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EXISTENCE AND STABILITY OF MEASURE DIFFERENTIAL EQUATIONS

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1. INTRODUCTION

When a system described by ordinary differential equation

$$\frac{dx}{dt} = f(t, x) \tag{1.1}$$

is acted upon by perturbation, the perturbed system is generally given by ordinary differential equation of the form $dx/dt = f(t, x) + G(t, x)$ where the perturbation term $G(t, x)$ is assumed to be continuous or integrable and as such the state of the system changes continuously with respect to time. But in physical systems one cannot expect the perturbations to be well-behaved and it is therefore important to consider the case when the perturbations are impulsive. This will give rise to equations of the form

$$Dx = f(t, x) + G(t, x)\, Du \tag{1.2}$$

where Du denotes the distributional derivative of function u. If u is a function of bounded variation, Du can be identified with a Stieltjes measure and will have the effect of suddenly changing the state of the system at the points of discontinuity of u. Equations of the form (1.2) are called *measure differential equations* [1]. By a solution $x(.)$ of (1.2) is meant a function of bounded variation whose distributional derivative Dx satisfies the equation (1.2). Schmaedeke [1] has considered the theory of the equation (1.2) in the special case when G is independent of x, i.e. $G(t, x) = G(t)$, and is assumed to be a continuous function of t in order to apply the methods of Riemann-Stieltjes integrals in the subsequent analysis. In [2], the authors have generalized the results in [1] for the following functional differential equation

$$Dx = f(t, x_t) + G(t, x_t)\, Du \tag{1.3}$$

where x_t represents the restriction of the function $x(.)$ on the interval $[p(t), q(t)]$, p and q being functions with the property $p(t) \leqq q(t) \leqq t$. In this case, the methods of R.S. integrals are unapplicable because of the possibility that $G(t, x_t)$ and $u(t)$ may have common discontinuities, and Lebesgue-Stieltjes integrals are therefore used.

In this paper we shall consider the equation (1.2) which is a particular case of equation (1.3). The equation (1.2) is treated through an equivalent integral equation. The proof of the equivalence is given in the Appendix, and is on the same lines as that of Theorem 1 in [2]. It differs essentially from the proof of Theorem 1 in [1] in two facts: firstly, the result in Lemma (given in the Appendix) is used since G is not necessarily continuous functions of t (as in [1]), and secondly, 'integration by parts' is used carefully since it is not valid in general for L.S. integrals. A local existence theorem is established using a method of successive approximation. This theorem, in fact, relaxes considerably the hypotheses of f and G in a similar theorem in [2]. It also shows that for local existence of solution of the equation considered in [1], f need not be Lipschitzian (in x).

The primary aim of considering (1.2) is the following. When the equation (1.2) is regarded as perturbed system of equation (1.1), a natural question arises under what conditions the stability properties of (1.1) are shared by the solutions of (1.2). It seems very difficult to get a satisfactory answer to this question. It may be so because differential and integral inequalities play a very important role in the stability theory; but when we consider the stability of solutions of (1.2), the fact that its solutions are discontinuous renders many of differential inequalities unapplicable while the integral inequalities are not available for the Stieltjes integrals. However, this paper deals with a stability theorem under conditions which may be regarded to be restrictive. Such conditions are natural to be expected, since otherwise the discontinuities of u may give considerable impulsive changes in the state variables to make the system unstable. The stability of systems with respect to impulsive perturbations has also been considered by BARBASHIN [3] and ZABALISHCHIN [4].

2. EXISTENCE OF SOLUTION

The n-dimensional Euclidean space will be denoted by R^n and the norm of a vector $x = (x^1, \ldots, x^n) \in R^n$ will be defined by

$$|x| = \sum_{i=1}^{n} |x_i| . \tag{2.1}$$

Let $\mathfrak{M}$ denote the set of all $n \times m$ matrices of real numbers. The norm of a matrix $M = (M_j^i) \in \mathfrak{M}$ will be defined by

$$|M| = \sum_{i=1}^{n} \sum_{j=1}^{m} |M_j^i| . \tag{2.2}$$

Let I be an interval, finite or infinite, of the real line. By $BV(I)$ is denoted the Banach space of all scalar functions f defined and of bounded variation on I, with the norm defined by

$$|f|_I = v(f, I) + |f(a+)| \tag{2.3}$$

where a is the left end point of I. By $BV(I)_n$ or $BV(I, R^n)$ will be denoted the space of all vector functions f defined on I with values in R^n whose individual components $\in BV(I)$. $BV(I)_n$ is a Banach space with norm of $f \in BV(I)_n$ defined by

$$\|f\|_I = \sum_{i=1}^{n} |f^i|_I = \sum_{i=1}^{n} \{v(f^i, I) + |f^i(a+)|\} = v(f, I) + f(a+) \tag{2.4}$$

where a is the left end point of I.

