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SECOND ORDER DIFFERENTIAL EQUATIONS WITH COMPLEX-VALUED COEFFICIENTS

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Introduction

In this paper oscillatory and asymptotic properties of solutions of the differential equation

$$[P(x)\,y']' + Q(x)\,y = 0 \tag{1}$$

are studied, where $P(x)$ and $Q(x)$ are complex-valued functions defined on an interval J with endpoints a, b, $a \geqq -\infty$, $b \leqq \infty$. In the whole paper we always suppose that the functions P, Q can be written in a "polar form" $P = p(x)\,e^{i\varphi(x)}$, $Q = q(x)\,e^{i\psi(x)}$ where

$$\begin{gathered} p(x),\ q(x),\ \varphi(x),\ \psi(x) \quad \text{are real functions,} \\ p(x) > 0\,; \quad p(x),\ q(x),\ \psi(x) \in C^0(J)\,; \quad \varphi(x) \in C^1(J)\,. \end{gathered} \tag{2}$$

Let $x_0 \in J$ and give two complex numbers y_0, y'_0; under the above assumptions the equation (1) has a unique solution $y(x)$ which is defined on J and satisfies $y(x_0) = y_0$, $y'(x_0) = y'_0$. In the sequel by a solution of (1) is always meant a nontrivial solution.

The paper is devided into three parts. The first part is devoted to the study of the logarithmic derivative $y'(x)/y(x)$ of a solution $y(x)$ of (1), in particular, to the study of its behaviour in a neighbourhood of a zero of $y(x)$. In the second part it is shown that every solution of (1) can be written in the form $y = r(x)\,e^{i\Theta(x)}$ where $r(x)$, $\Theta(x)$ are real functions satisfying the system

$$\begin{gathered} [p(x)\,r']' + \{\tfrac{1}{4}p(x)\,\varphi'^2(x) + q(x)\cos[\psi(x) - \varphi(x)] - \\ - p(x)\,[\Theta' + \tfrac{1}{2}\varphi'(x)]^2\}\,r = 0\,, \end{gathered} \tag{3}$$

$$[p(x)\,r\Theta']' + p(x)\,[\Theta' + \varphi'(x)]\,r' + q(x)\sin[\psi(x) - \varphi(x)]\,r = 0\,. \tag{4}$$

Since the functions $y(x)$ and $r(x)$ have exactly the same zeros and $|y(x)| = |r(x)|$, the equations (3), (4) reflect both the distribution of the zeros and growth properties of the solutions of (1). The device, to derive a pair of simultaneous differential equations which are satisfied by the modulus and argument of any solution $y(x)$ was first used in the case $P(x) \equiv 1$ by CHOY TAK TAAM [1] and, in the case $Q^{-1} = \bar{P}$ where $\bar{P}$ is the complex conjugate of P, by J. H. BARRET [2].

It is important to note that the equation

$$(5) \qquad [p(x)\, s']' + \{\tfrac{1}{4}p(x)\, \varphi'^2(x) + q(x) \cos [\psi(x) - \varphi(x)]\}\, s = 0$$

is a Sturm majorant for the equation (3) and that its left-hand side is independent upon the function Θ. This circumstance enables us, in the final section, to derive some oscillatory and asymptotic properties of solutions of (1) using Sturm comparis theorems.

1. Logarithmic derivative of solutions

Lemma 1. *Let $y(x)$ be a solution of (1) defined on J. The function $\sigma(x) =$ Re $y'(x)$: : $y(x)$ is continuous on J as long as $y(x) \neq 0$. If $y(c) = 0$, we have*

$$\lim_{x \to c+} \sigma(x) = \infty \quad \textit{if} \quad c < b\,, \qquad \lim_{x \to c-} \sigma(x) = -\infty \quad \textit{if} \quad c > a\,.$$

