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AN ORTHOGONAL THEORY OF A SET-VALUED BIFUNCTOR

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INTRODUCTION

The purpose of this paper is to extend the concept of a pure and a torsion theory [3] and [2] with respect to a set-valued bifunctor. In the section 1., the proposition 1.1. justifies the Fieldhouse's definition ([3], 1) of a pure theory with respect to the usual one defined by a proper class of short exact sequences of an abelian category. In the section 2., there is given the complete description of the orthogonal theories of the tensor and the torsion product together with an important example of the orthogonal theories of the bifunctor Ext^1 in abelian groups.

1. ORTHOGONAL THEORIES

Let $\mathscr{C}_1$ and $\mathscr{C}_2$ be two categories and $\mathbf{F} : \mathscr{C}_1 \times \mathscr{C}_2 \to Sets$ be a bifunctor into the category of sets. An *orthogonal theory* or the bifunctor $\mathbf{F}$ consists of an ordered pair $(\mathscr{M}, \mathscr{L})$ of classes of objects from $|\mathscr{C}_1|$ and $|\mathscr{C}_2|$ respectively, which are orthogonally closed with respect to the bifunctor $\mathbf{F}$. In other words, if we denote

$$\mathscr{L}^+ = \{M \in |\mathscr{C}_1| \mid \operatorname{card} \mathbf{F}(M, L) \leqq 1, \ \forall L \in \mathscr{L}\}$$

and

$$\mathscr{M}^* = \{L \in |\mathscr{C}_2| \mid \operatorname{card} \mathbf{F}(M, L) \leqq 1, \ \forall M \in \mathscr{M}\}$$

then

$$\mathscr{M} = \mathscr{L}^+ \quad \text{and} \quad \mathscr{L} = \mathscr{M}^* .$$

In such a case we shall say that $\mathscr{M}$ is the left orthogonal class and $\mathscr{L}$ the right orthogonal class. The orthogonal theory $(\mathscr{M}, \mathscr{L})$ is trivial if either $\mathscr{M}$ or $\mathscr{L}$ consists of $|\mathscr{C}_1|$ or $|\mathscr{C}_2|$ respectively. If $|\mathscr{C}_1| = |\mathscr{C}_2|$ and $(\mathscr{L}, \mathscr{M})$ is also an orthogonal theory, we shall say that the orthogonal theory $(\mathscr{M}, \mathscr{L})$ is commutative, furthermore the class $\mathscr{M} \cap \mathscr{L}$ is called the kernel of the orthogonal theory $(\mathscr{M}, \mathscr{L})$ regardless of the commutativity. Unless otherwise specified, we assume throughout this paper that the bifunctor $\mathbf{F}$ is covariant in both variables since the contravariant case can be obtained by the dualization of categories. If the bifunctor $\mathbf{F} : \mathscr{C}^0 \times \mathscr{C} \to \mathit{Sets}$ ($\mathscr{C}^0$ is the dual category of $\mathscr{C}$) can be factorized as $\mathbf{F}(X, Y) = \mathrm{Mor}_{\mathscr{C}'}(\mathbf{K}(X), \mathbf{K}(Y))$, where $\mathscr{C}'$ is a quotient category of $\mathscr{C}$, $\mathbf{K} : \mathscr{C} \to \mathscr{C}'$ the corresponding functorial epimorphism and $\mathrm{Mor}_{\mathscr{C}'}(A, B)$ the set of morphisms from the object A into the object B in the category $\mathscr{C}'$, then every orthogonal theory of $\mathbf{F}$ is called the *pure theory* for $\mathscr{C}$ with respect to the functorial epimorphism $\mathbf{K}$. The pure theory for $\mathscr{C}$, where $\mathscr{C} = \mathscr{C}'$ and $\mathbf{K}$ is the identity functor, is called the *torsion theory* for $\mathscr{C}$.