Consider the measure differential equation

$$\mathrm{D}x = f(t, x) + G(t, x)\,\mathrm{D}u \tag{2.5}$$

where f and G are defined on $R^+ \times R^n$ (R^+ is positive real line) with values in R^n and $\mathfrak{M}$ respectively, and u is a right continuous function $\in BV(R^+, R^m)$. Let S be an open connected set in R^n and I an interval with left end point $t_0 \geqq 0$. A function $x(.) = x(.; t_0, x_0)$ will be called a solution of (2.5) through (t_0, x_0) on the interval I if $x(.)$ is right continuous function $\in BV(I, S)$, $x(t_0) = x_0$ and the distributional derivative of $x(.)$ on (t_0, T) for any arbitrary $T \in I$ satisfies (2.5). Assume that for each $x(.) \in BV(I, S)$, $f(t, x(t))$ is Lebesgue integrable and $G(t, x(t))$ is integrable with respect to the L. S. measure du. Then, as in [2], $x(.)$ is a solution of (2.1) through (t_0, x_0) on I if and only if it satisfies the integral equation

$$x(t) = x_0 + \int_{t_0}^{t} f(s, x(s))\,\mathrm{d}s + \int_{t_0}^{t} G(s, x(s))\,\mathrm{d}u(s) \tag{2.6}$$

for $t \in I$. The integral $\int_{t_0}^{t} G(s, x(s))\,\mathrm{d}u(s)$ is considered over the half-open interval $(t_0, t]$. The proof is given in the Appendix.

Define

$$S_b(x_0) = \{x \in R^n : |x - x_0| < b\} \tag{2.7}$$

$$E = \{(t, x) : t \in [t_0, t_1],\ x \in S_b(x_0)\} \tag{2.8}$$

where $x_0 \in S$ and $b > 0$ is so small that $S_b(x_0) \subset S$.

Theorem 1 (Local Existence). *Suppose that f and G satisfy the following conditions on E:*

(i) *$f(t, x)$ is measurable in t for each x;*
(ii) *$f(t, x)$ is continuous in x for each t;*

(iii) *there exists a Lebesgue integrable function r such that*

$$|f(t, x)| \leqq r(t), \quad (t, x) \in E;$$

(iv) *$G(t, x(t))$ is* du*-integrable for each $x(.) \in BV([t_0, t_1], S_b(x_0))$;*
(v) *$G(t, x)$ is continuous in x for each t;*
(vi) *there exists a* dv_u*-integrable function w such that*

$$|G(t, x)| \leqq w(t), \quad (t, x) \in E,$$

where v_u denotes the total variation function of u.

Then there exists a solution $x(.)$ of (2.5) on some interval $[t_0, t_0 + a]$ satisfying the initial condition $x(t_0) = x_0$.

Proof. Choose a $(0 < 2a < t_1 - t_0)$ such that

$$\int_{t_0}^{t_0+2a} r(s)\,\mathrm{d}s + \int_{t_0}^{t_0+2a} w(s)\,\mathrm{d}v_u(s) \leqq c \tag{2.9}$$

where $0 < c < b$. Since $\int_{t_0}^{t} r(s)\,\mathrm{d}s$ is continuous function of t and $\int_{t_0}^{t} w(s)\,\mathrm{d}v_u(s)$ is right continuous function of t, it is possible to choose such an a.

Now consider the following integral equation

$$x^{(k)}(t) = \begin{cases} x_0 \quad \text{for} \quad t \in [t_0 - 2a/k, t_0] \\ x_0 + \int_{t_0}^{t} f(s, x^{(k)}(s - 2a/k))\,\mathrm{d}s + \int_{t_0}^{t} G(s, x^{(k)}(s - 2a/k))\,\mathrm{d}u(s) \\ \qquad\qquad\qquad \text{for} \quad t \in (t_0, t_0 + 2a] \end{cases} \tag{2.10}$$

For any k, the first expression in (2.10) defines $x^{(k)}$ on $[t_0 - 2a/k, t_0]$ where $x^{(k)}(t) = x_0$, and since $(t, x_0) \in E$ for $t \in [t_0, t_0 + 2a/k]$ the second expression of (2.10) defines $x^{(k)}$ as a function of bounded variation on the interval $(t_0, t_0 + 2a/k]$. Let us assume that $x^{(k)}$ is defined on $[t_0 - 2a/k, t_0 + 2aj/k]$, $1 \leqq j < k$. Then we have for $t \in [t_0 - 2a/k, t_0 + 2aj/k]$,

$$\begin{aligned} |x^{(k)}(t) - x_0| &\leqq v(x^{(k)}(.) - x_0, [t_0 - 2a/k, t_0 + 2aj/k]) \leqq \\ &\leqq \int_{t_0}^{t_0+2aj/k} |f(s, x^{(k)}(s - 2a/k)| \,\mathrm{d}s + \int_{t_0}^{t_0+2aj/k} |G(s, x^{(k)}(s - 2a/k)| \,\mathrm{d}v_u(s) \leqq \\ &\leqq c < b, \quad \text{by (2.9)}, \end{aligned} \tag{2.11}$$

and therefore the second expression of (2.10) defines $x^{(k)}$ on $(t_0 + 2aj/k, t_0 + 2a(j + 1)/k]$. Thus $x^{(k)}$ is defined on $[t_0 - 2a/k, t_0 + 2a]$ and it can be seen, as in (2.11), that

$$|x^{(k)}(t) - x_0| \leqq v(x^{(k)}(.) - x_0, [t_0 - 2a/k, t_0 + 2a]) < b. \tag{2.12}$$

