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SEQUENCES OF DISJOINT OPEN SETS

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The purpose of this paper is to discuss the validity of the statement:

(S): *Given an infinite subset A of a space X, there exists an infinite sequence $G_1, G_2, \ldots$ of mutually disjoint open subsets of X each of which intersects A.*

It is known that (S) holds in arbitrary regular T_1-space (see the proof of Theorem 2 in [7]). It also holds whenever X is Hausdorff and A has at least one accumulation point (see the proof of 5.2.4 in [3] or 27.A.12 in [4]). On the other hand, (S), in its full generality, fails for Hausdorff spaces; an example can be found in [5]. In this paper – in addition to extending (S) to other classes of spaces – we shall give an example similar to Frolík's [5], but simpler (our example is countable and second countable). We shall also indicate examples showing that a space satisfying (S) need not to be Urysohn; the question, however, if every Urysohn space satisfies (S) remains open.

Denote by $S(A)$ the part of the statement (S) following the first coma. Denote by $S'(A)$ the following statement:

$S'(A)$: *There exists a nonempty class $\mathfrak{R}$ of subsets of A and a function φ defined on $\mathfrak{R}$ such that for each B in $\mathfrak{R}$, $\varphi(B)$ is a pair (p, B') where $p \in B$, $B' \subseteq B$, $B' \in \mathfrak{R}$ and p and B' can be separated by open sets.*

Proposition 1. *Statements* $S(A)$ *and* $S'(A)$ *are equivalent.*

Proof. Assume $S(A)$, let $G_1, G_2, \ldots$ be a sequence of mutually disjoint open sets with $G_n \cap A \neq \emptyset$ for each n. Let $p_n \in G_n \cap A$ and $A' = \{p_1, p_2, \ldots\}$. Let $\mathfrak{R}$ be the collection of all infinite subsets of A'. For each $B \in \mathfrak{R}$, define $\varphi(B) = (p_k, B \setminus \{p_k\})$, where k is the smallest integer such that $p_k \in B$. Then $B \setminus \{p_k\} \in \mathfrak{R}$ and p_k and $B \setminus \{p_k\}$ can be separated by open sets.

Conversely, assume $S'(A)$, let $B \in \mathfrak{R}$. Then $\varphi(B) = (p_1, B_1)$ for some $p_1 \in B$, $B_1 \subseteq B$, $B_1 \in \mathfrak{R}$ and p_1 and B_1 can be separated by open sets, say G_1 and H_1, $p_1 \in G_1$, $B_1 \subseteq H_1$. Now, $B_1 \in \mathfrak{R}$ implies $\varphi(B_1) = (p_2, B_2)$ for some $p_2 \in B_1$, $B_2 \subseteq B_1$, $B_2 \in \mathfrak{R}$

and p_2 and B_2 can be separated by open sets, say G_2 and H_2, $p_2 \in G_2$, $B_2 \subseteq H_2$. We may assume $G_2, H_2 \subseteq H_1$. By induction, suppose we have $p_{n-1} \in G_{n-1}$, $B_{n-1} \subseteq \subseteq H_{n-1}$, $B_{n-1} \in \mathfrak{R}$ with G_{n-1}, H_{n-1} disjoint open and $G_{n-1}, H_{n-1} \subseteq H_{n-2}$. Then $\varphi(B_{n-1}) = (p_n, B_n)$ for some $p_n \in B_{n-1}$ and p_n and B_n can be separated by open sets, say G_n and H_n, $p_n \in G_n$, $B_n \subseteq H_n$. We may again assume $G_n, H_n \subseteq H_{n-1}$. In this way, we obtain a sequence $G_1, G_2, \ldots$ of mutually disjoint open sets each of which intersects A; in fact, $p_n \in G_n \cap A$ for each n.

Note that members of $\mathfrak{R}$ in S$'(A)$ have to be infinite sets. Thus, if for every infinite subset A of X, S$'(A)$ holds, then every infinite subset has a point and an infinite subset which can be separated by open sets. Conversely, if the last statement holds, then we can define a function φ which works for every infinite subset of A. Hence, from Proposition 1, we obtain:

Proposition 2. (S) *is equivalent to*

(S'): *Given an infinite subset A of X, there exist a point p in A and an infinite subset S of A such that p and S can be separated by open sets.*

We shall now examine spaces which satisfy (S).

Proposition 3. *If $f : X \to Y$ is a continuous function from a Hausdorff space X into a space Y satisfying* (S) *such that $f^{-1}(y)$ is countably compact for each $y \in Y$, then X satisfies* (S).

