x)|\, e^{-t/2\varepsilon^2} < \delta(t/\varepsilon) \leqq \delta(2D\varepsilon),$$

and for $t/2\varepsilon^2 \geqq D$ we have

$$|f(x)|\, e^{-t/2\varepsilon^2} \leqq He^{-D}.$$

By (3.4) we have

$$|f(x \pm t/\varepsilon)|\, e^{-t/2\varepsilon^2} \leqq H \exp\left[\frac{(t/\varepsilon)^2}{2(a + b)} - \frac{t}{2\varepsilon^2}\right] \leqq H \exp\left[-\frac{t}{2\varepsilon^2}\frac{a}{a + b}\right] \leqq He^{-D/2}$$

so that

$$|f(x \pm t/\varepsilon) - f(x)|\, e^{-t/2\varepsilon^2} \leqq He^{-D/2}.$$

Given $\eta > 0$ we choose D so large that

$$He^{-D/2} < \eta/2.$$

With D chosen, δ is determined, and we choose ε so small that

$$\delta(2D\varepsilon) < \eta/2.$$

Thus, for $|x| \leqq M$, $0 < t \leqq b$ we have

$$|f(x \pm t/\varepsilon) - f(x)|\, e^{-t/2\varepsilon^2} < \eta/2$$

if ε is sufficiently small. In particular

$$\tfrac{1}{2}[f(x + t/\varepsilon) + f(x - t/\varepsilon)]\, e^{-t/2\varepsilon^2} = f(x)\, e^{-t/2\varepsilon^2} + o(1),$$

and so

$$(3.6) \qquad u(x, t; \varepsilon) = \frac{1}{2}\int_{-\infty}^{\infty} f(\xi)\,[k_0(x - \xi, t; \varepsilon) + k_1(x - \xi, t; \varepsilon)]\, d\xi + \\ + f(x)\, e^{-t/2\varepsilon^2} + o(1)$$

where $o(1)$ is small for small ε, uniformly for $|x| \leqq M$, $0 < t \leqq b$.

We have been at pains to elaborate this simple argument for the following reason. The main body of the argument follows this one in outline: we estimate the difference $u(x, t; \varepsilon) - u(x, t)$ first for $t/2\varepsilon^2 < D$, then for $t/2\varepsilon^2 \geqq D$, then choose D large and ε small. While in outline our argument now proceeds to repeat itself, the details are more complicated and the estimates more tedious.

We assume now $t/2\varepsilon^2 < D$, and estimate $u(x, t; \varepsilon)$, as given by (3.6), by the mean value theorem and (2.4) and (2.5) to get

$$u(x, t; \varepsilon) = e^{-t/2\varepsilon^2}[(1/B)\sinh(Bt/2\varepsilon^2) + \cosh(Bt/2\varepsilon^2) - 1] f(\bar{x}) + \\ + f(x)\, e^{-t/2\varepsilon^2} + o(1)$$

where $|x - \bar{x}| < t/\varepsilon < 2D\varepsilon$. Since $1/B = 1 + O(\varepsilon^2)$ we calculate

$$u(x, t; \varepsilon) = e^{(B-1)t/2\varepsilon^2} f(\bar{x}) + [f(x) - f(\bar{x})]\, e^{-t/2\varepsilon^2} + O(\varepsilon^2)\, e^{O(D\varepsilon^2)} + o(1)\,.$$

After some reduction this is easily seen to yield

$$u(x, t; \varepsilon) = f(x) + O(D\varepsilon^2) + O(\delta(2D\varepsilon))\,[e^{O(D\varepsilon^2)} + 1] + O(\varepsilon^2)\, e^{O(D\varepsilon^2))} + o(1)\,.$$

The given $\eta > 0$, for arbitrary but fixed D, we have

$$|u(x, t; \varepsilon) - f(x)| < \eta/2$$

if ε is sufficiently small, $t/2\varepsilon^2 < D$, and $|x| \leqq M$.

It is a well known property of the kernel $k(x, t)$ that

$$u(x, t) = \int_{-\infty}^{\infty} k(x - \xi, t) f(\xi)\, d\xi$$

tends to $f(x)$ uniformly on compact sets as $t \to 0$. Hence

$$|u(x, t) - f(x)| < \eta/2$$

if $t < 2D\varepsilon^2$, and ε sufficiently small, and $|x| \leqq M$. This, together with the previous estimate yields

$$|u(x, t; \varepsilon) - u(x, t)| < \eta$$

for arbitrary but fixed D with $t/2\varepsilon^2 < D$, $|x| \leqq M$, and ε sufficiently small.

We now assume $t/2\varepsilon^2 > D$, and consider

$$u(x, t; \varepsilon) - u(x, t) = \frac{1}{2}\int_{-\infty}^{\infty} [k_0(x - \xi, t; \varepsilon) - k(x - \xi, t)] f(\xi)\, d\xi + \\ + \frac{1}{2}\int_{-\infty}^{\infty} [k_1(x-\xi, t; \varepsilon) - k(x-\xi, t)] f(\xi)\, d\xi + f(x)\, e^{-t/2\varepsilon^2} + o(1) \equiv \\ \equiv T_1 + T_2 + T_3 + o(1) \quad \text{respectively}\,.$$

Clearly

$$(3.7) \qquad |T_3| \leqq |f(x)|\, e^{-t/2\varepsilon^2} \leqq He^{-D}$$

for $|x| \leqq M$, and we proceed to estimate T_1.