Define

$$\tilde{x}^{(k)}(t + 2a/k) = x^{(k)}(t), \quad t_0 - 2a/k \leqq t \leqq t_0 + 2(1 - 1/k)\, a, \quad k \geqq 2. \tag{2.13}$$

From the right continuity of $x^{(k)}$, it follows that $\tilde{x}^{(k)}$ is also right continuous. Also, by (2.12), $\tilde{x}^{(k)}$ are of uniform bounded variation on $[t_0, t_0 + a]$. By Helly's selection principle, there exists a subsequence $\tilde{x}^{(k_j)}$ and a function x^* such that

$$\lim_{j\to\infty} \tilde{x}^{(k_j)}(t) = x^*(t), \quad t \in [t_0, t_0 + a]. \tag{2.14}$$

Therefore, by virtue of conditions (ii) and (iv), and (2.14),

$$\lim_{j\to\infty} f(t, \tilde{x}^{(k_j)}(t)) = f(t, x^*(t)), \tag{2.15}$$

$$\lim_{j\to\infty} G(t, \tilde{x}^{(k_j)}(t)) = G(t, x^*(t)). \tag{2.16}$$

By Lebesgue's dominated convergence theorem, we have

$$\lim_{j\to\infty} \int_{t_0}^{t} f(s, \tilde{x}^{(k_j)}(s))\, ds = \int_{t_0}^{t} f(s, x^*(s))\, ds.$$

Also,

$$\lim_{j\to\infty} \int_{t_0}^{t} G(s, \tilde{x}^{(k_j)}(s))\, du(s) = \lim_{j\to\infty} \left[\int_{t_0}^{t} G(s, \tilde{x}^{(k_j)}(s))\, du^+(s) - \int_{t_0}^{t} G(s, \tilde{x}^{(k_j)}(s))\, du^-(s) \right] =$$

[by Halmos [4, Sec. 29 (7)], where du^+ and du^- are positive and negative variations of L. S. measure du]

$$= \int_{t_0}^{t} G(s, x^*(s))\, du^+(s) - \int_{t_0}^{t} G(s, x^*(s))\, du^-(s) =$$

[by (2.16) using Lebesgue's dominated convergence theorem]

$$= \int_{t_0}^{t} G(s, x^*(s))\, du(s).$$

From (2.10), we write

$$\tilde{x}^{(k_j)}(t + 2a/k_j) = \begin{cases} x_0 & \text{for } t \in [t_0 - 2a/k, t_0] \\ x_0 + \displaystyle\int_{t_0}^{t} f(s, \tilde{x}^{(k_j)}(s))\, ds + \int_{t_0}^{t} G(s, \tilde{x}^{(k_j)}(s))\, du(s) & \\ & \text{for } t \in (t_0, t_0 + a]. \end{cases}$$

Taking limit as $j \to \infty$, we have

$$x^*(t) = x_0 + \int_{t_0}^{t} f(s, x^*(s))\, ds + \int_{t_0}^{t} G(s, x^*(s))\, du(s)$$

for $t \in [t_0, t_0 + a]$. Hence x^* is a solution of (2.5) on $[t_0, t_0 + a]$ through (t_0, x_0).

We begin by recalling certain definitions.

Let $x(.; t_0, x_0)$ be any solution of the differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(t, x), \quad x(t_0) = x_0, \quad t_0 \geqq 0 \tag{3.1}$$

where f is defined and continuous on $R^+ \times S_\varrho$, S_ϱ being the set

$$S_\varrho = \{x \in R^n : |x| < \varrho\}.$$

We assume that $f(t, 0) = 0$, $t \in R^+$, so that $x = 0$ is a (trivial) solution of (3.1) through $(t_0, 0)$.

Definition [5, (S_3), p. 136]. The trivial solution $x = 0$ of (3.1) is quasi-equi asymptotically stable if, for each $\varepsilon > 0$, $t_0 \in R^+$, there exist positive numbers $\delta_0 = \delta_0(t_0)$ and $T = T(t_0, \varepsilon)$ such that, for $t \geqq t_0 + T$ and $|x_0| < \delta_0$,

$$|x(t; t_0, x_0)| < \varepsilon.$$

Definition [5, p. 158]. The trivial solution of (3.1) is said to be exponentially asymptotically stable if there exist a $c > 0$ and a $K > 0$ such that

$$|x(t; t_0, x_0)| \leqq K e^{-c(t-t_0)} |x_0|.$$

A scalar function $V(t, x)$ defined on $R^+ \times S_\varrho$ is called a Lyapunov function if it is continuous in (t, x) and locally Lipschitz in x. If $V(t, x)$ is a Lyapunov function then we define

$$V'_{(3.1)}(t, x) = \overline{\lim_{h \to 0+}} \frac{1}{h} \{V(t + h, x + hf(t, x)) - V(t, x)\} =$$

$$= \overline{\lim_{h \to 0+}} \frac{1}{h} \{V(t + h, x(t + h)) - V(t, x)\}$$

where $x(.)$ is the solution of (3.1) through (t, x).