Examples. (i) Let $\mathscr{T}_P$ be the category of topological pairs and continuous maps preserving the base subspaces. Since the homotopy as an equivalence relation on sets of morphisms of $\mathscr{T}_P$ is compatible with the category structure of $\mathscr{T}_P$, it induces the desired quotient category and consequently the pure theory $(\mathscr{T}_P, \mathscr{C})$, where $\mathscr{C}$ is the class of all contractible spaces.

(ii) Let $\dot{\mathscr{T}}$ be the category of pointed topological spaces and continuous maps preserving base points. If we denote by $\mathscr{D}$ the full subcategory of $\dot{\mathscr{T}}$ having one point connected base component, we get the torsion theory $(\mathscr{K}, \mathscr{D})$ for $\dot{\mathscr{T}}$, where $\mathscr{K}$ is the full subcategory of pointed connected topological spaces.

The following proposition justifies the Fieldhouse's definition ([3], 1) of the pure theory with respect to the usual one defined by a proper class of short exact sequences of an abelian category ([7], 368). Let $\mathscr{C}$ be an abelian category. Denote by $\mathscr{E}(\mathscr{C})$ the additive category of short exact sequences of $\mathscr{C}$([7], 375).

Proposition 1.1. *Let $\mathscr{C}$ be an abelian category. Then any proper class $\mathscr{P} \subset |\mathscr{E}(\mathscr{C})|$ induces the pure theory $(|\mathscr{E}(\mathscr{C})|, \mathscr{P})$ for $\mathscr{E}(\mathscr{C})$.*

Proof. Let $E, E' \in |\mathscr{E}(\mathscr{C})|$ and $f_1, f_2 \in \mathrm{Mor}_{\mathscr{E}(\mathscr{C})}(E, E')$ with the structure $E : 0 \to A \to B \to C \to 0$, $E' : 0 \to A' \to B' \to C' \to 0$ and $f_i = (\alpha_i, \beta_i, \gamma_i)$, $i = 1, 2$. Now, we shall define an equivalence relation on each $\mathrm{Mor}_{\mathscr{E}(\mathscr{C})}(E, E')$ which is compatible with the category structure of $\mathscr{E}(\mathscr{C})$, since such a relation determines a quotient category of $\mathscr{E}(\mathscr{C})$. According to [7], 372, we have the commutative diagrams

$$\begin{array}{lccccccc} & 0 \to \mathrm{Mor}_{\mathscr{C}}(X, A) & \to & \mathrm{Mor}_{\mathscr{C}}(X, B) & \to & \mathrm{Mor}_{\mathscr{C}}(X, C) & \underset{\delta}{\to} & \mathrm{Ext}^1(X, A) \\ (1) & \downarrow{\scriptstyle \alpha_i} & & \downarrow{\scriptstyle \beta_i} & & \downarrow{\scriptstyle \gamma_i} & & \downarrow{\scriptstyle \bar{\alpha}_i} \\ & 0 \to \mathrm{Mor}_{\mathscr{C}}(X, A') & \to & \mathrm{Mor}_{\mathscr{C}}(X, B') & \to & \mathrm{Mor}_{\mathscr{C}}(X, C') & \underset{\delta'}{\to} & \mathrm{Ext}^1_{\mathscr{C}}(X, A') \end{array}$$

$i = 1, 2$, for $\forall X \in |\mathscr{C}|$.

Since the proper monomorphism are closed under push-outs, the morphism $\bar{\alpha}_i$, $i = 1, 2$, carry the subgroup of proper exact sequences $\mathrm{Ext}^1_{\mathscr{P}}(X, A)$ into the subgroup $\mathrm{Ext}^1_{\mathscr{P}}(X, A')$ ([7], 369) and hence there are the induced homomorphisms $\bar{\bar{\alpha}}_i$, $i = 1, 2$, which make the following diagram commutative (π and π' are the canonical projections).

