Hari Ballabh Mittal

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SOME GENERATING FUNCTIONS FOR POLYNOMIALS

HARI BALLABH MITTAL, Lucknow

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1. Introduction. Recently, J. W. BROWN [1] proved, that for the Laguerre polynomials

$$\sum_{n=0}^{\infty} L_n^{(a+mn)}(x)\, t^n = \frac{(1+v)^{1+a}}{1-mv}\, e^{-xv}, \tag{1.1}$$

where $v = t(1+v)^{1+m}$, m being an integer. Also, assuming

$$\sum_{n=0}^{\infty} L_n^{(a+mn)}(x)\, t^n = A(t)\, e^{xB(t)}, \tag{1.2}$$

he proved that

$$\sum_{n=0}^{\infty} L_n^{(-a-[1+m]n)}(x)\, t^n = \frac{A(-t)}{1-B(-t)} \exp\left\{\frac{-xB(-t)}{1-B(-t)}\right\}. \tag{1.3}$$

L. CARLITZ [2], extended the results of Brown and proved that (1.1) and (1.3) hold for any m.

A natural question arises as to whether we can find results of the forms (1.1) and (1.3) for other known polynomials, m being a constant.

The present paper is an answer to this question. We have found results of the forms (1.1) and (1.3) for generalized Laguerre, generalized Gould-Hopper, generalized Bessel and Jacobi polynomials. These results hold for all values of m. The results of this paper also extend the results, which we found on another occasion [9]. There [9], we found explicit expressions of the form (1.1) for $m = -1, 0, 1, 2$, for generalized Laguerre, generalized Gould-Hopper, generalized Bessel and Jacobi polynomials. The treatment being formal, we shall obtain our results quite heuristically.

2. Operational formulae. Recently [8], we considered a class of polynomials $F_n(x, a, m, p_r(x))$, defined by the Rodrigues formula

$$F_n(x, a, m, p_r(x)) = x^{-a} e^{p_r(x)}\, D^n\left[x^{mn+a} e^{-p_r(x)}\right], \tag{2.1}$$

a, m, being constants, $p_r(x)$ being a polynomial in x of degree r. It is immediate that the above polynomial reduces to the generalized Laguerre polynomial [7] for $m = 1$ and to the generalized Gould-Hopper polynomial [8] for $m = 0$. If, however, $p_r(x) = b/x$, then (2.1) reduces to the Bessel polynomial of KRALL and FRINK [6]. If $p_r(x)$ be a polynomial in $1/x$ of degree r, then for $m = 2$, we get a generalization of the Bessel polynomials.

In [7], we dealt with an operator T_k, defined by $T_k = x(k + xD)$, and hence such that

$$(2.2) \qquad T_k^n\{x^{b+r}\} = (b + r + k)_n \, x^{b+r+n} \,,$$

k being a constant. It is easily seen that

$$(2.3) \qquad T_k^n = x^n \prod_{j=0}^{n-1} (\delta + k + j)\,, \quad \delta = xD\,.$$

Since,

$$D^n[x^{mn+a}e^{-p_r(x)}] = x^{-n} \prod_{j=0}^{n-1} (\delta - j)\,\{x^{mn+a}e^{-p_r(x)}\} =$$

$$= \prod_{j=1}^{n} (\delta + j)\,\{x^{mn+a-n}e^{-p_r(x)}\} = x^{k-1} \prod_{j=0}^{n-1} (\delta + j + k)\,\{x^{mn+a-n-k+1}e^{-p_r(x)}\} =$$

$$= x^{k-n-1}\, T_k^n\{x^{mn+a-n-k+1}e^{-p_r(x)}\}\,,$$

we have from (2.1), the class of operational formulae

$$(2.4) \qquad F_n(x, a, m, p_r(x)) = x^{k-n-a-1}e^{p_r(x)}\, T_k^n\{x^{mn+a-n-k+1}e^{-p_r(x)}\}\,.$$

Giving different values to k in (2.4), we get different operational representations for the polynomial $F_n(x, a, m, p_r(x))$. If, however, we take $k = mn + a - n + 1$, we get the interesting result

$$(2.5) \qquad T^n_{(m-1)n+a+1}\{e^{-p_r(x)}\} = x^{2n-mn}e^{-p_r(x)}\, F_n(x, a, m, p_r(x))\,,$$

from which, by giving different values to m, we get operational formulae for different polynomials.