We shall now consider the perturbed system

$$\mathrm{D}x = f(t, x) + G(t, x)\, \mathrm{D}u. \tag{3.2}$$

The following theorem will be proved.

Theorem 2. *Let the trivial solution of* (3.1) *be exponentially asymptotically stable, i.e. there exist a* $c > 0$ *and a* $K > 0$ *such that*

$$|x(t; t_0, x_0)| \leqq K e^{-c(t-t_0)} |x_0|.$$

Suppose that

(1) $f(t, x)$ *satisfies Lipschitz condition in* x *for a constant* $M = M(\varrho) > 0$;

(2) $|G(t, x)| \leqq g(t)\,|x|$;

(3) $\int_0^\infty \omega(t)\,\mathrm{d}t < \infty$, *where*

$$\omega(t) = \overline{\lim_{h \to 0+}} \frac{1}{h} \int_t^{t+h} g(s)\,\mathrm{d}v_u(s)$$

is the upper right Dini derivative of the indefinite integral $\int_0^t g(s)\,\mathrm{d}v_u(s)$;

(4) *the discontinuities* $t_1 < t_2 < \ldots < t_k < \ldots$ *of* u *are isolated, and are such that*

$$|u(t_k) - u(t_k -)| \leqq \alpha e^{-c(t_k - t_0)}\,.$$

Then if $\sum_{k=1}^{\infty} g(t_k)$ *converges and* α *is sufficiently small, the trivial solution* $x = 0$ *of* (3.2) *is quasi-equi asymptotically stable.*

Proof. By [5, Th. 3.6.2 and Cor. 3.6.1] or [6, Cor. of Th. 19.2], there exists a Lyapunov function $V(t, x)$ defined on $R^+ \times S_\varrho$ with the following properties

(i) $|V(t, x) - V(t, y)| \leqq L|x - y|$;

(ii) $|x| \leqq V(t, x) \leqq K|x|$;

(iii) $V'_{(3.1)}(t, x) \leqq -\beta c V(t, x)$, where $0 < \beta < 1$.

Consider a function $W(t, x)$ defined on $R^+ \times S_\varrho$ by

$$W(t, x) = V(t, x) \exp\left\{-L\int_0^t \omega(s)\,\mathrm{d}s\right\}. \tag{3.3}$$

Let $x(.\,; t, x)$ and $x^*(.\,; t, x)$ be solutions through (t, x) of (3.1) and (3.2) respectively. We have,

(3.4)

$$W'_{(3.2)}(t, x) = \overline{\lim_{h \to 0+}} \frac{1}{h}\{W(t + h, x^*(t + h; t, x)) - W(t, x)\} =$$

$$= \overline{\lim_{h \to 0+}} \frac{1}{h}\{W(t + h, x^*(t + h; t, x)) - W(t + h, x(t + h; t, x))\} +$$

$$+ \overline{\lim_{h \to 0+}} \frac{1}{h}\{W(t + h, x(t + h; t, x)) - W(t, x)\} =$$

$$= \overline{\lim_{h \to 0+}} \frac{1}{h} \exp\left(-L\int_0^{t+h} \omega(s)\,\mathrm{d}s\right)\{V(t + h, x^*(t + h; t, x)) -$$

$$- V(t + h, x(t + h; t, x))\} + W'_{(3.1)}(t, x) =$$

$$= \overline{\lim_{h\to 0+}} \frac{1}{h} \exp\left(-L\int_0^{t+h} \omega(s)\,\mathrm{d}s\right)\left\{V\left[t + h, x + \right.\right.$$

$$\left. + \int_t^{t+h} f(s, x^*(s; t, x))\,\mathrm{d}s + \int_t^{t+h} G(s, x^*(s; t, x))\,\mathrm{d}u(s)\right] -$$

$$\left.- V\left[t + h, x + \int_t^{t+h} f(s, x(s; t, x))\,\mathrm{d}s\right]\right\} +$$

$$+ \exp\left(-L\int_0^t \omega(s)\,\mathrm{d}s\right)\{V'_{(3.1)}(t, x) - L\,\omega(t)\,V(t, x)\} \leqq$$

$$\leqq \overline{\lim_{h\to 0+}} \frac{1}{h} \exp\left(-L\int_0^{t+h} \omega(s)\,\mathrm{d}s\right)\left\{L\int_t^{t+h} |f(s, x^*(s; t, x)) - \right.$$

