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ON JUMPING FUNCTIONS BY CONNECTED SETS

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1. INTRODUCTION

In recent years, a number of authors have considered various questions concerning Darboux functions and functions with connected graphs. A survey of these studies can be found in [1]. Every function having a connected graph possesses the Darboux (intermediate value) property, but the converse is not true in general, although it is true for functions in the first class of Baire. In the present article we incorporate the class $\mathscr{D}$ of Darboux functions and the class $\mathscr{K}$ of functions with connected graphs into a hierarchy of four classes of functions, including also the continuous functions and the "weakly continuous functions". Each of these classes can be characterized in a natural, unifying way by separation or "jumping" properties. As a class of functions is replaced with a smaller class, the class of connected sets which are used for separation or jumping is enlarged. Thus, for example, a Darboux function can be looked upon as one whose graph cannot be separated over any interval by a horizontal segment, while a function in $\mathscr{K}$ can be looked upon [4] as one whose graph cannot be separated over any interval by a continuum.

The main purpose of this article is to study the four classes of functions in terms of separation and jumping properties. We do the same for the uniform closures of these classes [Section 3]. We also consider the inclusion relationships which exist among these four classes (and their closures) [Section 4], and certain approximation and representation theorems for arbitrary functions in terms of functions in the classes [Section 5]. In this connection, we extend certain results known for functions in $\mathscr{D}$.

2. PRELIMINARIES

Throughout the sequel we deal exclusively with real-valued functions of a real variable. Moreover we will regard a function as identical with its graph.

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A function f defined over an interval I is said to be *bilaterally dense in itself* (resp. *bilaterally c-dense in itself*) provided for each $z \in f$, each open square, which has a vertical side bisected by z and whose x-projection is contained in I, contains infinitely many points (resp. c points) of f.

If f is defined over an interval I and $\varepsilon > 0$, then the *ε-strip of f over I* is defined to be the set

$$S_\varepsilon(f) = \{(x, y) : x \in I \text{ and } |y - f(x)| < \varepsilon\}.$$

A *segment* of a strip $S_\varepsilon(f)$ is the non-empty intersection of $S_\varepsilon(f)$ with a vertical line.

For $(c, d) \in R^2$ define vertical open rays as follows:

$$R_+(c, d) = \{(x, y) : x = c \text{ and } y > d\}$$
$$R_-(c, d) = \{(x, y) : x = c \text{ and } y < d\}.$$

We say that a connected set C *jumps the ε-strip $S_\varepsilon(f)$ over $J = [a, b]$* provided:

(1) the x-projection of C contains J

and

(2) $R_+(a, f(a) + \varepsilon)$ and $R_-(b, f(b) - \varepsilon)$ both hit C

or

$R_-(a, f(a) - \varepsilon)$ and $R_+(b, f(b) + \varepsilon)$ both hit C.

We say a connected set C *jumps a function f over $J = [a, b]$* provided:

(1) the x-projection of C contains J

and

(2) $R_+(a, f(a))$ and $R_-(b, f(b))$ both hit C

or

$R_-(a, f(a))$ and $R_+(b, f(b))$ both hit C.

We say that a function f *cannot be separated over an interval I by a certain class $\mathscr{A}$* of connected sets provided for each closed subinterval J of I, no $A \in \mathscr{A}$ separates f in the space $J \times R$ (that is, $f \cap A = \emptyset$ implies f does not intersect more than one component of $(J \times R) - A$ in the space $J \times R$). A set B is said to be *weakly connected* over an interval I if it cannot be separated over I by an arc (i.e., a homeomorph of $[0, 1]$). The class of weakly connected functions over a given interval is denoted by $\mathscr{U}$. The class of continuous functions is denoted by $\mathscr{C}$. The class of *connected* functions (i.e., functions whose graphs are connected in R^2) is denoted by $\mathscr{K}$. A *Darboux* function is one which has the intermediate value property. The class of Darboux functions is denoted by $\mathscr{D}$. Clearly $\mathscr{C} \subset \mathscr{K} \subset \mathscr{U} \subset \mathscr{D}$. (Also see Theorem 5.)

3. CHARACTERIZATIONS OF THE CLASSES AND THEIR CLOSURES

First we establish the basic unifying relationship between these four families in terms of separations by connected sets and jumpings. Part (C) is a restatement of Theorem 2 of GARRETT, NELMS and KELLUM [4].

Theorem 1. *Let f be defined on an interval I.*

(A) *f is Darboux* $\Leftrightarrow$ *f cannot be separated over I by a horizontal line segment*
$\Leftrightarrow$ *over each closed subinterval any horizontal line segment which jumps f must hit f.*

(B) *f is weakly connected* $\Leftrightarrow$ *f cannot be separated over I by an arc*
$\Leftrightarrow$ *over each closed subinterval any arc which jumps f must hit f.*

(C) *f is connected* $\Leftrightarrow$ *f cannot be separated over I by a continuum*
$\Leftrightarrow$ *over each closed subinterval any continuum which jumps f must hit f.*

(D) *f is continuous* $\Leftrightarrow$ *over each closed subinterval any connected set which jumps f must hit f.*

Proof. (A) is obvious from the definition of a Darboux function. The proof of part (B) is similar to the proof of part (C) (see [4]).

Hence, it suffices to prove the non-trivial implication of (D). Suppose each connected set which jumps f hits f. Then f must be bilaterally c-dense in itself. To see this, suppose S is a square having $(x, f(x))$ as the center of one of its vertical sides. If $S \cap f$ has cardinality less than c, there clearly exists a line segment M missing f and having end points in the vertical sides of S. This M jumps f but does not hit it. Hence, f is bilaterally c-dense in itself. Moreover, $R^2 - f$ is disconnected because if not, it is a connected set which jumps f but does not hit it.