(2)

$$\begin{array}{ccccccc}
0 \to \mathrm{Mor}_{\mathscr{C}}(X, A) & \to & \mathrm{Mor}_{\mathscr{C}}(X, B) & \to & \mathrm{Mor}_{\mathscr{C}}(X, C) & \xrightarrow[\pi \circ \delta]{} & \mathrm{Ext}^1_{\mathscr{C}}(X, A)/\mathrm{Ext}^1_{\mathscr{P}}(X, A) \\
\downarrow{\scriptstyle \alpha_i} & & \downarrow{\scriptstyle \beta_i} & & \downarrow{\scriptstyle \gamma_i} & & \downarrow{\scriptstyle \bar{\bar{\alpha}}_i} \\
0 \to \mathrm{Mor}_{\mathscr{C}}(X, A') & \to & \mathrm{Mor}_{\mathscr{C}}(X, B') & \to & \mathrm{Mor}_{\mathscr{C}}(X, C') & \xrightarrow{\pi' \circ \delta'} & \mathrm{Ext}^1_{\mathscr{C}}(X, A')/\mathrm{Ext}^1_{\mathscr{P}}(X, A')
\end{array}$$

Since f_i, $i = 1, 2$, were chosen arbitralily, we can define the desired relation on $\mathrm{Mor}_{\mathscr{E}(\mathscr{C})}(E, E')$ by $f_1 \sim f_2 \Leftrightarrow \pi' \circ \delta' \circ \gamma_1 = \pi' \circ \delta' \circ \gamma_2$, $\forall X \in |\mathscr{C}|$. It is easy to show that it is the equivalence relation compatible with the category structure of $\mathscr{E}(\mathscr{C})$. Let us find $|\mathscr{E}(\mathscr{C})|^*$. Consider again the diagram (2). If E runs through the whole $|\mathscr{E}(\mathscr{C})|$, then $E' \in |\mathscr{E}(\mathscr{C})|^* \Leftrightarrow \pi' \circ \delta' = 0$ and since the equality $\pi' \circ \delta' = 0$ holds for $\forall X \in |\mathscr{C}|$, $E' \in \mathscr{P}$. Conversely, the implication $E' \in \mathscr{P} \Rightarrow \pi' \circ \delta' = 0$ follows immediately by [7], 372, q.e.d.

As to the properties of orthogonal theories one can prove the similar propositions as in [2], 228–9, including the fact that both left and right orthogonal classes are closed under isomorphic copies.

2. AN APPLICATION

Unless otherwise specified, we assume throughout this section that all the mentioned groups are abelian. First, let us introduce some notation important for this part of the paper.

$\mathscr{AB}$ – the category of abelian groups,
$\mathscr{T}$ – the full subcategory of ordinary torsion groups,
$\mathscr{F}$ – the full subcategory of ordinarily torsion-free groups,
$\mathscr{D}$ – the full subcategory of divisible groups,
$\mathscr{R}$ – the full subcategory of reduced groups,
G_t – the ordinary torsion subgroup of the $G \in |\mathscr{AB}|$,
G_R – the reduced subgroup of the $G \in |\mathscr{AB}|$, i.e.
$G_R \cong G/D$, where D is the maximal divisible subgroup of G,
G_{Rt} – the ordinary torsion part of G_R,
G_{tR} – the reduced subgroup of G_t,
G_{RF} – the ordinarily torsion-free group of G_R, i.e. $G_{RF} = G_R/G_{Rt}$,
Q – the group of rational integers,
N – the set of natural numbers,
P – the set of natural prime numbers,

$\mathbf{Z}(p^k) = \mathbf{Z}/p^k\mathbf{Z},\ p \in \mathbf{P},\ k \in \mathbf{N}$,
$\mathscr{C} = \{G \in |\mathscr{A}\mathscr{B}|\ |\mathrm{Ext}^1(\mathbf{Q}, G) = 0\}$ – the cotorsion groups,
$\mathbf{Z}(p^\infty)$ – the Prüfer p-group, $p \in \mathbf{P}$,
$\mathbf{Q}(J) = \{m/n \in \mathbf{Q} \mid n = \prod p_i^{\alpha_i},\ p_i \in J \subset \mathbf{P},\ \alpha_i \in \mathbf{N}\}$,
$\coprod\limits_\alpha G_\alpha$ – the coproduct of the groups G_α,
$\prod\limits_\alpha G_\alpha$ – the product of the groups G_α,
$G^{(I)}$ – the coproduct of copies of G with the index set I.