$$\left.- f(s, x(s; t, x))|\,\mathrm{d}s + L\int_t^{t+h} |G(s, x^*(s; t, x))|\,\mathrm{d}v_u(s)\right\} +$$

$$+ \exp\left(-L\int_0^t \omega(s)\,\mathrm{d}s\right)\{-\beta c V(t, x) - L\,\omega(t)\,V(t, x)\} \leqq$$

$$\leqq \overline{\lim_{h\to 0+}} \frac{1}{h} \exp\left(-L\int_0^{t+h} \omega(s)\,\mathrm{d}s\right)\left\{LM \sup_{t\leqq s\leqq t+h} |x^*(s; t, x) - x(s; t, x)|\,h + \right.$$

$$\left. + L \sup_{t\leqq s\leqq t+h} |x^*(s; t, x)| \int_t^{t+h} g(s)\,\mathrm{d}v_u(s)\right\} +$$

$$+ \exp\left(-L\int_0^t \omega(s)\,\mathrm{d}s\right)\{-\beta c V(t, x) - L\,\omega(t)\,V(t, x)\} =$$

$$= \exp\left(-L\int_0^t \omega(s)\,\mathrm{d}s\right)\{L\,\omega(t)\,|x| - \beta c V(t, x) - L\,\omega(t)\,V(t, x)\} \leqq$$

$$\leqq \exp\left(-L\int_0^t \omega(s)\,\mathrm{d}s\right) L\,\omega(t)\,V(t, x) - \beta c V(t, x) - L\,\omega(t)\,V(t, x) =$$

$$= -\beta c W(t, x)\,.$$

Since $x^*(.\,; t_0, x_0)$ is continuous on $[t_{k-1}, t_k)$, $k = 1, 2, \ldots$, we have, by [6, Th. 4.1],

$$W(t, x^*(t; t_0, x_0)) \leqq W(t_{k-1}, x^*(t_{k-1}, t_0, x_0)) \exp(-\beta c(t - t_{k-1})) \quad \text{for} \quad t \in [t_{k-1}, t_k)\,. \tag{3.5}$$

We have,

$$(3.6) \qquad x^*(t; t_0, x_0) - x^*(t-; t_0, x_0) =$$

$$= \lim_{h\to 0+} \left\{ \int_{t-h}^{t} f(s, x^*(s; t_0, x_0))\, ds + \int_{t-h}^{t} G(s, x^*(s; t_0, x_0))\, du(s) \right\}.$$

The first limit on the right is zero; and we shall prove that

$$(3.7) \quad \lim_{h\to 0+} \left| \int_{t-h}^{t} G(s, x^*(s; t_0, x_0))\, du(s) \right| = |G(t, x^*(t; t_0, x_0))\,(u(t) - u(t-))|.$$

Consider the positive set function μ defined by

$$\mu(A) = \left| \int_A G(s, x^*(s; t_0, x_0))\, du(s) \right|.$$

Let $h_1 \geqq h_2 \geqq h_3 \geqq \ldots > 0$ and let $A_k = [t - h_k, t]$ and $h_k \to 0$ as $k \to \infty$. Then $A_1 \supset A_2 \supset A_3 \supset \ldots$ and $\bigcap_{k-1}^{\infty} A_k = A_0$ where $A_0 = \{t\}$. Therefore, $\mu(A_k) \to \mu(A_0)$ (RUDIN [7], Th. 1.19(e)). But $\mu(A_0) = |G(t, x^*(t; t_0, x_0))\,(u(t) - u(t-))|$, by MUNROE [8, Ex. s, p. 199]. Therefore (3.7) is established; and we have, from (3.6),

$$(3.8) \qquad |x^*(t; t_0, x_0) - x^*(t-; t_0, x_0)| = |G(t, x^*(t; t_0, x_0))\,(u(t) - u(t-))|.$$

Now,

$$(3.9) \qquad |W(t, x^*(t; t_0, x_0)) - W(t, x^*(t-; t_0, x_0))| =$$

$$= \lim_{h\to 0+} |W(t, x^*(t; t_0, x_0)) - W(t-h, x^*(t-h; t_0, x_0))| =$$

[since $W(t, x)$ is continuous and $x^*(.)$ is right continuous]

$$= \exp\left(-L\int_0^t \omega(s)\, ds\right) |V(t, x^*(t; t_0, x_0)) - V(t, x^*(t-; t_0, x_0))| \leqq$$

$$\leqq \exp\left(-L\int_0^t \omega(s)\, ds\right) L|x^*(t; t_0, x_0) - x^*(t-; t_0, x_0)| \leqq$$

$$\leqq \exp\left(-L\int_0^t \omega(s)\, ds\right) L|G(t, x^*(t; t_0, x_0))|\, |u(t) - u(t-)| \leqq$$

[by (3,8)]

$$\leqq \exp\left(-L\int_0^t \omega(s)\, ds\right) L\, g(t)\, |x^*(t; t_0, x_0)|\, |u(t) - u(t-)| \leqq$$

$$\leqq \exp\left(-L\int_0^t \omega(s)\, ds\right) L\varrho\, g(t)\, |u(t) - u(t-)|.$$