So assume that $R^2 - f$ is disconnected by disjoint open sets A and B, both of which hit $R^2 - f$ and whose union contains $R^2 - f$. Without loss of generality we may assume bndry A = bndry B, which set we denote by ∂. Clearly $\partial \subseteq f$.

Suppose $x \in A$ and $y \in B$ with $x_1 \neq y_1$ and J is the open interval determined by x_1 and y_1. Since ∂ separates x and y we know that there exists a component M of ∂ which separates x and y. But M is closed and $M \subseteq f$ so that each point (except possibly two) of M is a cut point. This implies that M is an arc whose x-projection contains J. Therefore, $f \mid J$ is continuous and is contained in ∂.

Let $\lambda = \inf\{x_1 : x \in A\}$ and $\mu = \sup\{x_1 : x \in B\}$. From the above, f will be continuous on $(-\infty, +\infty)$ if $\lambda = -\infty$ and $\mu = +\infty$. So suppose λ is finite. Then $f \mid (\lambda, \mu)$ is continuous and is contained in ∂.

Next we need the following easily proved results: (1) for each $z \in f$, $R_+(z)$ and $R_-(z)$ belong entirely to A or B and (2) if $z_n \to z$ with $z_n \in f$ and $z \in f$ and $R_+(z_n) \subseteq A$, then $R_+(z) \subseteq A$.

Now let $z_0 = (\lambda, f(\lambda))$. Since $\{x : x_1 < \lambda\} \subseteq B$ by assumption, we have by (2) $R_+(z_0) \cup R_-(z_0) \subseteq B$. Because $f \mid (\lambda, \mu) \subseteq \partial$ and f is bilaterally dense in itself, we have $z_0 \in \partial = \text{bndry } A$. Hence, there exists $z_n \in f \mid (\lambda, \mu)$ such that $z_n \to z_0$ and $R_+(z_n) \subseteq A$ for all n (or $R_-(z_n) \subseteq A$ for all n). Then by (1) we have $R_+(z_0) \subseteq A$, which is a contradiction.

Therefore, $\lambda = -\infty$ and analogously $\mu = +\infty$ which finishes the proof. Note that the above proof can be shortened and simplified considerably in case f is bounded.

In Theorem 1, part (D), we cannot assert that f is continuous if and only if f cannot be separated over I by a connected set. For example, the function f given by $f(x) = 0$ when $x \leqq 0$ and $f(x) = \sin(1/x)$ when $x > 0$ is discontinuous yet possesses the property that it cannot be separated by a connected set.

Therefore, the separation property is appropriate when dealing with closed, connected sets, but not with connected sets in general. The jumping property, on the other hand, is appropriate to all connected sets and yields characterizations of the given families (Theorem 1) and of the uniform limits of some of the given families (Theorems 2, 3, and 4).

We now attempt to obtain similar characterizations for the uniform limits of sequences in $\mathscr{D}$, $\mathscr{U}$, $\mathscr{K}$, and $\mathscr{C}$, or equivalently to characterize the closure $\overline{\mathscr{D}}$, $\overline{\mathscr{U}}$, $\overline{\mathscr{K}}$, and $\overline{\mathscr{C}}$ (in the metric space of uniform convergence) of these families. It turns out that $\overline{\mathscr{D}} = \overline{\mathscr{U}}$ (Theorem 5) and, of course, $\mathscr{C} = \overline{\mathscr{C}}$. The characterizations we obtain for $\overline{\mathscr{D}}$, $\overline{\mathscr{U}}$, and $\overline{\mathscr{C}}$ are similar and parallel in the sense of the jumping relationship expressed in Theorem 1.

We begin by stating the characterization of $\overline{\mathscr{D}}$ which was established, although expressed in different terms, in BRUCKNER, CEDER and WEISS [3].

Theorem 2. *Let f be defined on an interval I.*

$f \in \overline{\mathscr{D}} \Leftrightarrow (\alpha_1)$: *over any closed subinterval J any horizontal line segment which jumps any $S_\varepsilon(f)$ hits it in c points*

$\Leftrightarrow (\alpha_2)$: *$f$ is bilaterally c-dense in itself and over each closed subinterval any horizontal line segment which jumps $S_\varepsilon(f)$ hits it.*

The analogue of Theorem 2 for weakly connected functions is the following theorem. In view of the fact that $\overline{\mathscr{D}} = \overline{\mathscr{U}}$ (Theorem 5) (α_i) is equivalent to (β_i), $i = 1, 2$.

Theorem 3. *Let f be defined on an interval I.*

$f \in \overline{\mathscr{U}} \Leftrightarrow (\beta_1)$: *over each closed subinterval each arc which jumps any $S_\varepsilon(f)$ hits it in c distinct segments of $S_\varepsilon(f)$ or contains an entire segment of $S_\varepsilon(f)$*

$\Leftrightarrow (\beta_2)$: *$f$ is bilaterally c-dense in itself and over each closed subinterval any arc which jumps any $S_\varepsilon(f)$ hits it (i.e., any $S_\varepsilon(f)$ is weakly connected).*

Before proving Theorem 3 we need the following lemma.

Lemma 1. *Let $\mathscr{M}$ be the family of all arcs which intersect c distinct segments of $S_\varepsilon(f)$. Then there exists a g such that $\|f - g\| < \varepsilon$ and for each $M \in \mathscr{M}$, $g \cap \cap M \neq \emptyset$.*

Proof. Well-order $\mathscr{M}$ as $\{M_\alpha\}_{\alpha<c}$ and well-order the domain interval I by a relation T. Let π be the x-projection mapping.