We shall say that $G_1 \in \mathscr{T}$ has the primary decomposition disjoint with the primary decomposition of $G_2 \in \mathscr{T}$ if $G_1 = \coprod\limits_{p \in I_1} G_{1,p}$ and $G_2 = \coprod\limits_{p \in I_2} G_{2,p}$ and $I_1 \cap I_2 = \emptyset$ provided that $I_1, I_2 \subset \mathbf{P}$.

A formal symbol $\prod\limits_{p \in \mathbf{P}} p^{e(p)}$, where $e(p) = 0$ or 1 or ∞ is called the restricted Steinitz number (RSN). If $e(p)$ can have only the value 0 or 1, it is called the strongly restricted Steinitz number (SRSN).

Proposition 2.1. *Let $A, B \in |\mathscr{A}\mathscr{B}|$. Then*

$$A \otimes B = 0 \Leftrightarrow \{A_t \otimes B_t = A_t \otimes B/B_t = A/A_t \otimes B_t = A/A_t \otimes B/B_t = 0\}.$$

Proof. ($\Rightarrow$) The sequence $0 \to A_t \to A \to A/A_t \to 0$ is exact and since the tensor product $\otimes$ is the right-exact functor in both variables, $B \otimes A/A_t = 0$. Consider the exact sequence $0 \to B_t \to B \to B/B_t \to 0$. Since A/A_t is the flat $\mathbf{Z}$-module, $B_t \otimes A/A_t = B/B_t \otimes A/A_t = 0$ and by symmetry $A_t \otimes B/B_t = 0$.

Hence we have the exact sequences $\mathrm{Tor}^1(A/A_t, B_t) = 0 \to A_t \otimes B_t \to A \otimes B_t \to 0$ and $\mathrm{Tor}^1(B/B_t, A) = 0 \to B_t \otimes A \to 0$ which imply the desired result. ($\Leftarrow$) The right hand side implies $A_t \otimes B = 0$ and consequently $A \otimes B = 0$, q.e.d.

Corollary 2.2. *If $A \otimes B = 0$ then either A or B is the ordinary torsion group. If $A \in \mathscr{T}$ then $A \otimes B = 0 \Leftrightarrow \{A_R \otimes B_{Rt} = A \otimes B_R/B_{Rt} = 0.\}$.*

Proof. The first part of the proof follows directly from the proposition 2.1. since A/A_t is flat and $A/A_t \otimes \mathbf{Z} \cong A/A_t \subset A/A_t \otimes B/B_t$. The rest of the proof is an immediate consequence of the first part of the proof and the fact that the tensor product of the ordinary torsion and divisible group is always the zero-group, q.e.d.

Corollary 2.3. *Let $A \in \mathscr{T}$, $B \in |\mathscr{A}\mathscr{B}|$. Then $A \otimes B = 0$ iff both the following conditions are satisfied*:

(i) *B_{Rt} possesses the primary decomposition disjoint with the primary decomposition of A_R.*

(ii) *B_R/B_{Rt} is p-divisible for any prime p which determines the primary decomposition of A,*

Proof. (i) By [4], 255, $A_R \otimes B_{Rt} \cong \bar{A}_R \otimes \bar{B}_{Rt}$, where $\bar{A}_R$ and $\bar{B}_{Rt}$ are the basic subgroups of A_R and B_{Rt} respectively and since the basic subgroup is a direct sum of cyclic subgroups and the bifunctor $\otimes$ preserves coproducts, (i) immediately follows.

(ii) By [4], 255, $A \otimes B_R/B_{Rt} \cong \prod_p A_p^{(r(p))}$, where $r(p)$ is the rank of B_{RF}/pB_{RF} and p runs through the set of primes which determines the primary decomposition of A, q.e.d.

Remark. Notice that $B_R \notin \mathscr{T}$ implies the weaker condition in (ii) — it is sufficient to consider $B_R = pB_R$.