Therefore,

$$(3.10) \qquad W(t_k, x^*(t_k; t_0, x_0)) \leqq$$

$$\leqq W(t_k, x^*(t_k-; t_0, x_0)) + \exp\left(-L\int_0^{t_k} \omega(s)\,\mathrm{d}s\right) L\varrho\, g(t_k)\,|u(t_k) - u(t_k-)| \leqq$$

$$\leqq W(t_{k-1}, x^*(t_{k-1}; t_0, x_0))\, e^{-\beta c(t_k - t_{k-1})} + \exp\left(-L\int_0^{t_k} \omega(s)\,\mathrm{d}s\right) \alpha L\varrho\, g(t_k)\, e^{-c(t_k - t_0)} \leqq$$

[by (3.5) and condition (4) of the theorem]

$$\leqq e^{-\beta c(t_k - t_{k-1})}\, W(t_{k-1}, x^*(t_{k-1}; t_0, x_0)) + \alpha L\varrho\, g(t_k)\, e^{-\beta c(t_k - t_0)}, \quad \text{since} \quad 0 < \beta < 1\,.$$

Now, from (3.5),

$$(3.11) \qquad W(t, x^*(t; t_0, x_0)) \leqq W(t_0, x_0)\, e^{-\beta c(t - t_0)} \quad \text{for} \quad t \in [t_0, t_1)$$

and

$$W(t, x^*(t; t_0, x_0)) \leqq W(t_1, x^*(t_1; t_0, x_0))\, e^{-\beta c(t - t_1)} \quad \text{for} \quad t \in [t_1, t_2)$$

$$\leqq [W(t_0, x_0) + \alpha L\varrho\, g(t_1)]\, e^{-\beta c(t - t_0)} \quad \text{for} \quad t \in [t_0, t_2)\,,$$

by (3.10) and (3.11).

In general,

$$W(t, x^*(t; t_0, x_0)) \leqq [W(t_0, x_0) + L\varrho\alpha \sum_{k=1}^{\infty} g(t_k)]\, e^{-\beta c(t - t_0)}\,.$$

This implies

$$|x^*(t; t_0, x_0)| \exp\left(-L\int_0^t \omega(s)\,\mathrm{d}s\right) \leqq$$

$$\leqq \left[K|x_0| \exp\left(-L\int_0^{t_0} \omega(s)\,\mathrm{d}s\right) + L\varrho\alpha \sum_{k=1}^{\infty} g(t_k)\right] e^{-\beta c(t - t_0)}\,;$$

therefore,

$$(3.12) \quad |x^*(t; t_0, x_0)| \leqq [K|x_0| + L\varrho\alpha \sum_{k=1}^{\infty} g(t_k)] \exp\left(L\int_0^{\infty} \omega(s)\,\mathrm{d}s\right) e^{-\beta c(t - t_0)}.$$

Let $\delta_0 (\leqq \varrho)$ be such that if $|x_0| \leqq \delta_0$ then

$$K|x_0| \exp\left(L\int_0^{\infty} \omega(s)\,\mathrm{d}s\right) < \delta_1 \quad \text{where } 0 < \delta_1 < \varrho\,,$$

and let α be such that

$$L\varrho\alpha \sum_{k=1}^{\infty} g(t_k) \exp\left(L\int_0^{\infty} \omega(s)\,\mathrm{d}s\right) \leqq \varrho - \delta_1\,.$$

The theorem then follows from (3.12).

I. Distributions. Let Ω be an open set of R^n. By $C_c^\infty(\Omega)$ we denote the set of all complex functions on Ω whose support is compact and which have partial derivatives of all orders $< \infty$. $C_c^\infty(\Omega)$ is a normed linear space with addition, scalar multiplication and norm defined by

$$(\varphi_1 + \varphi_2)(x) = \varphi_1(x) + \varphi_2(x), \quad (\alpha\varphi)(x) = \alpha\, \varphi(x), \quad \|\varphi\| = \sup_{x\in\Omega} |\varphi(x)| .$$

A continuous linear functional defined on $C_c^\infty(\Omega)$ is called a distribution on Ω. It follows from Riesz representation theorem that the set of all complex regular Borel measures on Ω is, by $\mu \leftrightarrow F_\mu$, in one-one correspondence with the set of all distributions on Ω, where F_μ is the distribution defined by $F_\mu(\varphi) = \int_\Omega \varphi \, \mathrm{d}\mu$ $(\varphi \in C_c^\infty(\Omega))$.

Let a complex function f defined a.e. on Ω be locally integrable on Ω with respect to the Lebesgue measure in the sense that for any compact subset K of Ω, $\int_K |f(x)| \, \mathrm{d}x < < \infty$. Then

$$F_f(\varphi) = \int_\Omega f(x)\, \varphi(x) \, \mathrm{d}x , \quad \varphi \in C_c^\infty(\Omega) ,$$

defines a distribution F_f on Ω. Two distributions F_{f_1} and F_{f_2} are equal as functionals $(F_{f_1}(\varphi) = F_{f_2}(\varphi)$ for every $\varphi \in C_c^\infty(\Omega))$ if and only if $f_1(x) = f_2(x)$ a.e. (see [9], p. 48).