Choose x_0 to be the T-first point in $\pi(M_0 \cap S_\varepsilon(f))$ and define $g(x_0)$ so that $(x_0, g(x_0)) \in M_0 \cap S_\varepsilon(f)$. Now assume we have defined x_α for all $\alpha < \beta$ such that $x_\alpha \neq x_\gamma$ when $\alpha \neq \gamma$ and such that $(x_\alpha, g(x_\alpha)) \in M_\alpha \cap S_\varepsilon(f)$.

Consider β and let x_β be the first T-point in $\pi(M_\beta \cap S_\varepsilon(f)) - \{x_\xi : \xi < \beta\}$, which is non-empty by assumption. Then define $g(x_\beta)$ such that $(x_\beta, g(x_\beta)) \in M_\beta \cap S_\varepsilon(f)$. For $x \notin \{x_\alpha : \alpha < c\}$ we put $g(x) = f(x)$. Then it is easily checked that g is the desired function.

Proof of Theorem 3. We will prove the following chain of inequalities: $f \in \overline{\mathscr{U}} \Rightarrow \Rightarrow (\beta_2) \Rightarrow (\beta_1) \Rightarrow f \in \overline{\mathscr{U}}$.

(I) To prove $f \in \overline{\mathscr{U}} \Rightarrow (\beta_2)$, assume $f \in \overline{\mathscr{U}}$. First we show that f is bilaterally c-dense in itself. Suppose f is not bilaterally c-dense at x. Then there exist two open squares S_1 and S_2 of side length 3ε and ε respectively each having $(x, f(x))$ as the midpoint of its (say) right side, such that card $(S_1 \cap f) < c$. Now choose $g \in \mathscr{U}$ such that $\|f - g\| < \varepsilon$. Then card $(g \cap S_2) < c$ so g is not bilaterally c-dense at x. But each weakly connected function is bilaterally c-dense in itself. Hence, f is bilaterally c-dense in itself.

Now suppose M is an arc which jumps $S_\varepsilon(f)$ over $J = [a, b]$. Select $g \in \mathscr{U}$ with $g \subseteq S_\varepsilon(f)$. Then M must jump g so that $M \cap S_\varepsilon(f) \neq \emptyset$.

(II) To prove $(\beta_2) \Rightarrow (\beta_1)$ suppose M is an arc which jumps $S_\varepsilon(f)$ over the interval $[a, b]$ and which hits less than c segments of $S_\varepsilon(f)$. We may assume that $(a, f(a))$ and $(b, f(b))$ are the end points of M. Hence, there is a non-end point z of M which belongs to some segment V of $S_\varepsilon(f)$. We will show that $V \subseteq M$.

Let $z \in M \cap V$. Since M is an arc and z is not an end point of M, in each neighborhoood of z there is an open subarc of M containing z. However, since f is bilaterally c-dense in itself there exist two sequences of segments which miss M, one approaching V from the left and the other from the right (in case z projects to a or b, then there is only one sequence). Hence, any such open subarc must be contained entirely in V. Therefore, $M \cap V$ is open in V and since M is closed it follows that $V \subseteq M$.

(III) To prove $(\beta_1) \Rightarrow f \in \overline{\mathscr{U}}$, we first prove that f is bilaterally c-dense in itself. Suppose S is an open square having $(a, f(a))$ as the center of say the left vertical side and having the other vertical side with x-coordinate b. Suppose card $(S \cap f) < c$. Then clearly one can find a line segment M in S having ends in the vertical sides which jumps f over $[a, b]$. For a small enough ε it follows that M hits less than c segments of $S_\varepsilon(f)$ over $[a, b]$ which contradicts (β_1). Therefore f is bilaterally c-dense in itself.

To show $f \in \overline{\mathscr{U}}$ it suffices to show that each $S_\varepsilon(f)$ contains a member of $\mathscr{U}$. Given

$\varepsilon > 0$ construct $g \subseteq S_\varepsilon(f)$ by Lemma 1. Next we show that $g \in \mathscr{U}$. Let J be a subinterval and M be an arc in $J \times R$ such that $M \cap g = \emptyset$ in $J \times R$. It clearly suffices to show that g intersects only one component of $(J \times R) - M$ in $J \times R$. Suppose $(a, g(a))$ and $(b, g(b))$ belong to different components. We may assume that there is a point of M above $(a, g(a))$ and a point of M below $(b, g(b))$. Since $M \cap g = \emptyset$, M does not hit c segments of $S_\varepsilon(f)$ by construction of g. So M hits less than c segments of $S_\varepsilon(f)$, which by (β_1) implies that M does not jump $S_\varepsilon(f)$ over any subinterval of (a, b). Therefore there exists a set A with card $A < c$ such that either for all $x \in (a, b) - A$, $R_+(x, f(x) - \varepsilon) \cap M = \emptyset$ or for all $x \in (a, b) - A$, $R_-(x, f(x) + \varepsilon) \cap M = \emptyset$. Suppose the former case holds. Then there exists a sequence $\{x_n\}_{n=1}^\infty$ of points in $(a, b) - A$ such that $x_n \to a$ and $R_+(x_n, f(x_n) - \varepsilon) \cap M = \emptyset$ and, by virtue of the bilateral c-denseness of f, $f(x_n) \to f(a)$. Since M is a continuum and there exists a point (a, m) above $(a, g(a))$, it is easily seen that the vertical segment from (a, m) to $(a, f(a) - \varepsilon)$ lies entirely in M and contains a point of g. This is a contradiction, so that g hits only one component of $(J \times R) - M$ and hence $g \in \mathscr{U}$.

The analogue of Theorems 2 and 3 for continuous functions will, because $\overline{\mathscr{C}} = \mathscr{C}$, yield another characterization of continuity which is formally stronger than the one obtained in Theorem 1. First we need the following lemma.