Theorem 2.4. *There is a one-one correspondence between the orthogonal theories of the tensor product in abelian groups and the set of restricted Steinitz numbers. In this correspondence if $\prod_{p \in \mathbf{P}} p^{e(p)}$ is a RSN, the corresponding left orthogonal class is $\mathscr{M} = \{\bigcup_{p \in \mathbf{P}} G_p\}^{*+}$, where*

$$G_p = \begin{cases} 0\,, & \text{if } e(p) = 0\,, \\ \mathbf{Z}(p)\,, & \text{if } e(p) = 1\,, \\ \mathbf{Z}(p^\infty)\,, & \text{if } e(p) = \infty\,. \end{cases}$$

Moreover, $\mathscr{M}$ precisely consists of the ordinary torsion groups $G = G_R \oplus D$, $D \in \mathscr{D}$, where the primary decomposition of G_R consists of the p-groups with $e(p) = 1$ and the primary decomposition of D consists of the p-groups with $e(p) = 1$ or ∞. On the other hand, the corresponding right orthogonal class $\mathscr{M}^*$ consists of the groups B such that B_{Rt} is p-divisible for each $p \in \mathbf{P}$ with $e(p) = 1$ and B_R/B_{Rt} is p-divisible for each $p \in \mathbf{P}$ with $e(p) = 1$ or ∞.

Proof. We shall construct the inverse map from the set of orthogonal theories into the set of RSN's. Let $(\mathscr{M}, \mathscr{X})$ be an orthogonal theory of $\otimes$. By the corollary 2.2. either $\mathscr{M} \subset \mathscr{T}$ or $\mathscr{X} \subset \mathscr{T}$. Since $\otimes$ is the commutative bifunctor, we can assume that $\mathscr{M} \subset \mathscr{T}$. If there exists a non-zero p-group $M \in \mathscr{M}$ but $M_R = 0$ for every such a group, we shall set $e(p) = \infty$. If there exists a p-group $M \in \mathscr{M}$ such that $M_R \neq 0$, we shall set $e(p) = 1$. Otherwise, set $e(p) = 0$. Now, if $e(p) = 1$, then $\mathscr{M}$ contains all the p-groups. For, $M \in \mathscr{M}$ being the p-group with $M_R \neq 0$ implies by [4], 80 that M_R possesses a non-zero direct summand $\mathbf{Z}(p^k) \in \mathscr{M}$. Since the bifunctor $\otimes$ is right-exact $\mathbf{Z}(p^m) \in \mathscr{M}$, for each $1 \leqq m \leqq k$ and the exact sequence $0 \to \mathbf{Z}(p^k) \to \to \mathbf{Z}(p^{k+1}) \to \mathbf{Z}(p) \to 0$ implies that $\mathbf{Z}(p^m) \in \mathscr{M}$, for each $1 \leqq m < \infty$. Let G be an arbitrary p-group with $e(p) = 1$. There exists the universal epimorphism $\varphi : : \coprod_{g \in G} \mathbf{Z}(p^{k(g)}) \to G$ induced by the injective monomorphisms $\mathbf{Z}(p^{k(g)}) \to G$, where $k(g)$

is the exponent of the order of the element $g \in G$. Hence by the right exactness of $\otimes$ which preserves coproducts, $G \in \mathcal{M}$. To finish the proof it is sufficient to show that for a given RSN the corresponding left orthogonal class $\mathcal{M}$ precisely consists of the groups $G = G_R \oplus D$ satisfying the hypothesis, since the rest immediately follows from the corollary 2.3. According to what we have just proved, every such a group $G = G_R \oplus D \in \mathcal{M}$. First, let there exist a non-zero q-group $M \in \mathcal{M}$ with $e(q) = 0$. Since $M_R \neq 0$ implies $\mathbf{Z}(q) \in \mathcal{M}$ and by the corollary 2.3. $\mathbf{Z}(q) \in \mathcal{M} \cap \mathcal{M}^* \subset \mathcal{D} \cap \mathcal{T}$, we have $M_R = 0$. The corollary 2.3. also implies that $\mathbf{Q}(J) \in \mathcal{X}$, where $J = \{p \in \in \mathbf{P} \mid e(p) = 1 \text{ or } \infty\}$ since by [4] 255, $\mathbf{Z}(q^\infty) \otimes \mathbf{Q}(J) \cong \mathbf{Z}(q^\infty)^{(r(q))}$, where $r(q)$ is the rank of $\mathbf{Q}(J)/q\,\mathbf{Q}(J) \neq 0$, it yields the contradiction $M = 0$. Secondly, let there exist a non-zero q-group $M \in \mathcal{M}$ with $e(q) = \infty$ and $M_R \neq 0$. Then similarly, the corollary 2.3. implies the contradiction $\mathbf{Z}(q) \in \mathcal{M} \cap \mathcal{M}^* \subset \mathcal{D} \cap \mathcal{T}$, q.e.d.