The derivative of a distribution F with respect to x^i, denoted by $\mathrm{D}_i F$ or $\partial F/\partial x^i$, is defined by

$$\mathrm{D}_i F(\varphi) = -F(\partial\varphi/\partial x^i) , \quad \varphi \in C_c^\infty(\Omega)$$

and is also a distribution on Ω. A distribution is infinitely differentiable in the sense of above definition.

Since a locally integrable function f on an open interval I of real line can be identified with the distribution F_f on I, $\mathrm{D}F_f(\equiv \mathrm{d}F_f/\mathrm{d}t)$ will be denoted by $\mathrm{D}f$ and called distributional derivative of f to distinguish from its ordinary derivative $f'(\equiv \mathrm{d}f/\mathrm{d}t)$. If f is absolutely continuous, then $\mathrm{D}f$ is the ordinary derivative f' (which is defined a.e.), f' being considered equivalent to the distribution $F_{f'}$. If f is of bounded variation then $\mathrm{D}f$ is the L. S. measure $\mathrm{d}f$.

II. Integral Form of Measure Differential Equation. We shall show the equivalence of the equations (2.5) and (2.6). For this we shall use the following lemma.

Lemma. *If g is a function integrable with respect to μ, and F is a distribution on Ω given by*

$$F(\varphi) = \int_\Omega \varphi \, \mathrm{d}\mu , \quad \varphi \in C_c^\infty(\Omega) , \tag{1}$$

then the product gF defined by

$$(2)\qquad (gF)(\varphi) = \int_\Omega g\varphi \, \mathrm{d}\mu\,, \quad \varphi \in C_c^\infty(\Omega)$$

is also a distribution on Ω.

Proof. Since $\varphi \in C_c^\infty(\Omega)$, it is bounded and μ-measurable and g is given to be μ-integrable. Therefore, by Munroe [8, Ex. i, p. 184], $g\varphi$ is μ-integrable. Thus the right hand side of (2) is meaningful. gF defined by (2) is obviously a linear functional on $C_c^\infty(\Omega)$. Furthermore,

$$|(gF)|(\varphi) \leqq \int_\Omega |g|\,|\varphi|\, d|\mu| \leqq \|\varphi\| \int_\Omega |g| \, \mathrm{d}|\mu|$$

where $|\mu|$ denotes the total variation measure of μ.

Therefore,

$$\|gF\| = \sup\{|(gF)(\varphi)| : \|\varphi\| \leqq 1\} \leqq \int_\Omega |g| \, d|\mu| < \infty\,,$$

since μ-integrability of g implies $|\mu|$-integrability of $|g|$, by DUNFORD-SCHWARTZ [10, Lemma 18, p. 113]. Thus gF is bounded (and hence continuous) linear functional on $C_c^\infty(\Omega)$, and is, therefore, a distribution on Ω.

Theorem. $x(.)$ *is a solution of* (2.5) *through* (t_0, x_0) *on an interval* I, *with left end point* t_0, *if and only if* $x(.)$ *satisfies* (2.6) *for* $t \in I$.

Proof. Let $x(.)$ satisfy (2.6) for $t \in I$. The integral $\int_{t_0}^t f(s, x(s))\, \mathrm{d}s$ is absolutely continuous (and hence continuous and of bounded variation) function of t on I. The integral $\int_{t_0}^t G(s, x(s))\, \mathrm{d}u(s)$ is a function of bounded variation on I and the right continuity of u implies that it is also right continuous function of t. Therefore, $x(.) \in BV(I, S)$ and is right continuous. Obviously, $x(t_0) = x_0$. Let T be any arbitrary point in I, and let F^i be the distribution on (t_0, T) to be identified with the i-th component $x^i(.)$ of $x(.)$. Then

$$(3)\qquad F^i(\varphi) = \int_J \left(x_0^i + \int_{t_0}^t f^i(s, x(s))\, \mathrm{d}s + \int_{t_0}^t [G(s, x(s))\, \mathrm{d}u(s)]^i \right) \varphi(t)\, \mathrm{d}t$$

for all $\varphi \in C_c^\infty(J)$ where $J = (t_0, T)$. The distributional derivative is

$$(4)\qquad \mathrm{D}F^i(\varphi) = -F^i(\varphi') =$$

$$= -\int_J \left[x_0^i + \int_{t_0}^t f_i(s, x(s))\, \mathrm{d}s + \int_{t_0}^t \Big(\sum_{j=1}^m G_j^i(s, x(s))\, \mathrm{d}u^j(s)\Big) \right] \varphi'(t)\, \mathrm{d}t$$

where $G_j^i(t, x)$ is i, j-th element of $G(t, x)$ and $u^j(t)$ is j-th component of $u(t)$.