Lemma 2. *Suppose f is bilaterally dense in itself and each $S_\varepsilon(f)$ contains a closed connected set M_ε whose x-projection is I, then f is continuous over I.*

Proof. Suppose f is discontinuous at $x_0 \in I$. Then there exists a cluster value λ at x_0 which is different from $f(x_0)$. Suppose λ is finite. Take $\varepsilon = \frac{1}{3}|\lambda - f(x_0)|$. Then clearly both ε-vertical open intervals about (x_0, λ) and $(x_0, f(x_0))$ contain a point of the closed set M_ε which contradicts the fact that $M_\varepsilon \subseteq S_\varepsilon(f)$. So there are no finite cluster values at x_0 other than $f(x_0)$.

Now suppose λ is an infinite cluster value, say $\lambda = +\infty$, and there exists a sequence $\{x_n\}$ such that $x_n > x_0$ and $f(x_n) \to \infty$. Since f is bilaterally dense in itself there exist $y_n > x_0$ such that $y_n \to x_0$ and $f(y_n) \to f(x_0)$. Since there are no finite cluster values there exists a closed square S of length r less than 1 having left vertical side centered at $(x_0, f(x_0) + 4)$ such that $S \cap f = \emptyset$. Pick k such that x_k and y_k are both in $(x_0, x_0 + r)$ and $f(x_k) > f(x_0) + 8$ and $f(y_k) < f(x_0) + 2$. Then $S \cup R_+(y_k, f(y_k)) \cup R_-(x_k, f(x_k))$ is a connected, closed set separating M_1. Hence, M_1 is not connected, which is a contradiction. Hence, there is no infinite cluster value. Therefore f is continuous in I.

Theorem 4. *Let f be defined on an interval I.*

$f \in \mathscr{C} \Leftrightarrow (\xi_2)$: f is bilaterally dense in itself and over each closed subinterval any connected set which jumps any $S_\varepsilon(f)$ hits it.

Proof. Obviously $f \in \mathscr{C} \Rightarrow (\xi_2)$. To prove that (ξ_2) implies $f \in \mathscr{C}$ it will suffice to show that f is continuous whenever f is bilaterally dense in itself and $R^2 - S_\varepsilon(f)$ is disconnected for each $\varepsilon > 0$. The proof follows the pattern of part (D) of Theorem 1. Assume $R^2 - S_\varepsilon(f)$ is disconnected by open sets A and B with $\partial A = \partial B = \partial \subseteq S_\varepsilon(f)$. Letting $x \in A$ and $y \in B$ with $x_1 \neq y_1$ and J be the open interval determined by x_1 and y_1 we see that $S_\varepsilon(f)$ contains a closed, connected set M_ε whose x-projection contains J. Now defining λ and μ as before and proceeding as before we conclude that $S_\varepsilon(f)$ contains a closed, connected set M_ε. Now apply Lemma 2 to get f continuous.

The analogue of conditions (α_1) and (β_1) of Theorems 2 and 3 respectively for continuous functions does not hold. For example, consider the 0 function, $\varepsilon = 1$ and define M to be the union of the lines given by $y = 1$, $y = -1$ and the line segments $[(0, -1), (0, 0)]$ and $[(1/n, 0), (1/n, 1)]$ for positive n. Then M is a connected set which jumps $S_\varepsilon(f)$ but M neither intersects $S_\varepsilon(f)$ in c segments nor does it contain a whole segment.

We have been unable to prove completely the analogue of Theorems 2 and 3 for the class of connected functions $\mathscr{K}$. We state it as a conjecture.

Conjecture. *Let f be defined on an interval I. Then*

$f \in \mathscr{K} \Leftrightarrow (\gamma_1)$: *for sufficiently small $\varepsilon > 0$, for each closed subinteral each continuum which jumps $S_\varepsilon(f)$ hits it in c-segments or contains an entire segment of it*

$\Leftrightarrow (\gamma_2)$: *$f$ is bilaterally c-dense in itself and over each closed subinterval each continuum which jumps any $S_\varepsilon(f)$ hits it*

$\Leftrightarrow (\gamma_3)$: *$f$ is bilaterally c-dense in itself and each $S_\varepsilon(f)$ is connected.*

Using the methods of proof in Theorem 3 it is possible to prove that $(\gamma_1) \Rightarrow f \in \mathscr{K} \Rightarrow (\gamma_3) \Rightarrow (\gamma_2)$. Moreover it is easily seen that $(\gamma_2) \Leftrightarrow (\gamma_3)$ when f is bounded. The status of the missing link $(\gamma_2) \Rightarrow (\gamma_1)$ remains unsolved.

If (γ_2) does imply that $f \in \mathscr{K}$ it is necessary to have f bilaterally c-dense in itself (in contrast to the situation in Theorem 4). For example, let $f = 0$ on the irrationals and define f on the rationals so f is dense in the plane. Then f is bilaterally dense in itself but not bilaterally c-dense in itself. It is not difficult to verify that each $S_\varepsilon(f)$ is connected when f is dense in the plane. Moreover, $f \notin \mathscr{K}$, because, as the proof of $f \in \mathscr{U} \Rightarrow (\beta_2)$ shows, $f \in \mathscr{K}$ implies that f is bilaterally c-dense in itself.

It seems to be necessary to stipulate that the jumping condition of (γ_1) holds only for sufficiently small $\varepsilon > 0$. The Darboux function f constructed below does not satisfy the jumping condition of (γ_1) for all $\varepsilon > 0$, but it seems to satisfy (γ_1). (The analogous situation does not prevail in Theorem 2.)