In the integral defining T_1 substitute $\xi = x + y$ and write

$$T_1 = \frac{1}{2}\left\{\int_{-\infty}^{-t/2\varepsilon} + \int_{-t/2\varepsilon}^{t/2\varepsilon} + \int_{t/2\varepsilon}^{\infty}\right\} f(x + y)\,[k_0(y, t; \varepsilon) - k(y, t)]\,\mathrm{d}y \equiv$$

$$\equiv T_{11} + T_{12} + T_{13} \quad \text{respectively.}$$

By (2.1) and (3.4) we have

$$|T_{11}| + |T_{13}| \leqq H \int_{t/2\varepsilon}^{\infty} e^{y^2/2(a+b)}\, k(y, t)\,\mathrm{d}y = \frac{e^{ct}H}{\sqrt{t}} \int_{t/2\varepsilon}^{\infty} e^{y^2/2(a+b)}\, e^{-y^2/4t}\,\mathrm{d}y\,.$$

By (3.5) we estimate further:

$$|T_{11}| + |T_{13}| \leqq \frac{H}{\sqrt{t}} \int_{t/2\varepsilon}^{\infty} e^{-\alpha^2 y^2/4t}\,\mathrm{d}y \leqq \frac{H}{\sqrt{t}}\, e^{-\alpha^2 t/32\varepsilon^2} \int_0^{\infty} e^{-\alpha^2 y^2/8t}\,\mathrm{d}y \leqq$$

$$\leqq He^{-\alpha^2 t/32\varepsilon^2} \leqq He^{-\alpha^2 D/16}\,.$$

To estimate T_{12} we use $|I_0(z) - e^z/(2\pi z)^{1/2}| \leqq Ke^z/z^{3/2}$, $z \geqq 0$, which follows from the continuity of $I_0(z)$ and the standard asymptotic expansion. It is of interest only for z bounded away from zero, which is precisely the circumstances here. We get immediately

$$|T_{12}| \leqq \int_0^{t/2\varepsilon} e^{y^2/2(a+b)} \left|\frac{e^{-t/2\varepsilon^2} e^{BtR/2\varepsilon^2}}{2\varepsilon\,(2\pi BtR/2\varepsilon^2)^{1/2}} - \frac{e^{ct} e^{-y^2/4t}}{(4\pi t)^{1/2}}\right| \mathrm{d}y +$$

$$+ H \int_0^{t/2\varepsilon} \frac{e^{y^2/2(a+b)} e^{-t/2\varepsilon^2} e^{BtR/2\varepsilon^2}}{2\varepsilon(BtR/2\varepsilon^2)^{3/2}}\,\mathrm{d}y \equiv T_{121} + T_{122} \quad \text{respectively}\,,$$

where $R = (1 - \varepsilon^2 y^2/t^2)^{1/2}$, $B = (1 + 4c\varepsilon^2)^{1/2}$.

In the integrand we have $0 \leqq y \leqq t/2\varepsilon$, so $R \geqq \sqrt{3}/2$, and using $\sqrt{(1 + h)} \leqq \leqq 1 + h/2$ for $h \geqq -1$ we compute

$$T_{122} \leqq \frac{H\varepsilon^2}{t^{3/2}} \int_0^{t/2\varepsilon} e^{y^2/2(a+b)} e^{-t/2\varepsilon^2} e^{t/2\varepsilon^2 + ct - y^2/4t} e^{-c\varepsilon^2 y^2/2t}\,\mathrm{d}y \leqq$$

$$\leqq \frac{H\varepsilon^2}{t^{3/2}} \int_0^{t/2\varepsilon} e^{-\alpha^2 y^2/4t}\,\mathrm{d}y \leqq H\frac{\varepsilon}{\sqrt{t}} \leqq \frac{H}{\sqrt{D}}\,.$$

And in T_{121} we factor

$$e^{ct}e^{-y^2/4t}/\sqrt{(4\pi tBR)}$$

out from the absolute value, use (3.5) and get

$$T_{121} \leqq \frac{H}{\sqrt{t}} \int_0^{t/2\varepsilon} e^{-\alpha^2 y^2/4t} \left| \exp\left[y^2/4t - ct - t/2\varepsilon^2 + BtR/2\varepsilon^2\right] - \right.$$

$$\left. - \exp\left[\tfrac{1}{4}\log(1 + 4c\varepsilon^2)(1 - \varepsilon^2 y^2/t^2)\right] \right| \mathrm{d}y \,.$$