Integration by parts yields

$$(5) \qquad -\int_J \left[x_0^i + \int_{t_0}^t f^i(s, x(s))\, ds\right] \varphi'(t)\, dt = \int_J \varphi(t) f^i(t, x(t))\, dt$$

since $\varphi(t_0) = \varphi(T) = 0$.

The function $g(t) = \int_{t_0}^t G_j^i(s, x(s))\, du^j(s)$ is right continuous and is of bounded variation on the interval $J = (t_0, T)$. We have

$$\int_J g(t)\, \varphi'(t)\, dt = \int_{(t_0,T)} g(t)\, d\varphi(t) = \int_{(t_0,T]} g(t)\, d\varphi(t) - \int_{\{T\}} g(t)\, d\varphi(t)\,.$$

But $\int_{\{T\}} g(t)\, d\varphi(t) = 0$, since φ is continuous; and

$$\int_{(t_0,T]} g(t)\, d\varphi(t) = g(T)\, \varphi(T) - g(t_0)\, \varphi(t_0) - \int_{(t_0,T]} \varphi(t)\, dg(t) =$$

[by Munroe [8, Ex. n, p. 185]]

$$= -\int_{(t_0,T]} \varphi(t)\, dg(t) =$$

[since $\varphi(t_0) = \varphi(T) = 0$]

$$= -\int_{(t_0,T)} \varphi(t)\, dg(t) - \int_{\{T\}} \varphi(t)\, dg(t) = -\int_J \varphi(t)\, dg(t) -$$

$$- \varphi(T)\,(g(T) - g(T-)) = -\int_J \varphi(t)\, dg(t)\,,$$

[since $\varphi(T) = 0$]. Therefore,

$$\int_J g(t)\, \varphi'(t)\, dt = -\int_J \varphi(t)\, dg(t)\,.$$

That is,

$$\int_J \left\{\int_{t_0}^t G_j^i(s, x(s))\, du^j(s)\right\} \varphi'(t)\, dt = -\int_J \varphi(t)\, d\left\{\int_{t_0}^t G_j^i(s, x(s))\, du^j(s)\right\} =$$

$$= -\int_J \varphi(t)\, G_j^i(s, x(s))\, du^j(s)\,,$$

by Dunford-Schwartz [10, Cor. 6, p. 180]. Summation over j in the above equation yields

$$(6) \quad \int_J \left\{\int_{t_0}^t \left(\sum_{j=1}^m G_j^i(s, x(s))\, du^j(s)\right)\right\} \varphi'(t)\, dt = -\int_J \varphi(t) \left(\sum_{j=1}^m G_j^i(s, x(s))\, du^j(s)\right).$$

From (4), (5), and (6), we obtain

$$(7) \qquad DF^i(\varphi) = \int_J \varphi(t) f^i(t, x(t))\, dt + \int_J \varphi(t)\, [G(t, x(t))\, du(t)]^i\, dt\,.$$

By the above Lemma, the last continuous linear functional in (7) is identified with the measure $G(t, x(t))\, \mathrm{d}u(t)$. The first continuous linear functional in (7) is identified with $f(t, x(t))$. Thus the derivative $\mathrm{D}F(\varphi)$ is identified with $f(t, x(t)) + G(t, x(t))\, \mathrm{D}u$. Hence $x(.)$ is a solution of (2.5) through (t_0, x_0).

Conversely, let $x(.)$ be a solution of (2.5) through (t_0, x_0) on the interval I. Then for $J = (t_0, T)$, where T is an arbitrary point in I, we have

$$(8) \qquad \int_J \varphi(t)\, \mathrm{d}x^i(t) = \int_J \varphi(t) f^i(t, x(t))\, \mathrm{d}t + \int_J \varphi(t) \left[G(t, x(t))\, \mathrm{d}u(t)\right]^i \mathrm{d}t\,, \quad i = 1, 2, \ldots, n$$

for all $\varphi \in C_c^\infty(J)$. Integrating the left hand side of (8) by parts and using (5) and (6) we obtain

$$\int_J \varphi'(t) \left(x^i(t) - x_0^i\right) \mathrm{d}t = \int_J \varphi'(t) \left\{ \int_{t_0}^t f^i(s, x(s))\, \mathrm{d}s + \int_{t_0}^t \left[G(s, x(s))\, \mathrm{d}u(s)\right]^i \right\} \mathrm{d}t\,.$$

Therefore,

$$(9) \qquad x^i(t) = x_0^i + \int_{t_0}^t f^i(s, x(s))\, \mathrm{d}s + \int_{t_0}^t \left[G(s, x(s))\, \mathrm{d}u(s)\right]^i \mathrm{d}s$$

a.e. in J. But, since $x^i(.)$ is right continuous, $x(.)$ being solution of (2.5), and since the right hand side of (9) is a right continuous function of t, equality holds every where in J in (9). Thus $x(.)$ satisfies (2.6) for $t \in I$. This completes the proof.

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