Proposition 2.5. *Let $(\mathcal{M}, \mathcal{X})$ be an orthogonal theory of $\otimes$ in abelian groups with the RSN $\prod_{p \in \mathbf{P}} p^{e(p)}$.*

Then it possesses the following properties:

(i) *$(\mathcal{M}, \mathcal{X})$ is the commutative orthogonal theory.*

(ii) *The kernel of $(\mathcal{M}, \mathcal{X})$ is $\{D \in \mathcal{D} \cap \mathcal{T} \mid D = \coprod \mathbf{Z}(p^\infty), e(p) = 1 \text{ or } \infty\}$.*

In the following we shall investigate the orthogonal theories of the bifunctors Tor^1 and Ext^1 ([7], 150, 63).

Proposition 2.6. *Let $A, B \in |\mathcal{AB}|$. Then* $\mathrm{Tor}(A, B) = 0$ *iff the primary decomposition of A_t is disjoint with the primary decomposition of B_t.*

Proof. If we apply Tor on the exact sequences $0 \to A_t \to A \to A/A_t \to 0$ and $0 \to \to B_t \to B \to B/B_t \to 0$, we shall get the equality $\mathrm{Tor}(A, B) \cong \mathrm{Tor}(A_t, B_t)$ by the left exactness of Tor and the fact that any ordinary torsion-free group is flat.

Since Tor preserves coproducts and $\mathrm{Tor}(\mathbf{Z}(p^k), G) \cong \{g \in G \mid p^k g = 0\}$ ([7, 130), it is sufficient to prove that $\mathrm{Tor}(A_{tp}, B_{tq}) = 0$, for every $p, q \in \mathbf{P}$, $p \neq q$. Consider the exact sequence $0 \to \bar{A}_{tp} \to A_{tp} \to A_{tp}/\bar{A}_{tp} \to 0$, where $\bar{A}_{tp}$ is the basic subgroup of A_{tp}. Since $\bar{A}_{tp}$ is a direct sum of cyclic p-groups and $A_{tp}/\bar{A}_{tp}$ is a divisible p-group, it is sufficient to prove that $\mathrm{Tor}(\mathbf{Z}(p^\infty), B_{tq}) = 0$. But this is immediate since $\mathrm{Tor}(\mathbf{Z}(p^\infty), B_{tq}) \cong \{b \in B_{tq} \mid \exists (k \in \mathbf{N}) \text{ such that } p^k b = 0\} = 0$ q.e.d.

Theorem 2.7. *There is a one-one correspondence between the orthogonal theories of the torsion product* Tor *in abelian groups and the set of strongly restricted Steinitz numbers. In this correspondence, if $\prod_{p \in \mathbf{P}} p^{e(p)}$ is a SRSN, the corresponding left orthogonal class is $\mathcal{M} = \{\mathbf{Z}(p) \mid p \in \mathbf{P} \text{ and } e(p) = 1\}^{*+}$.*

Moreover $\mathcal{M}$ consists of the groups whose ordinary torsion subgroup has the primary decomposition from the set $J = \{p \in \mathbf{P} \mid e(p) = 1\}$. On the other hand,

$\mathcal{M}^*$ consists of the groups whose ordinary torsion subgroup has the primary decomposition disjoint with the set J.