Proof. Let $y_1(x) = \operatorname{Re} y(x)$, $y_2(x) = \operatorname{Im} y(x)$; it is

$$\sigma(x) = \operatorname{Re} \frac{y_1'(x) + iy_2'(x)}{y_1(x) + iy_2(x)} = \frac{y_1(x)\, y_1'(x) + y_2(x)\, y_2'(x)}{y_1^2(x) + y_2^2(x)}$$

so that $\sigma(x)$ is continuous on J as long as $y(x) \neq 0$. Let c be a zero of $y(x)$, $c < b$ and let x be fixed, $c < x < b$. By the law of the mean there exist numbers ξ, η, $c < \xi$, $\eta < x$ such that $y_1(x) = y_1'(\xi)\,(x - c)$, $y_2(x) = y_2'(\eta)\,(x - c)$. Now it is

$$\sigma(c+) = \lim_{x \to c+} \frac{[y_1'(\xi)\, y_1'(x) + y_2'(\eta)\, y_2'(x)]\,(x - c)}{[y_1'^2(\xi) + y_2'^2(\eta)]\,(x - c)^2}$$

and $\xi, \eta \to c+$ if $x \to c+$. Since $y_1'(c)$, $y_2'(c)$ cannot vanish simultaneously, we have

$$\lim_{x \to c+} \frac{y_1'(\xi)\, y_1'(x) + y_2'(\eta)\, y_2'(x)}{y_1'^2(\xi) + y_2'^2(\eta)} = 1$$

so that $\sigma(c+) = +\infty$.

In the same manner it can be proved $\sigma(c-) = -\infty$ if $c > a$.

Lemma 2. *Suppose that* (2) *holds. Let* $y(x)$ *be a solution of* (1) *defined on* J. *The function*

$$\tau(x) = \begin{cases} \operatorname{Im} \dfrac{y'(x)}{y(x)} & \text{for} \quad y(x) \neq 0 \\ -\dfrac{\varphi'(c)}{2} & \text{for} \quad y(c) = 0 \end{cases}$$

is continuous on J.

Proof. The continuity of the function $\tau(x)$ at the points different from zeros of $y(x)$ is evident. Let $y(x)$ vanish at a point c. Putting $u(x) = y(x)$, $v(x) = P(x)\, y'(x)$ equation (1), written as a system, is $P(x)\, u' = v$, $v' = -Q(x)\, u$. Note that $v(c) \neq 0$; otherwise, it would be $u(c) = 0$ and this is impossible because of $y(x) \not\equiv 0$. Now, let us compute

$$\lim_{x\to c} \tau(x) = \lim_{x\to c} \frac{\operatorname{Im} y'(x)\, \bar{y}(x)}{y(x)\, \bar{y}(x)} = \lim_{x\to c} \frac{1}{2i\, p(x)}\, \frac{e^{-i\varphi(x)}\, \bar{u}(x)\, v(x) - e^{i\varphi(x)}\, u(x)\, \bar{v}(x)}{u(x)\, \bar{u}(x)}\, .$$

Using l'Hospital's rule we obtain (for the sake of brevity the variable x is dropped)

$$\lim_{x\to c} \frac{e^{-i\varphi} \bar{u} v - e^{i\varphi} u \bar{v}}{u \bar{u}} = \lim_{x\to c} \frac{e^{-i\varphi}(-i\varphi' \bar{u} v + \bar{u} v' + \bar{u}' v) - e^{i\varphi}(i\varphi' u \bar{v} + u \bar{v}' + u' \bar{v})}{u' \bar{u} + u \bar{u}'} =$$

$$= \lim_{x\to c} \frac{e^{-i\varphi}(-i\varphi' \bar{u} v - Q u \bar{u} + \bar{P}^{-1} v \bar{v}) - e^{i\varphi}(i\varphi' u \bar{v} - \bar{Q} u \bar{u} + P^{-1} v \bar{v})}{P^{-1} \bar{u} v + \bar{P}^{-1} u \bar{v}} =$$

$$= \lim_{x\to c} \frac{-i\varphi' p(e^{-i\varphi} \bar{u} v + e^{i\varphi} u \bar{v})}{e^{-i\varphi} \bar{u} v + e^{i\varphi} u \bar{v}} + \lim_{x\to c} \frac{p(\bar{Q} e^{i\varphi} - Q e^{-i\varphi})\, u \bar{u}}{e^{-i\varphi} \bar{u} v + e^{i\varphi} u \bar{v}} =$$

$$= -i\, \varphi'(c)\, p(c) + \lim_{x\to c} pq \frac{(e^{i(\varphi-\psi)} - e^{-i(\varphi-\psi)})\, u \bar{u}}{e^{-i\varphi} \bar{u} v + e^{i\varphi} u \bar{v}}\, .$$

Using l'Hospital's rule once more it is easy to prove that

$$\lim_{x\to c} \frac{u \bar{u}}{e^{-i\varphi} \bar{u} v + e^{i\varphi} u \bar{v}} = 0\, .$$

Therefore, we have $\lim_{x\to c} \tau(x) = -\frac{1}{2}\varphi'(c)$ as asserted.