Proof. Let A be an infinite subset of X, we shall show that S(A) holds. If $f(A)$ is infinite, then S(A) is derived from the fact that S$(f(A))$ holds. If $f(A)$ is finite, then A is contained in a countably compact set, by the result of Čech quoted in the introduction, S(A) holds.

Corollary 1. *If X can be mapped into a regular T_1-space by a one-to-one continuous function, then X satisfies* (S).

Corollary 2. *Every functionally Hausdorff space satisfies* (S).

Proof. Since a space is functionally Hausdorff iff it admits a continuous one-to-one function onto a completely regular space (see, e.g., [6], 3.20 with $E = I = [0, 1]$), it suffices to apply the preceding corollary.

Proposition 4. *If X can be embedded in a Hausdorff space Y so that X is countably compact relative to Y (i.e. every infinite subset of X has an accumulation point in Y), then X satisfies* (S).

Proof. This is immediate from the result of Čech quoted in the introduction.

The following example is a Hausdorff space which does not satisfy (S).

Example. Let $l_k = \{(x, 1/k) \mid x$ rational, $x > 0\}$, $k = 1, 2, \ldots$ be subsets of the plane and let $X = (\bigcup_{k=1}^{\infty} l_k) \cup \{(n, 0) \mid n = 1, 2, \ldots\}$. Define a topology on X by taking all points $(a, b) \in X$ with $b > 0$ to be isolated. Neighborhoods of $(n, 0) \in X$ are of the form $\{T_n \cap (\bigcup_{m \geqq k} l_m)\} \cup \{$almost all points on $l_n\}$ for some positive integer k, where T_n is the equilateral triangle with vertex at $(n, 0)$ and base on l_1.

The space X thus obtained is Hasudorff. It does not satisfy (S). To see this, it suffices to show, by Proposition 2, that every point of the infinite set $A = \{(n, 0) \mid n = 1, 2, \ldots\}$ and every infinite subset of A cannot be separated by open sets. Indeed, if G is open in X such that $G \cap A$ is infinite, then $A \subseteq \bar{G}$. From this, it follows that if H_1, H_2 are open sets containing a point $p \in A$ and an infinite subset $F \subseteq A$ respectively, then H_1 and H_2 cannot be disjoint.

We give some remarks.

1. We have not been able to decide whether Urysohn spaces satisfy (S). But a space can satisfy (S) without being Urysohn. As an example, we take the space constructed by Bing ([2], Example 1) which we will denote by B. It is well-known that B is not Urysohn. To show that B satisfies (S), we shall show that it satisfies (S'). Given a point $p = (x, y) \in B$, we denote by $l(p)$ and $r(p)$ the points $(x - y/\sqrt{3}, 0)$ and $(x + y/\sqrt{3}, 0)$ respectively. (S') is shown by selecting from an infinite set $A \subseteq B$ a sequence of distinct points $p_1, p_2, \ldots$ such that both sequences $l(p_1), l(p_2), \ldots$ and $r(p_1), r(p_2), \ldots$ are convergent (to a number or to $\pm\infty$).

Another example of a non-Urysohn space satisfying (S) can be obtained as follows: write the set $S(\Omega)$ of all countable ordinals as the union of two disjoint cofinal subsets A_1 and A_2. Add two new points a_1 and a_2 to $S(\Omega)$. Define a topology on $X = S(\Omega) \cup \{a_1, a_2\}$ by taking neighborhoods of a_i to be all sets of the form $\{a_i\} \cup [(\xi, \Omega) \cap A_i]$, $i = 1, 2$, where $\xi \in S(\Omega)$; neighbourhoods of points of $S(\Omega)$ are the usual ones. X is a Hausdorff extension of $S(\Omega)$ and obviously X is countably compact; consequently X satisfies (S). On the other hand, X is not Urysohn; in fact, it can be shown that *every Urysohn extension of* $S(\Omega)$ *is either* $S(\Omega)$ *or* $S(\Omega) \cup \{\Omega\}$. (Note also that the above X is absolutely closed — we can produce, in a similar way, other absolutely closed extensions X' of $S(\Omega)$ so that $X' \setminus S(\Omega)$ has more than two points; indeed, the Katětov extension $\varkappa S(\Omega)$ is of this type.)

Note that a space which can be mapped into a Urysohn (in particular, regular) space by a one-to-one continuous function is also Urysohn. Hence, any non-Urysohn space satisfying (S) can serve as an example showing that the converse of Corollary 1 to Proposition 3 need not hold.