To construct such a function f, let C be the Cantor set in $[0, 1]$. Decompose the family of intervals contiguous to C into two disjoint classes $\mathscr{A}$ and $\mathscr{B}$ such that $C \subseteq \overline{\bigcup \mathscr{A}}$ and $C \subseteq \overline{\bigcup \mathscr{B}}$. Define a continuum K by putting

$$K = (C \times [1, 8]) \cup \{[(a, 8), (b, 8)] : (a, b) \in \mathscr{A}\} \cup \{[(a, 1), (b, 1)] : (a, b) \in \mathscr{B}\}\,.$$

Define the function f as follows: for $(a, b) \in \mathscr{A}$ put $f(a) = 0 = f(b)$ and define f so that it takes on all numbers in $(0, 7)$ on each subinterval of (a, b); for $(a, b) \in \mathscr{B}$ put $f(a) = 9 = f(b)$ and define f so that it takes in all numbers in $(2, 9)$ on each subinterval of (a, b). On $D = [0, 1] - \bigcup\{[a, b] : (a, b) \in \mathscr{A} \cup \mathscr{B}\}$ define f so it takes in each rational value in $(1, 2) \cup (7, 8)$ in each relative subinterval of D $\aleph_0$ times, otherwise $f = 0$ on D.

Clearly $f \in \mathscr{D}$ and also $f \notin \mathscr{K}$ because the continuum K "when squeezed down" to 7 and up to 2 separates f. For $\varepsilon = 1$, K jumps $S_1(f)$ yet intersects only countably many segments of it and never contains an entire segment so that the jumping statement of (γ_1) is not satisfied for $\varepsilon = 1$. We conjecture that condition (γ_1) is satisfied for f whenever $\varepsilon < 1$.

The implication $(\gamma_2) \Rightarrow f \in \bar{\mathscr{K}}$ is true for bounded functions if the following question has an affirmative answer.

Question 1. *Suppose f_1, f_2 and f_3 are functions defined on the interval I such that $f_1 \cup f_2 \cup f_3$ is connected. Does there exist a connected function $h \subseteq f_1 \cup f_2 \cup f_3$ with domain I?*

To show that $(\gamma_2) \Rightarrow f \in \bar{\mathscr{K}}$ is true when I is bounded and the above question has an affirmative answer, let $\varepsilon > 0$ and construct g as in Lemma 1 for $\mathscr{M}$ the class of continua and for the strip $S_{\varepsilon/2}(f)$. Consider the set $A = g \cup (f + \frac{1}{2}\varepsilon) \cup (f - \frac{1}{2}\varepsilon)$. If A is connected, then there exists a connected function $h \subseteq S_\varepsilon(f)$.

If A is not connected there exists, since A is bounded, a continuum M separating A. Since $M \cap g = \emptyset$ we have by Lemma 1 that M intersects less than c segments of $S_{\varepsilon/2}(f)$. If $M \cap S_{\varepsilon/2}(f) = \emptyset$, we have by (γ_2) that M does not jump $S_{\varepsilon/2}(f)$ so every point of M is either below or above $S_{\varepsilon/2}(f)$, so that M cannot separate A. Therefore, some segment V of $S_{\varepsilon/2}(f)$ intersects M at (x, y). Then the segment joining (x, y) to $(x, f(x) + \frac{1}{2}\varepsilon)$ lies in M or the segment joining (x, y) to $(x, f(x) - \frac{1}{2}\varepsilon)$ lies in M. If this is not true then there exist disks D_1 and D_2 centred at points above (x, y) with the same radius such that $D_1 \cap M = \emptyset = D_2 \cap M$. Since f is bilaterally c-dense in itself and M hits less than c segments of $S_{\varepsilon,2}(f)$, it follows that there are two segments V_1 and V_2 both missing M and hitting D_1 and D_2 one on each side of V. Then $D_1 \cup D_2 \cup$ $\cup V_1 \cup V_2$ separates the continuum M, a contradiction. Since M is closed this means that either $(x, f(x) + \frac{1}{2}\varepsilon)$ or $(x, f(x) - \frac{1}{2}\varepsilon)$ belongs to M which contradicts the fact that $M \cap A = \emptyset$. Hence, A must be connected, in which case $S_\varepsilon(f)$ contains a member of $\mathscr{U}$ so that $f \in \bar{\mathscr{U}}$.

The implication $(\gamma_3) \Rightarrow f \in \bar{\mathscr{K}}$ is true if the following question can be answered affirmatively.

Question 2. *Suppose f is bilaterally c-dense in itself and $S_\varepsilon(f)$ is connected for all ε and suppose N is a set which intersects less than c segments of a given $S_\varepsilon(f)$ yet contains no entire segment of $S_\varepsilon(f)$. Is $S_\varepsilon(f) - N$ connected?*

To show how an affirmative answer to Question 2 yields $(\gamma_3) \Rightarrow f \in \bar{\mathscr{K}}$, construct $g \in S_\varepsilon(f)$ according to Lemma 1 so that g hits each continuum which hits $S_\varepsilon(f)$ in c-segments or which contains an entire segment of $S_\varepsilon(f)$. To show g is connected it suffices to show by Theorem 1 (C) that each continuum M which jumps g hits g. If M hits c segments of $S_\varepsilon(f)$ then $M \cap g \neq \emptyset$ by the construction of g. Also if M contains a segment of $S_\varepsilon(f)$, then $M \cap g \neq \emptyset$. So assume that M hits less than c segments and contains no segment. Then by hypothesis $S_\varepsilon(f) - M$ is connected.