Theorem 1. *Suppose that* (2) *holds. Let* $y(x)$ *be a solution of* (1) *defined on* J. *The trajectory* $z(x) = y'(x)/y(x)$ *is continuous on* J *as long as* $y(x) \neq 0$; *at every zero of* $y(x)$ *there exists an asymptote parallel to the real axis and passing through the point* $z = -\frac{1}{2}i\varphi'(c)$.

Proof. The assertion follows immediatelly from Lemmas 1 and 2.

The proofs of the following two lemmas are straightforward and we omit them.

Lemma 3. *Suppose that* (2) *holds. Let* $y = y(x)$ *be a function defined on* J. *Put* $y_1(x) = \operatorname{Re} y(x)$, $y_2(x) = \operatorname{Im} y(x)$ *and suppose* $y_1, y_2 \in C^1$; $py'_1, py'_2 \in C^1$. *Then* $y(x)$ *is a solution of* (1) *if and only if*

$$[p(x)\, y'_1]' - p(x)\, \varphi'(x)\, y'_2 + \tag{6}$$
$$+ q(x)\, \{y_1 \cos[\psi(x) - \varphi(x)] - y_2 \sin[\psi(x) - \varphi(x)]\} = 0\,,$$

$$[p(x)\, y'_2]' + p(x)\, \varphi'(x)\, y'_1 + \tag{7}$$
$$+ q(x)\, \{y_2 \cos[\psi(x) - \varphi(x)] + y_1 \sin[\psi(x) - \varphi(x)]\} = 0\,.$$

Lemma 4. *Suppose that* (2) *holds. Let* $r = r(x)$, $\Theta = \Theta(x)$ be real functions defined on J, $r, \Theta \in C^1$; $pr', pr\Theta' \in C^1$. *Then the function* $y = r(x)\, e^{i\Theta(x)}$ *is a solution of* (1) *if and only if the functions* $r(x)$, $\Theta(x)$ *satisfy* (3) *and* (4).

2. Existence and uniqueness for the system (3), (4)

Theorem 2. *Suppose that the functions* $p = p(x)$, $q = q(x)$, $\varphi = \varphi(x)$, $\psi = \psi(x)$ *defined on* J *satisfy* (2). *Let* $x_0, r_0, r'_0, \Theta_0, \Theta'_0$ *be real numbers,* $r_0 \neq 0$ *or* $r_0 = 0$, $\Theta'_0 = -\frac{1}{2}\varphi'(x_0)$. *Then the system* (3), (4) *has a unique solution* $(r(x), \Theta(x))$ *satisfying the initial conditions*

$$r(x_0) = r_0\,, \quad r'(x_0) = r'_0\,, \quad \Theta(x_0) = \Theta_0\,, \quad \Theta'(x_0) = \Theta'_0 \tag{8}$$

and existing on J.

Proof. Let $r_0 e^{i\Theta_0} = y_0$, $(r'_0 + ir_0\Theta'_0)\, e^{i\Theta_0} = y'_0$. Thus

$$|r_0| = |y_0|\,, \quad |y'_0| = \sqrt{(r_0'^2 + r_0^2\Theta_0'^2)} \tag{9}$$

and for $r_0 \neq 0$

$$r_0 \operatorname{Re} \frac{y'_0}{y_0} = r'_0\,, \quad \operatorname{Im} \frac{y'_0}{y_0} = \Theta'_0\,. \tag{10}$$