Proof. We shall construct the inverse map from the set of orthogonal theories into the set of SRSN's. Let $(\mathcal{M}, \mathcal{X})$ be an orthogonal theory of Tor. If there exists a non-zero p-group $M \in \mathcal{M}$, we shall set $\mathrm{e}(p) = 1$ and otherwise $\mathrm{e}(p) = 0$. First, we have to show that if $\mathcal{M}_1 = \{\mathbf{Z}(p) \mid \mathrm{e}(p) = 1\}^{*+}$ then $\mathcal{M}_1$ contains no non-zero q-group for $q \in \mathbf{P}$, $q \notin J = \{p \in \mathbf{P} \mid \mathrm{e}(p) = 1\}$.

Since $\mathrm{Tor}(\mathbf{Z}(p), G) \cong \{g \in G \mid pg = 0\}$ ([7], 130), $\mathcal{M}_1^*$ consists of the groups whose ordinary torsion part has the primary decomposition disjoint with the set J and hence $\mathcal{M}_1$ contains no non-zero q-group M, for $q \in \mathbf{P}$, $q \notin J$ (Otherwise $M \in \mathcal{M}_1 \cap \mathcal{M}_1^*$ and $\mathrm{Tor}(M, M) \neq 0$ by the proposition 2.6.) To finish the proof, it is sufficient to show that if $(\mathcal{M}, \mathcal{X})$ is an orthogonal theory of Tor with the SRSN $\prod_{p \in \mathbf{P}} p^{\mathrm{e}(p)}$, then $\mathcal{M}$ consists of the groups whose ordinary torsion part has the primary decomposition from the set J. Since Tor is left exact, $M \in \mathcal{M}$ implies that $M_t \in \mathcal{M}$ has the primary decomposition from the set J. By the proposition 2.6., $\mathcal{X}$ consists of the groups whose ordinary torsion part has the primary decomposition disjoint with J and consequently $\mathcal{M}$ precisely consists of the groups whose ordinary torsion part has the primary decomposition from the set J, q.e.d.

Proposition 2.8. *Let $(\mathcal{M}, \mathcal{X})$ be an orthogonal theory of* Tor^1. *Then it possesses the following properties*:

(i) *$(\mathcal{M}, \mathcal{X})$ is the commutative orthogonal theory.*

(ii) *The kernel of $(\mathcal{M}, \mathcal{X})$ is $\mathcal{F}$.*

Theorem 2.9. *There is a one-one correspondence between the orthogonal theories $(\mathcal{M}, \mathcal{X})$ of* Ext^1, *where $\mathbf{Q} \in \mathcal{M}$ and the set of SRSN's. In this correspondence, if $\prod_{p \in \mathbf{P}} p^{\mathrm{e}(p)}$ is s SRSN, the corresponding left-orthogonal class $\mathcal{M}$ is $\{\{\mathbf{Q}\} \cup \{\mathbf{Z}(p) \mid p \in \mathbf{P}, \mathrm{e}(p) = 1\}\}^{*+}$.*

Moreover, $\mathcal{M}$ consists of the groups whose ordinary torsion part has the primary decomposition from the set $J = \{p \in \mathbf{P} \mid \mathrm{e}(p) = 1\}$. On the other hand, $\mathcal{M}^*$ consists of the cotorsion groups which are p-divisible for $\forall p \in J$.

Proof. First, let us define the inverse map from the set of orthogonal theories $(\mathcal{M}, \mathcal{X})$, where $\mathbf{Q} \in \mathcal{M}$ into the set of SRSN's. Let $(\mathcal{M}, \mathcal{X})$ be such an orthogonal theory of Ext. If there exists a non-zero p-group $M \in \mathcal{M}$, we shall set $\mathrm{e}(p) = 1$ and otherwise $\mathrm{e}(p) = 0$. We have to show that $\mathcal{M}_1 = \{\{\mathbf{Q}\} \cup \{\mathbf{Z}(p) \mid p \in \mathbf{P}, \mathrm{e}(p) = 1\}\}^{*+}$ contains no non-zero q-group for $q \in \mathbf{P}$, $q \notin J$. According to [4], 243, $\mathrm{Ext}(\mathbf{Z}(p), G) \cong G/pG$ and since $\mathbf{Z}(q)$ is the cotorsion group ([4], 85) the existence of such a non-zero group $M \in \mathcal{M}$ and the right exactness of Ext imply the contradiction $\mathbf{Z}(q) \in \mathcal{M}_1 \cap \mathcal{M}_1^*$.