2. The converse of Proposition 4 need not hold. Indeed, as an example, we can take any non-countably compact, absolutely closed, functionally Hausdorff space. Such a space can be found, for example, in [1] (5; 1°; p. 5). The verification that this space is functionally Hausdorff and absolutely closed will follow from the following remarks.

Let us observe that the construction used in [1] (5; 1°; p. 5) as well as in [1] (5; 2°; p. 5) can be generalized as follows. Let X be a Hausdorff space, let $A \subseteq X$ and assume that for every $p \in A$, we have a closed nowhere dense subset F_p of X. Enlarge the topology of X by adding to it all sets of the form $(U \setminus F_p) \cup \{p\}$ where p is an arbitrary point of A and U is an arbitrary open subset of X containing p. Denote the space so obtained by X'. The identity map f_0 from X' onto X is continuous. Hence X' is Hausdorff (respectively functionally Hausdorff) if X is such. We can show that

(i) $\mathrm{Int}_{X'} (\mathrm{cl}_{X'} U)$ *is open in* X *for every* U *open in* X'.

Furthermore, f_0 has the following property.

(ii) *For every* $x \in X$ *and every open set* U *of* X' *with* $f_0^{-1}(x) \subseteq U$, *there exists an open set* V *of* X *with* $x \in V$ *and* $f_0^{-1}(\mathrm{cl}_X V) \subseteq \mathrm{cl}_{X'} U$.

Indeed, we may take $V = \mathrm{Int}_{X'} (\mathrm{cl}_{X'} U)$.

Observe that if $p \in \mathrm{cl}_X (F_p \setminus \{p\})$ for at least one $p \in A$, then X' is not compact. Furthermore, if we start with a Hausdorff compact X, then X' is absolutely closed. In fact, we have a stronger result.

(iii) *If* X *is Hausdorff and absolutely closed, then* X' *is absolutely closed.*

(iii) follows from (ii) and the following lemma.

Lemma. *Let* S *and* T *be Hausdorff spaces,* T *absolutely closed. If there is a function* f *from* S *onto* T *with the following two properties*

(a) $f^{-1}(t)$ *is compact for every* $t \in T$,

(b) *for every* $t \in T$ *and every* U *open in* S *with* $f^{-1}(t) \subseteq U$, *there exists an open set* V *of* T *with* $t \in V$ *and* $f^{-1}(\overline{V}) \subseteq \overline{U}$, *then* S *is absolutely closed.*

Proof. Let $\mathfrak{U} = \{U_\xi \mid \xi \in \Xi\}$ be an cover of S. For each $t \in T$, $\mathfrak{U} \cap f^{-1}(t) = \{U_\xi \cap f^{-1}(t) \mid \xi \in \Xi\}$ is an open cover of $f^{-1}(t)$. There exist $\xi_{1(t)}, \xi_{2(t)}, \ldots, \xi_{n(t)}$ in Ξ such that

$$f^{-1}(t) \subseteq U_{\xi_{1(t)}} \cup U_{\xi_{2(t)}} \cup \ldots \cup U_{\xi_{n(t)}}.$$

By (b), there exists V_t open in T such that $t \in V_t$ and

$$f^{-1}(\overline{V}_t) \subseteq \overline{U_{\xi_{1(t)}}} \cup \overline{U_{\xi_{2(t)}}} \cup \ldots \cup \overline{U_{\xi_{n(t)}}}.$$

Now $\{V_t\}_{t \in T}$ is an open cover for T; hence there exist $t_1, t_2, \ldots, t_m$ in T such that $\overline{V}_{t_1} \cup \overline{V}_{t_2} \cup \ldots \cup \overline{V}_{t_m} = T$. Then

$$S = f^{-1}(T) = f^{-1}\Big(\bigcup_{i=1}^{m} \overline{V}_{t_i}\Big) = \bigcup_{i=1}^{m} f^{-1}(\overline{V}_{t_i}) \subseteq \bigcup_{i=1}^{m} \Big(\overline{U_{\xi_{1(t_i)}}} \cup \overline{U_{\xi_{2(t_i)}}} \cup \ldots \cup \overline{U_{\xi_{n(t_i)}}}\Big).$$

To conclude this remark, we observe that absolutely closed Hausdorff spaces need not satisfy (S). Indeed, we can take an absolutely closed extension of any Hausdorff space which does not satisfy (S).

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