Let $y(x)$ be a solution of (1) defined on J and satisfying the initial conditions $y(x_0) = y_0$, $y'(x_0) = y'_0$. If $y(x)$ possesses zeros, let us denote a_0 the zero of $y(x)$ situated nearest to the left of the point x_0. If $y(x_0) = 0$, put $a_0 = x_0$. Further, let $a_1 < a_2 < \ldots$ be the zeros of $y(x)$ situated to the right of a_0 and $a_{-1} > a_{-2} > \ldots$ the zeros situated to the left of a_0. Denote $J_k = (a_k, a_{k+1})$, $k = 0, \pm 1, \ldots$ If $y(x)$ has no zeros, let $J_0 = J$. If $y(x)$ has a finite number l of zeros in the interval (a, x_0) $\{(x_0, b)\}$, let $J_{-l-1} = (a, a_{-l})$ $\{J_l = (a_l, b)\}$. Define

$$r(x) = \begin{cases} \varepsilon_k |y(x)| & \text{for} \quad x \in J_k\,, \\ 0 & \text{for} \quad x = a_k \end{cases} \tag{11}$$

where ε_k denotes $(-1)^k \operatorname{sgn} r_0$ if $r_0 \neq 0$ and, $(-1)^k \operatorname{sgn} r_0'$ if $r_0 = 0$; next, define

$$\Theta'(x) = \begin{cases} \operatorname{Im} \dfrac{y'(x)}{y(x)} & \text{for} \quad x \neq a_k\,, \\ -\frac{1}{2}\varphi'(a_k) & \text{for} \quad x = a_k\,. \end{cases} \tag{12}$$

By Lemma 2, the function $\Theta'(x)$ is continuous on J; for that reason there exists a primitive function $\int_{x_0}^{x} \Theta'(t)\,\mathrm{d}t$. Let

$$\Theta(x) = \Theta_0 + \int_{x_0}^{x} \Theta'(t)\,\mathrm{d}t\,. \tag{13}$$

In every interval J_k the pair of functions $(r(x), \Theta(x))$ is a solution of (3), (4). It is sufficient to prove this assertion for the functions $r = |y(x)|$, $\Theta = \operatorname{Im} y'(x)/y(x)$ since the system (3), (4) is homogeneous in r. Setting $y_1 = \operatorname{Re} y$, $y_2 = \operatorname{Im} y$ we have

$$r' = (y_1y_1' + y_2y_2')\,r^{-1}\,,$$

$$(pr')' = r^{-1}[py_1'^2 + py_2'^2 + y_1(py_1')' + y_2(py_2')'] - pr^{-2}r'(y_1y_1' + y_2y_2')\,.$$

Using the relations (6), (7) we get after an easy calculation

$$(pr')' = -qr\cos(\psi - \varphi) + pr^{-1}[\varphi'(y_1y_2' - y_1'y_2) + y_1'^2 + y_2'^2 - r'^2]\,.$$

Finally,

$$y_1y_2' - y_1'y_2 = r^2\Theta'\,, \quad y_1'^2 + y_2'^2 = r'^2 + r^2\Theta'^2\,, \tag{14}$$

so that

$$(pr')' = -qr\cos(\psi - \varphi) + p\varphi'\Theta'r + pr\Theta'^2\,,$$

and (3) holds. From

$$(rp\Theta')' = \left(\frac{y_1py_2' - y_2py_1'}{r}\right)' = r^{-1}[y_1(py_2')' - y_2(py_1')'] - r^{-2}r'p(y_1y_2' - y_1'y_2)$$

and in virtue of (6), (7) and (14) we get

$$(rp\Theta')' = -p\varphi'r' - qr\sin(\psi - \varphi) - pr'\Theta'\,,$$

and (4) holds too. We see that the pair of functions $(r(x), \Theta(x))$ defined above satisfy the equations (3), (4) on every interval J_k. It remains to verify the statement of the theorem at any a_k. This will be proved in several steps.

First of all note that the function $r(x)$ is continuous at a_k because of $r(a_k) = 0$, $\lim_{x\to a_k} r(x) = 0$. Since

$$r' = \varepsilon_k(y_1y_1' + y_2y_2')\,(y_1^2 + y_2^2)^{-1/2}$$

on every J_k, there exist, by the law of the mean, numbers ξ, η, $a_k < \xi, \eta < x$ so that $y_1(x) = y_1'(\xi)(x - a_k)$, $y_2(x) = y_2'(\eta)(x - a_k)$. Thus

$$\lim_{x \to a_k +} r'(x) = \frac{\varepsilon_k[y_1'(\xi)\, y_1'(x) + y_2'(\eta)\, y_2'(x)]\,(x - a_k)}{\sqrt{(y_1'^2(\xi) + y_2'^2(\eta))}\,|x - a_k|} = \varepsilon_k |y'(a_k)| \neq 0\,.$$