To finish the proof it is sufficient to show that if $(\mathscr{M}, \mathscr{L})$ is an orthogonal theory, where $\mathbf{Q} \in \mathscr{M}$ and with SRSN $\prod_{p \in \mathbf{P}} p^{e(p)}$ then $\mathscr{M}$ precisely consists of the groups whose ordinary torsion part has the primary decomposition from J. Since Ext is right exact, $\mathscr{M}$ is closed under subobjects and according to what we have just proved, $M \in \mathscr{M}$ implies that the primary decomposition of M_t is from J. By [5], 370, $\mathbf{Q} \in \mathscr{M}$ implies $\mathscr{F} \subset \mathscr{M}$, hence it is sufficient to show that $\mathscr{M}$ contains all the p-groups, for $\forall p \in J$. Since $\mathscr{M}$ is closed under subgroups and group extensions ([2], 224), $\mathbf{Z}(p^k) \in \mathscr{M}$, for $\forall(k \in \mathbf{N})$ and $\forall p \in J$. Hence with respect to the existence of basic subgroups it remains to show that $\mathbf{Z}(p^\infty) \in \mathscr{M}$, for $\forall p \in J$. Consider the exact sequence $0 \to \mathrm{Hom}(\mathbf{Z}(p^\infty), L) \to \mathrm{Hom}(\mathbf{Q}(p), L) \to \mathrm{Hom}(\mathbf{Z}, L) \cong L \to \mathrm{Ext}(\mathbf{Z}(p^\infty), L) \to \mathrm{Ext}(\mathbf{Q}(p), L) \to \mathrm{Ext}(\mathbf{Z}, L) = 0$, for $\forall L \in \mathscr{L}$. Since each divisible group is injective, it is sufficient to consider only the groups $L \in \mathscr{L} \cap \mathscr{R}$ and this immediately implies the short exact sequence $0 \to \mathrm{Hom}(\mathbf{Q}(p), L) \xrightarrow[\varphi]{} \mathrm{Hom}(\mathbf{Z}, L) \to \mathrm{Ext}(\mathbf{Z}(p^\infty), L) \to 0$, since $\mathbf{Q}(p)$ as the torsion-free group is contained in $\mathscr{M}$. Hence, to prove that $\mathrm{Ext}(\mathbf{Z}(p^\infty), L) = 0$, for $\forall L \in \mathscr{L}$, all we have to show is that the homomorphism

$$\begin{aligned} \varphi : \mathrm{Hom}(\mathbf{Q}(p), L) &\to \mathrm{Hom}(\mathbf{Z}, L) \\ \mathbf{f} &\longmapsto \mathbf{f}/\mathbf{Z} \end{aligned}$$

is surjective, for $\forall L \in \mathscr{L}$. Let $\mathbf{g} \in \mathrm{Hom}(\mathbf{Z}, L)$. Since $\mathrm{Ext}(\mathbf{Z}(p), G) \cong G/pG$ ([4], 243), $\mathscr{L}$ consists of the p-divisible cotorsion groups, for $\forall p \in J$ and the existence of the homomorphism $\mathbf{f} \in \mathrm{Hom}(\mathbf{Q}(p), L)$ such that $p^n\mathbf{f}(1/p^n) = \mathbf{g}(1)$, for $\forall n \in \mathbf{N}$ immediately follows, q.e.d.

Remark. A method of generation of orthogonal theories of the bifunctor $\mathrm{Mor}_{\mathscr{C}}$, where $\mathscr{C}$ is an abelian category is described in [6].

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