The exponents of both exponentials inside the absolute value signs are bounded above. Then, since $|e^z - e^\zeta| \leqq K|\zeta - z| \leqq K(|\zeta| + |z|)$ if ζ and z are bounded above, we get

$$T_{121} \leqq \frac{H}{\sqrt{t}} \int_0^{t/2\varepsilon} e^{-\alpha^2 y^2/4t} \left\{ \left| \frac{y^2}{4t} - ct - \frac{t}{2\varepsilon^2} + \frac{t}{2\varepsilon^2}\left(1 + 4c\varepsilon^2 - \frac{y^2\varepsilon^2}{t^2} + \frac{4cy^2\varepsilon^4}{t^2}\right)^{1/2} \right| + \right.$$

$$\left. + \tfrac{1}{4}\left|\log(1 + 4c\varepsilon^2)\right| + \tfrac{1}{4}\log\left|\left(1 - \frac{\varepsilon^2 y^2}{t^2}\right)\right| \right\} \mathrm{d}y \,.$$

We estimate the square root by

$$\sqrt{(1 + h)} = 1 + h/2 + E \quad \text{where} \quad 0 \leqq E \leqq h^2/2$$

if $h > -1$, and

$$|\log(1 + h)| \leqq K|h| \quad \text{for} \quad |h| < \tfrac{1}{2} \,.$$

We observe that in the first absolute value, the first three terms arising from the square root cancel with the three standing in front of the square root. Then estimating y/t by $1/2\varepsilon$, and t by a we get finally,

$$T_{121} \leqq \frac{H}{\sqrt{t}} \int_0^{t/2\varepsilon} e^{-\alpha^2 y^2/4t} \left[\varepsilon^2 + \frac{\varepsilon^2 y^2}{t^2}\right] \mathrm{d}y$$

where, as usual, H has a new, but constant value. Substituting $z = \alpha y/2\sqrt{t}$, and integrating from 0 to ∞ we get

$$T_{121} \leqq H[\varepsilon^2 + \varepsilon^2/t] = H\varepsilon^2/t \leqq H/D \,.$$

The integral T_2 can be estimated in a similar way. Then our argument is as before: We choose D sufficiently large that

$$|u(x, t; \varepsilon) - u(x, t)| < \eta$$

for $t/2\varepsilon^2 > D$. Then choose ε sufficiently small that the inequality holds for $t/2\varepsilon^2 < D$.

To complete the proof we have only to discuss the case where $f \equiv g \equiv 0$ and F satisfies (3.2). Then

$$|u(x, t; \varepsilon) - u(x, t)| \leqq A \int_0^t d\tau \int_{-\infty}^{+\infty} |k_0(x - \xi, t - \tau; \varepsilon) - k(x - \xi, t - \tau)| \, e^{\xi^2/4a} \, d\xi =$$

$$= A \int_0^{t-\delta} d\tau \int_{-\infty}^{\infty} |k_0(x - \xi, t - \tau; \varepsilon) - k(x - \xi, t - \tau)| \, e^{\xi^2/4a} \, d\xi +$$

$$+ A \int_{t-\delta}^{t} d\tau \int_{-\infty}^{\infty} |k_0(x - \xi, t - \tau; \varepsilon) - k(x - \xi, t - \tau)| \, e^{\xi^2/4a} \, d\xi \equiv$$

$$\equiv T_4 + T_5 \quad \text{respectively} .$$

We estimate T_5 first.

$$T_5 \leqq A(C_0 + 1) \, e^{|c|a} \int_{t-\delta}^{t} d\tau \int_{-\infty}^{\infty} k(x - \xi, t - \tau) \, e^{\xi^2/4a} \, d\xi =$$

$$= A(C_0 + 1) \, e^{|c|a} \int_{t-\delta}^{t} e^{c(t-\tau)} \left(\frac{a}{a - (t - \tau)} \right)^{1/2} e^{x^2/4[a-(t-\tau)]} \, d\tau \leqq$$

$$\leqq A(C_0 + 1) \, e^{|c|(a+\delta)} \left(\frac{a}{a - b} \right)^{1/2} e^{M^2/4(a-b)} \, \delta$$

for $|x| \leqq M$, $0 < t \leqq b$.

As for T_4, we remark that the previous calculation (for $t/2\varepsilon^2 \geqq D$), contains a proof that

$$\int_{-\infty}^{\infty} |k_0(x - \xi, t - \tau; \varepsilon) - k(x - \xi, t - \tau)| \, e^{\xi^2/4a} \, d\xi$$

converges uniformly to zero for $0 < \delta < t - \tau < a$, and hence so does T_4.

4. THE HIGHER DIMENSIONAL CASE

We comment very briefly about the higher dimensional case. The solution formula (1.5) is available there. For example, these are given explicitly for dimensions 2 and 3 for the equation

$$u_{tt} + u_t = \Delta u$$

in [3] p. 695, and can be written out for any given dimension by the methods used there. Then a change of scale of ε^2 on the t axis and ε in the x space reduces this equation to

$$\varepsilon^2 u_{tt} + u_t = \Delta u .$$

The solution of the initial value problem for this equation can then be written out, and the argument, only slightly more complicated, in detail can be carried out, yielding the same result.

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