In the same manner it can be shown $\lim_{x \to a_k -} r'(x) = \varepsilon_k |y'(a_k)|$. Hence at every point a_k there exists the limit $\lim_{x \to a_k} r'(x) = \varepsilon_k |y'(a_k)|$. Consequently $r'(a_k) = \lim_{x \to a_k} [r(x) - - r(a_k)]\,(x - a_k)^{-1} = \lim_{\xi \to a_k} r'(\xi)$ so that the function $r'(x)$ is continuous at a_k. Since the function pr' is continuously differentiable on every J_k, we see from the law of the mean that

$$[p(x)\, r'(x)]'_{x=a_k} = \lim_{\xi \to a_k} [p(\xi)\, r'(\xi)]'\,.$$

Hence, in virtue of (2), $[p(x)\, r'(x)]'$ exists and is continuous at a_k. Finally from the law of the mean and the equation (4), we have

$$[p(x)\, r(x)\, \Theta'(x)]'_{x=a_k} = \lim_{\xi \to a_k} [p(\xi)\, r(\xi)\, \Theta'(\xi)]' = -p(a_k)\, r'(a_k)\, \frac{\varphi'(a_k)}{2}\,.$$

Thus the pair of functions $(r(x), \Theta(x))$ satisfies the equations (3), (4) on the whole interval J.

Now let us show that the functions $r(x)$, $\Theta(x)$ satisfy initial conditions. If $r_0 \neq 0$, in view of (11), (9), (13), (12) and the fact that $x_0 \in J_0$, $r(x_0) = \varepsilon_0 |y(x_0)| = |y_0| \operatorname{sgn} r_0 = = |r_0| \operatorname{sgn} r_0 = r_0$,

$$r'(x_0) = \operatorname{sgn} r_0 [y_1(x_0)\, y_1'(x_0) + y_2(x_0)\, y_2'(x_0)]\, [y_1^2(x_0) + y_2^2(x_0)]^{-1/2} =$$

$$= r_0 \operatorname{Re} \frac{y_0'}{y_0} = r_0 \frac{r_0'}{r_0} = r_0'\,,$$

$\Theta(x_0) = \Theta_0$, $\Theta'(x_0) = \operatorname{Im}(y_0'/y_0) = \Theta_0'$. If $r_0 = 0$, it is $r(x_0) = r_0$, $r'(x_0) = = \lim_{x \to a_0} r'(x) = \operatorname{sgn} r_0' |y'(a_0)| = \operatorname{sgn} r_0' |r_0'| = r_0'$, $\Theta(x_0) = \Theta_0$, $\Theta'(x_0) = -\frac{1}{2}\varphi'(a_0) = = \Theta_0'$.

To prove uniqueness suppose that the system (3), (4) has two different solutions $(r_j(x), \Theta_j(x))$, where $j = 1, 2$, satisfying (8). Then the functions $y_j(x) = r_j(x)\, e^{i\Theta_j(x)}$ being solutions of (1) satisfy at x_0 the same initial conditions and $y_1(x) \not\equiv y_2(x)$ in contradiction to the uniqueness of solutions of the equation (1). The proof is complete.

Theorem 3. *Suppose that* (2) *holds. Let* $y(x)$ *be a solution of* (1) *defined on* J *and satisfying at* $x_0 \in J$ *initial conditions* $y(x_0) = y_0$, $y'(x_0) = y_0'$. *Then there exists a solution* $(r(x), \Theta(x))$ *of the system* (3), (4) *defined on* J *such that* $y(x) = r(x)\, e^{i\Theta(x)}$.

Proof. If $y_0 \neq 0$, let $r_0 = |y_0|$, $r'_0 = r_0 \operatorname{Re}(y'_0/y_0)$, $\Theta'_0 = \operatorname{Im}(y'_0/y_0)$ and Θ_0 be determined by means of conditions

$$\cos \Theta_0 = \frac{\operatorname{Re} y_0}{|y_0|}, \quad \sin \Theta_0 = \frac{\operatorname{Im} y_0}{|y_0|}.$$