Therefore, $S_\varepsilon(f) - M$ is contained in a component G of the complement of M. Since G is polygonally connected there exists a polygonal arc P joining $(a, g(a))$ and $(b, g(b))$ such that $P \cap M = \emptyset$. Hence, M is above P or below P which contradicts the assumption that M jumps g. Hence, the last case is impossible and $g \cap M \neq \emptyset$.

4. RELATIONSHIPS AMONG THE CLASSES AND THEIR CLOSURES

The next result gives inclusion relationships between the four families $\mathscr{C}, \mathscr{K}, \mathscr{U}$, and $\mathscr{D}$ and their uniform closures. We use $\subset$ to mean "$\subseteq$ but not $=$".

Theorem 5. *The following relationships hold.*

$$\mathscr{C} = \bar{\mathscr{C}} \subset \mathscr{K} \subset \bar{\mathscr{K}} \subset \bar{\mathscr{U}} = \bar{\mathscr{D}}\,,$$

$$\mathscr{K} \subset \mathscr{U} \subset \mathscr{D} \subset \bar{\mathscr{U}} = \bar{\mathscr{D}}\,,$$

and

$$\bar{\mathscr{K}} \nsubseteq \mathscr{D}\,.$$

Proof. That $\bar{\mathscr{U}} = \bar{\mathscr{D}}$ will be shown below; $\mathscr{C} = \bar{\mathscr{C}}$ is obvious as well as all the $\subseteq$ relationships. We may construct functions $f_1, f_2, \ldots, f_5$ such that: $f_1(x) = \sin(1/x)$, $x > 0$ and $f_1(0) = 0$; $f_2(x) \neq 0$ yet f_2 takes on all non-zero numbers in each interval; f_3 misses the line $y = x$ and takes on all numbers in each subinterval; f_4 misses the pseudo arc but hits each nonvertical arc; f_5 is bilaterally c-dense in the plane and f_5 misses the continuum K of the example following the conjecture about $\bar{\mathscr{K}}$. Finally, let f be the function of the same example. Then, we have $f_1 \in \mathscr{K} - \mathscr{C}$, $f_2 \in \bar{\mathscr{K}} - \mathscr{D}$, $f_3 \in \mathscr{D} - \mathscr{U}$, $f_4 \in \mathscr{U} - \mathscr{K}$, $f_5 \in \bar{\mathscr{U}} - \bar{\mathscr{K}}$, and $f \in \bar{\mathscr{K}} - \mathscr{K}$.

In order to show $\bar{\mathscr{U}} = \bar{\mathscr{D}}$ we need only show that $f \in \bar{\mathscr{D}}$ implies that any arc M which jumps any $S_\varepsilon(f)$ over an interval I hits $S_\varepsilon(f)$.

Suppose that $M \cap S_\varepsilon(f) = \emptyset$. In $I \times R$ let O_A and O_B be two separated open sets whose union is $(I \times R) - M$. Let $V(x)$ denote the open vertical segment of $S_\varepsilon(f)$, $\{(x, y) : |y - f(x)| < \varepsilon\}$. Then each $V(x)$ belongs entirely to O_A or O_B. Let $A = \{x : V(x) \subseteq O_A\}$ and $B = \{x : V(x) \subseteq O_B\}$. Then $A \cup B = I$ and $A \cap B = \emptyset$ and $A \neq \emptyset \neq B$ so that there exists a point $x \in \partial A \cap \partial B \cap I^0$.

Since $f \in \bar{\mathscr{D}}$ the bilateral cluster set $C(f, x)$ of f at x considered as a subset of the plane is a closed vertical segment, possibly infinite in length (see [1], [3]). Next we show that $C(f, x) \cap M \neq \emptyset$. If $C(f, x) \cap M = \emptyset$, then since a limit of a sequence

$\{V(x_n)\}_{n=1}^{\infty}$ in O_A (or O_B) belongs to $O_A \cup M$ (resp. $O_B \cup M$), it follows that $C(f, x) \subseteq \subseteq O_A$ or O_B. Assume $C(f, x) \subseteq O_A$. Since $x \in \partial B$ there exists $\{x_n\}_{n=1}^{\infty}$ in B such that $x_n \to x$ and $f(x_n) \to +\infty$ (or $-\infty$). Assume that $f(x_n) \to +\infty$ and $f(x_n) > n$. Then the line segment joining $(x, n) \in O_A$ with $(x_n, f(x_n)) \in O_B$ intersects M for sufficiently large n. Therefore, M is unbounded, a contradiction. Hence, $C(f, x) \cap M \neq \emptyset$.

Now consider the non-empty compact set $M \cap C(f, x)$ in the interval $C(f, x)$. By assumption we have $(x, f(x)) \in C(f, x) - M$. Hence we can find an open interval U in $C(f, x) - M$ one of whose end points, say z, belongs to M. Since $z \in C(f, x) \cap M$ there exist sequences $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ converging to x for which $x_n < x < y_n$ and $\{V(x_n)\}_{n=1}^{\infty}$ and $\{V(y_n)\}_{n=1}^{\infty}$ converge to the line segment $L = \{(x, y) : |y - z_2| < \varepsilon\}$. Since M is an arc and z can be assumed to not be an end point of M, there exists a closed disk D of radius $< \varepsilon$ centered at z such that $M \cap D$ is a subarc of M. However, $V(x_n)$ and $V(y_n)$ miss M and for large n they separate $M \cap D$ (in the space D), if $M \cap D \nsubseteq L$. Hence, $M \cap D \subseteq L$. Since M has no end point on L (we can assume that the end points of M project to give the end points of I) it follows that z is interior to $M \cap L$, a contradiction. This finishes the proof that $\overline{\mathscr{U}} = \overline{\mathscr{D}}$.

We turn now to a consideration of how $\mathscr{K}$ is distributed as a subspace of the metric space $\overline{\mathscr{D}}$ furnished with the metric of uniform convergence. (Specifically, we define the metric ϱ as follows: for given functions f and g let $\sigma(f, g) = \sup \{|f(x) - g(x)| : : x \in R\}$. If $\sigma(f, g) = \infty$, put $\varrho(f, g) = 1$ otherwise $\varrho(f, g) = \sigma(f, g)/(1 + \sigma(f, g))$. We note the diameter of $\overline{\mathscr{D}}$ is 1.)

We shall see, on the one hand, if $f \in \overline{\mathscr{D}}$ has even one point of continuity, then there are spheres in $\overline{\mathscr{D}} - \overline{\mathscr{K}}$ arbitrarily close to f, while at the other extreme, if f is dense in the plane then $\mathscr{K}$ is dense in every open sphere of f of radius not exceeding 1.

We begin with an example which is a simplified version of the second example directly following the conjecture in Section 3.

Let C be the Cantor set in $[0, 1]$ and decompose the family of intervals contiguous to C into two disjoint classes $\mathscr{A}$ and $\mathscr{B}$ such that $C \subseteq \overline{\bigcup \mathscr{A}}$ and $C \subseteq \overline{\bigcup \mathscr{B}}$. We define a continuum M as follows

$$M = (C \times [1, 9]) \cup (\bigcup \mathscr{A} \times \{9\}) \cup (\bigcup \mathscr{B} \times \{1\}) .$$

Then we may easily construct a function f having the following properties: f is continuous on the complement of C, $f(A) = [0, 8]$ for $A \in \mathscr{A}$, $f(B) = [2, 10]$ for $B \in \mathscr{B}$, $f(x) = 0$ (resp. 10) when x is an end point of a $A \in \mathscr{A}$ (resp. $B \in \mathscr{B}$), and finally, $f(x) = 0$ whenever x is a non-end point of C.

It is easy to verify that f is a Darboux function which is separated by M (or jumped by M and $M \cap f = \emptyset$). In fact, any function $g \subset S_1(f)$ is separated by M. We note also that f is in the second class of Baire (equivalently Borel) and, in fact, differs from a function in the first class of Baire on only a denumerable set. This is in contrast with the fact that any Darboux function in the first class of Baire is connected and thus in $\mathscr{K}$.

Employing this example we may prove

Theorem 6. *Let $g \in \overline{\mathscr{D}}$ have a point of continuity. Then there are spheres in $\overline{\mathscr{D}}$ arbitrarily close to g and containing no members of $\mathscr{K}$.*

Proof. Let g be continuous at x_0 and let $\varepsilon > 0$ where $\varepsilon < 1$. Choose $\delta > 0$ such that $|g(x) - g(x_0)| < \varepsilon$ whenever $|x - x_0| < \delta$. Let M_0 be a scaled-down version of the continuum M above, that is, having "domain" and "range" as the intervals $[x_0 - \delta, x_0 + \delta]$ and $[g(x) - \varepsilon, g(x) + \varepsilon]$ instead of R and $[1, 9]$. Let f_0 be defined relative to M_0 as f was defined relative to M and extend the domain of f_0 to $[0, 1]$ in such a way that $f_0 \in \mathscr{D}$ and $|f_0(x) - g(x)| < \varepsilon$ for all x. Then, it is clear that there is an entire sphere in $\overline{\mathscr{D}}$ centered at f_0 which misses $\mathscr{K}$, which proves the theorem.

Theorem 7. *If $f \in \overline{\mathscr{D}}$ is dense in the plane, then $\mathscr{K}$ is dense in each open sphere of radius $\leqq 1$.*

Proof. First we show that any function $g \in \overline{\mathscr{D}}$ dense in the plane belongs to $\overline{\mathscr{K}}$. According to the proven parts of the conjecture of Section 3 it suffices to verify condition (γ_1). Suppose M is a continuum which jumps $S_\varepsilon(g)$ over an interval I. Let M_ε be the set obtained by replacing each point in M by an open vertical segment of height 2ε centred at the point. It follows readily from the Baire Category Theorem that M_ε contains a rectangle B (with sides parallel to the axes) of height ε. Since g is dense in the plane and $g \in \overline{\mathscr{D}}$, it follows that card $(g \cap B) = c$ and thus card $(g \cap M_\varepsilon) = c$, which is obviously equivalent to card $(S_\varepsilon(g) \cap M) = c$, so that g satisfies (γ_1).

Now if $f_0 \in \overline{\mathscr{D}}$ such that $f_0 - f$ is bounded (i.e., $\varrho(f_0, f) < 1$ in the metric ϱ of $\overline{\mathscr{D}}$) then f_0 is also dense in the plane, so $f_0 \in \overline{\mathscr{K}}$ by the argument above. In other words, $\mathscr{K}$ is dense in any sphere of radius not exceeding 1 centered at f.

5. APPROXIMATIONS AND REPRESENTATIONS BY FUNCTIONS IN THE CLASSES

Next we present some miscellaneous results illustrating the common properties shared by Darboux and connected functions.