If $y_0 = 0$, let $r_0 = 0$, $r'_0 = |y'_0|$, $\Theta'_0 = -\frac{1}{2}\varphi'(x_0)$ and, Θ_0 be such that

$$\cos \Theta_0 = \frac{\operatorname{Re} y'_0}{|y'_0|}, \quad \sin \Theta_0 = \frac{\operatorname{Im} y'_0}{|y'_0|}.$$

Then there exists a unique solution $(r(x), \Theta(x))$ of the system (3), (4) satisfying the initial conditions $r(x_0) = r_0$, $r'_0(x_0) = r'_0$, $\Theta(x_0) = \Theta_0$, $\Theta'(x_0) = \Theta'_0$. In view of Lemma 4 the function $r(x)\, e^{i\Theta(x)}$ is a solution of (1) and since $r(x_0)\, e^{i\Theta(x_0)} = y(x_0)$, $[r'(x_0) + ir(x_0)\,\Theta'(x_0)]\, e^{i\Theta(x_0)} = y'(x_0)$, we have $y(x) \equiv r(x)\, e^{i\Theta(x)}$. This completes the proof.

3. Oscillatory and asymptotic properties of solutions

Theorem 4. *Let the functions* $P(x) = p(x)\, e^{i\varphi(x)}$, $Q(x) = q(x)\, e^{i\psi(x)}$ *satisfy the conditions* (2) *and, in addition, let* $p\varphi' \in C^1(J)$,

$$(15) \qquad [p(x)\,\varphi'(x)]' - 2q(x) \sin [\psi(x) - \varphi(x)] \neq 0\,.$$

Then the differential equation (1) *is disconjugate on* J.

Proof. Suppose that there exists a solution $y(x)$ of (1) having at least two zeros $\alpha < \beta$ in J and $y(x) \neq 0$ for $\alpha < x < \beta$. By Theorem 3 $y(x)$ can be written in the form $y = r(x)\, e^{i\Theta(x)}$, $r(\alpha) = r(\beta) = 0$ and $r(x) \neq 0$ for $\alpha < x < \beta$. Multiplying the equation (4) by $r(x)$ it can be reduced to

$$[p(x)\, r^2(x)\, \Theta'(x)]' + p(x)\, r(x)\, r'(x)\, \varphi'(x) + q(x)\, r^2(x) \sin [\psi(x) - \varphi(x)] = 0\,.$$

Integrating this equation from α to x we get

$$p(x)\, r^2(x)\, \Theta'(x) = -\int_\alpha^x \{p(t)\, r(t)\, r'(t)\, \varphi'(t) + q(t)\, r^2(t) \sin [\psi(t) - \varphi(t)]\}\, dt$$

and the integration by parts in the first integral on the right-hand side leads to the relation

$$p(x)\, r^2(x)\, [\Theta'(x) + \tfrac{1}{2}\varphi'(x)] = \frac{1}{2}\int_\alpha^x r^2(t)\, \{[p(t)\, \varphi'(t)]' - 2q(t) \sin [\psi(t) - \varphi(t)]\}\, dt\,.$$

Since the right-hand side is different from zero for $\alpha < x \leqq \beta$ we have a contradiction with $r(\beta) = 0$. Theorem is proved.

Let the assumptions of the preceding theorem be satisfied and, instead of (15), let be $[p(x)\,\varphi'(x)]' - 2q(x)\sin[\psi(x) - \varphi(x)] \equiv 0$. Then the system (3), (4) can be simplified as follows

$$p(x)\,r^2(x)\,[\Theta'(x) + \tfrac{1}{2}\varphi'(x)] = C\,,$$

$$[p(x)\,r'(x)]' + \{\tfrac{1}{4}p(x)\,\varphi'^2(x) + q(x)\cos[\psi(x) - \varphi(x)]\}\,r(x) = \frac{C^2}{p(x)\,r^3(x)}\,.$$

The solutions of (1) form two groups corresponding to the values $C \neq 0$ and $C = 0$; denote them A and B.

The group A contains the solutions of (1) satisfying at a point $x_0 \in J$ the initial conditions $y(x_0) = y_0$, $y'(x_0) = y'_0$ and it is $y_0 \neq 0$, $\operatorname{Im}(y'_0/y_0) \neq -\tfrac{1}{2}\varphi'(x_0)$. Such a solution is of the form

$$y = r(x)\exp\left\{-\frac{i}{2}\varphi(x) + \int_{\alpha_0}^{x}\frac{dt}{p(t)\,r^2(t)}\right\}$$

and the function $r(x)$ is a solution of the equation

$$(16)\qquad [p(x)\,r']' + [\tfrac{1}{4}p(x)\,\varphi'^2(x) + q(x)\cos[\psi(x) - \varphi(x)]\}\,r = \frac{C^2}{p(x)\,r^3}$$

where

$$C = p(x_0)\,|y_0|^2\left[\operatorname{Im}\frac{y'_0}{y_0} + \tfrac{1}{2}\varphi'(x_0)\right] \neq 0\,.$$

All other solutions of (1) belong to the group B and each of them is of the form $y = r(x)\,e^{-i\varphi(x)/2}$. The function $r(x)$ is a solution of the equation

$$(17)\qquad [p(x)\,r']' + \{\tfrac{1}{4}p(x)\,\varphi'^2(x) + q(x)\cos[\psi(x) - \varphi(x)]\}\,r = 0\,.$$

We see that the solutions of the latter group are determined by the solutions of the linear equation (17) with real coefficients. Every solution of the equation (16) is of the form $r = [r_1^2(x) + r_2^2(x)]^{1/2}$ where r_1, r_2 are suitable independent solutions of (17) whose wronskian $W(r_1, r_2)$ has the value $C \neq 0$. Especially every solution of the group B has no zeros.

The advantage in considering these special cases is that they provide a set of much needed examples to give insight into the behaviour of solutions of (1). In both cases A, B the behaviour of solutions (1) is determined by means of the linear differential equation (17) with real valued coefficients, theory of which is developed in depth because it was very much in the center of attention during the past decades.

Theorem 5. *Assume the conditions* (2) *and that a solution* $s(x)$ *of* (5) *has exactly* $n \geqq 1$ *zeros* $a < x_1 < \ldots < x_n \leqq b$. *Let* $y(x)$ *be a solution of* (1) *satisfying*

$$(18)\qquad \operatorname{Re}\frac{y'(a)}{y(a)} \geqq \frac{s'(a)}{s(a)}\,.$$

(*The expression on the right (or left) of* (18) *is considered to be* $+\infty$ *if* $s(a) = 0$ (*or* $y(a) = 0$).) *Then* $y(x)$ *has at most n zeros on* $(a, x_n]$.

Proof. Suppose that there exists a solution $y(x)$ of (1) having $n + 1$ zeros $a < x'_1 < \ldots < x'_{n+1} \leqq x_n$. In view of Theorem 3 the solution $y(x)$ can be written in the form $y = r(x)\,e^{i\Theta(x)}$. If $y(a) \neq 0$, we have $\operatorname{Re}(y'(a)/y(a)) = r'(a)/r(a)$; further $r(a) = 0$ if $y(a) = 0$. Since the equation (5) is a Sturm majorant of (1), Sturm's first comparison theorem {[3], pp. 334} guarantees the existence of a solution $s(x)$ of (5) having at least $n + 1$ zeros on (a, x'_{n+1}) which contradicts to the assumption. This completes the proof.

Consequence. If the differential equation (5) is nonoscillatory on J then the same is true for the equation (1).

Theorem 6. *Let the assumptions* (2) *be satisfied. Let* $m(x) \geqq 0$ *be a continuous function on* $[a, b]$ *and*

$$\gamma_m = \inf_{x\in[a,b]} \frac{m(x)}{\displaystyle\int_a^x \frac{dt}{p(t)} \int_x^b \frac{dt}{p(t)}} \,. \tag{19}$$

If

$$\int_a^b m(t)\,[\tfrac{1}{4}p(t)\,\varphi'^2(t) + q(t)\cos[\psi(t) - \varphi(t)]]^+\,dt \leqq \gamma_m \int_a^b \frac{dt}{p(t)}$$

where $[\;]^+ = \max([\;], 0)$ *then the differential equation* (1) *is disconjugate on* $[a, b]$.