We begin by characterizing the pointwise limits of sequences of connected functions. It is well known that any function is a pointwise limit of a sequence of Darboux functions, each of which may be taken to be Lebesgue measurable (resp. of Borel class α) when the original function is Lebesgue measurable (resp. of Borel class $\alpha + 1$). (See [1] and [2].) Part of this characterization carries through for $\mathscr{K}$ and hence $\mathscr{U}$ as follows:

Theorem 8. *Each function is a pointwise limit of a sequence of connected functions. Moreover, the sequential members may be taken to be measurable if the original function is measurable.*

Proof. By a result in [3] we may express the domain interval I as $\bigcup_{n=0}^{\infty} A_n$ where $A_n \cap A_m = \emptyset$ when $n \neq m$ and for $n \geqq 1$ each A_n is a 1st Category, null F_σ set which is c-dense in I and A_0 is a residual G_δ of full measure. Let $\mathscr{M}$ be the collection of all planar continua having non-degenerate projection in I. Let f be the given function. On each A_n define a function f_n which intersects each member of $\mathscr{M}$ and put

$$g_n = f_n \cup f \mid (I - A_n) .$$

Then each g_n is connected and for each x we have $g_n(x) = f(x)$ except possibly for one n so that $\lim_{n \to \infty} g_n = f$. Since

$$[g_n < \lambda] = ([f < \lambda] \cap (I - A_n)) \cup ([f_n < \lambda] \cap A_n)$$

and A_n has measure 0, g_n is measurable whenever f is measurable.

We have been unable to extend Theorem 6 to assert that the functions g_n may be taken to be Borel when f is Borel. Since the "lower envelope" of members of $\mathscr{M}$ are Borel 1 functions, this assertion would follow from an affirmative answer to the following open question.

Question 3. *Let A be a F_σ set which is c-dense in R. Does there exist a Borel function on R which intersects over A each Borel function of class one?*

It is also well known that each function is a sum of two Darboux functions where the summand functions can be taken to be measurable (resp. of Borel class α) if the original function is measurable (resp. of Borel class α) (see [1] and [2]). We can strengthen this result for the non-Borel case to the following:

Theorem 9. *Each function is a sum of two connected functions. Moreover, the summand functions may be taken to be measurable if the original function is measurable.*

Proof. First we may find disjoint c-dense measurable sets A, B and C such that $I = A \cup B \cup C$ and A and B are null sets. Let f be the given function. Define h and g on A and B respectively such that h and g intersect each $M \in \mathscr{M}$ (as defined in the proof of Theorem 8). Now define

$$f_1 = h \cup (f - g) \mid B \cup (\tfrac{1}{2}f) \mid C$$

and

$$f_2 = (f - h) \mid A \cup g \cup (\tfrac{1}{2}f) \mid C .$$

Then f_1 and f_2 are connected functions such that $f = f_1 + f_2$. Since A and B are null sets f_1 and f_2 are clearly measurable whenever f is measurable.

If the answer to Question 3 is affirmative, then h and g can be taken to be Borel functions so that f_1 and f_2 are Borel whenever f is Borel.

It is also well known that each function is equal to a Darboux function except on a F_σ set of 1^{st} Category and measure 0. (See [2].) This may be strengthened to connected functions as follows:

Theorem 10. *Each function f is equal to a connected function h except on a 1^{st} Category null, F_σ set. If f is measurable, so is h.*

Proof. Let $I = \bigcup_{n=0}^{\infty} A_n$ be the decomposition cited in the proof of Theorem 8. Pick g on A_1 so that g intersects each $M \in \mathscr{M}$. Then $h = f \mid (I - A_1) \cup g$ is connected and $f = h$ except on the 1^{st} Category, null F_σ set A_1. Obviously h is measurable whenever f is.

Again an affirmative answer to Question 3 would yield an extension of Theorem 10 to cover the Borel case.

In [1] a class of functions $\mathscr{S}$ is called *a maximal additive family of functions for $\mathscr{J}$* if $\mathscr{S}$ is the biggest family of functions $\mathscr{V}$ such that $f \in \mathscr{V}$ and $g \in \mathscr{J}$ imply that $f + g \in \mathscr{J}$. In [1], we proved that the class of constant functions is the maximally additive class for the Darboux functions. Clearly the maximally additive class of $\mathscr{C}$ is $\mathscr{C}$ itself. The analogoue for $\mathscr{K}$ and $\mathscr{U}$ is the following

Theorem 11. *The class of continuous functions is the maximally additive class of functions for both the class of connected functions and the class of weakly connected functions.*

Proof. We must show (1) $\mathscr{C} + \mathscr{K} \subseteq \mathscr{K}$ and $\mathscr{C} + \mathscr{U} \subseteq \mathscr{U}$; and (2) if $g + \mathscr{K} \subseteq \mathscr{K}$ or $g + \mathscr{U} \subseteq \mathscr{U}$, then $g \in \mathscr{C}$. If $g \in \mathscr{C}$, $f \in \mathscr{K}$ (or $\mathscr{U}$) and M is a continuum (resp. arc) jumping $f + g$, then $M - g$ is a continuum (resp. arc) which jumps f and hence f hits $M - g$ or equivalently $f + g$ hits M. Thus (1) is proved.

Let us now prove (2) for $\mathscr{K}$ (the proof for $\mathscr{U}$ is similar). Suppose $g + \mathscr{K} \subseteq \mathscr{K}$. Then $g = g + 0 \subseteq \mathscr{K} \subseteq \mathscr{D}$ so that g is a Darboux function so that the left and right cluster sets at each point are overlapping closed intervals. Suppose that g is discontinuous at say 0, and 0 is interior to the domain interval I (the proof for 0 an end point of I is similar). It follows that there exists a finite left cluster value, say 0, and a different right cluster value, say 1. Now define

$$f(x) = \begin{cases} -g(x) + 1\,, & x \geqq 0 \\ -g(x)\,, & x < 0\,. \end{cases}$$

Then $f \mid (-\infty, 0]$ and $f \mid [0, +\infty)$ are both connected and 0 is common to the right and left cluster sets of f at 0. Therefore, $f \in \mathscr{K}$. However, $(f + g)(x) = 1$ if $x > 0$ and $= 0$ if $x < 0$ so that $f + g \notin \mathscr{K}$, a contradiction.

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