Proof. The assertion of this theorem follows immediatelly from a simple modification of a theorem due to Hartman {[3], pp. 345} to the equation (5) which will be introduced like a

Lemma 5. *Let the assumptions* (2) *be satisfied and let* γ_m *be defined by* (19). *If there exists a solution* $s(x)$ *of* (5) *having at least two zeros on* $[a, b]$ *then it holds*

$$\int_a^b m(t)\,[\tfrac{1}{4}p(t)\,\varphi'^2(t) + q(t)\cos[\psi(t) - \varphi(t)]]^+\,dt > \gamma_m \int_a^b \frac{dt}{p(t)} \,.$$

Theorem 7. *Assume the conditions* (2) *on* $[0, l]$. *Let* $y(x)$ *be a solution of* (1) *and* N *the number of its zeros on* $0 < x \leqq l$. *Then*

$$N < 1 + \frac{1}{2}\left\{\int_0^l \frac{dt}{p(t)} \int_0^l [\tfrac{1}{4}p(t)\,\varphi'^2(t) + q(t)\cos[\psi(t) - \varphi(t)]]^+\,dt\right\}^{1/2} . \tag{20}$$

Proof. In order to prove this, let $N \geqq 2$ and let N zeros of s on $(0, l]$ be $x_1 < \ldots < x_N$. Since

$$\int_a^x \frac{dt}{p(t)} \int_x^b \frac{dt}{p(t)} \leqq \frac{1}{4}\left[\int_a^b \frac{dt}{p(t)}\right]^2$$

the choice $m(x) \equiv 1$ in Lemma 5 gives

$$\int_a^b [\tfrac{1}{4}p(t)\,\varphi'^2(t) + q(t)\cos[\psi(t) - \varphi(t)]]^+ \,\mathrm{d}t > 4\left[\int_a^b \frac{\mathrm{d}t}{p(t)}\right]^{-1}.$$

Also, since the harmonic mean of $N - 1$ positive numbers is majorized by their arithmetic mean, we have

$$(21)\quad \left\{\frac{1}{N-1}\sum_{k=1}^{N-1}\left[\int_{x_k}^{x_{k+1}}\frac{\mathrm{d}t}{p(t)}\right]^{-1}\right\}^{-1} \leqq \frac{1}{N-1}\sum_{k=1}^{N-1}\int_{x_k}^{x_{k+1}}\frac{\mathrm{d}t}{p(t)} = \frac{1}{N-1}\int_{x_1}^{x_N}\frac{\mathrm{d}t}{p(t)},$$

for $k = 1, \ldots, N - 1$.

Thus adding (21) for $k = 1, \ldots, N - 1$ gives

$$\int_0^l [\,]^+ \,\mathrm{d}t > \int_{x_1}^{x_N} [\,]^+ \,\mathrm{d}t \geqq 4\sum_{k=1}^{N-1}\left[\int_{x_k}^{x_{k+1}}\frac{\mathrm{d}t}{p(t)}\right]^{-1} \geqq$$

$$\geqq 4(N-1)^2\left[\sum_{k=1}^{N-1}\int_{x_k}^{x_{k+1}}\frac{\mathrm{d}t}{p(t)}\right]^{-1} \geqq 4(N-1)^2\left[\int_0^l\frac{\mathrm{d}t}{p(t)}\right]^{-1}$$

hence (20).

Theorem 8. *Let p, q, φ, ψ satisfy the conditions* (2) *on* $[x_0, +\infty)$ *and*

$$\tfrac{1}{4}p(x)\,\varphi'^2(x) + q(x)\cos[\psi(x) - \varphi(x)] \leqq 0 \quad on \quad J\,.$$

Then there exist solutions $s_0(x)$, $s_1(x)$ of the equation (5) *satisfying the conditions $s_0(x) > 0$, $s_0'(x) \leqq 0$, $s_1(x) > 0$, $s_1'(x) > 0$ for $x \geqq x_0$. The differential equation* (1) *has a pair of solutions y_0, y_1 with the properties*

$$\frac{s_1'(x)}{s_1(x)} \leqq \operatorname{Re}\frac{y_1'(x)}{y_1(x)}, \quad \frac{s_0'(x)}{s_0(x)} \geqq \operatorname{Re}\frac{y_0'(x)}{y_0(x)}\,.$$

Proof. The existence of s_0, s_1 is guaranteed by a well known theorem introduced for example in the monograph by P. Hartman {[3], pp 357}. Since the equation (5) is a Sturm majorant of (1) and in view to the relation $r'/r = \operatorname{Re} y'/y$ the rest of the assertion follows from the comparison theorem {[3], pp. 358–359}.

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