3. The generating functions. By making use of the Lagranges expansion formula

$$(3.1) \qquad (1 + v)^{a+1} = 1 + (a + 1) \sum_{n=1}^{\infty} \binom{a + (b + 1)\,n}{n - 1} \frac{t^n}{n}\,,$$

where

$$(3.2) \qquad v = t(1 + v)^{b+1}\,, \quad v(0) = 0\,,$$

L. CARLITZ [2], proved that

$$\frac{(1+v)^{a+1}}{1-bv} = \sum_{n=0}^{\infty} \binom{a+(b+1)n}{n} t^n , \tag{3.3}$$

where $v = t(1+v)^{b+1}$. Let

$$f(x) = \sum_{r=0}^{\infty} a_r x^r . \tag{3.4}$$

Then,

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} T^n_{(m-1)n+a+1}\{x^b f(x)\} = \sum_{n,r=0}^{\infty} \frac{t^n}{n!} a_r(b+r+(m-1)n+a+1)_n x^{b+r+n} =$$

$$= \sum_{n,r=0}^{\infty} a_r \binom{b+r+mn+a}{n} t^n x^{b+r+n} .$$

Hence,

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} T^n_{(m-1)n+a+1}\{x^b f(x)\} = \sum_{n,r=0}^{\infty} a_r \binom{b+r+mn+n}{n} t^n x^{b+r+n} . \tag{3.5}$$

Making use of (3.3), we get from (3.5)

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} T^n_{(m-1)n+a+1}\{x^b f(x)\} = \sum_{r=0}^{\infty} a_r x^{b+r} \frac{(1+v)^{b+r+a+1}}{1-(m-1)v} ,$$

where $v = xt(1+v)^m$. Hence, we have the general operational generating function

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} T^n_{(m-1)n+a+1}\{x^b f(x)\} = \frac{x^b(1+v)^{a+b+1}}{1-(m-1)v} f[x(1+v)] , \tag{3.6}$$

where $v = xt(1+v)^m$.

Again, from (3.1), we have by putting $b = m-1$,

$$(1+v)^{a+1} = 1 + (a+1)\sum_{n=1}^{\infty} \binom{a+mn}{n-1} \frac{t^n}{n} , \tag{3.7}$$

where $v = t(1+v)^m$. Putting $a = 0$ in (3.7), we get

$$v = \sum_{n=1}^{\infty} \binom{mn}{n-1} \frac{t^n}{n} . \tag{3.8}$$

Similarly, for $a = -2$, we get from (3.7)

$$\frac{v}{1+v} = \sum_{m=1}^{\infty} \binom{mn-2}{n-1} \frac{t^n}{n} . \tag{3.9}$$

Again, following Carlitz [2], we define

$$B(t, c) = -\sum_{n=1}^{\infty} \binom{(c+1)n}{n-1} \frac{t^n}{n} \tag{3.10}$$

and

$$A(t, a, c, d) = \frac{(1 - B(t, c))^{a+d+1}}{1 + c\, B(t, c)}. \tag{3.11}$$

In view of (3.8), (3.10) and (3.11), the generating function (3.6) becomes

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} T^n_{(m-1)n+a+1}\{x^b f(x)\} = \tag{3.12}$$

$$= x^b A(xt, a, m-1, b) f[x(1 - B(xt, m-1))].$$

Again, as is shown by Carlitz [2],

$$B(t, -c-1) = \frac{-B(-t, c)}{1 - B(-t, c)}, \tag{3.13}$$

and

$$A(t, -a, -c-1, -d) = \frac{A(-t, a, c, d)}{1 - B(-t, c)}. \tag{3.14}$$

Therefore, by (3.12), (3.13) and (3.14), we have the operational generating function

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} T^n_{-(m-1)-a+1}\{x^{-b} f(x)\} = \tag{3.15}$$

$$= \frac{x^{-b} A(-xt, a, m-2, b)}{1 - B(-xt, m-2)} f\left[x\left\{1 + \frac{B(-xt, m-2)}{1 - B(-xt, m-2)}\right\}\right].$$

Now, let $b = 0$ and $f(x) = e^{-p_r(x)}$ in (3.6), then

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} T^n_{(m-1)n+a+1}\{e^{-p_r(x)}\} = \frac{(1+v)^{a+1}}{1-(m-1)v} \exp\{-p_r[x(1+v)]\}, \tag{3.16}$$

where $v = xt(1+v)^m$. Using (2.5) in (3.16), we get the general result

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} F_n(x, a, m, p_r(x)) = \frac{(1+v)^{a+1}}{1-(m-1)v} e^{p_r(x)} e^{-p_r[x(1+v)]}, \tag{3.17}$$

where $v = x^{m-1} t(1+v)^m$.

4. Generalized Gould-Hopper Polynomial. Earlier [8], we considered a generalization of Gould-Hopper polynomial [5], defined by

$$H_n(x, a, p_r(x)) = (-1)^n x^{-a} e^{p_r(x)} D^n[x^a e^{-p_r(x)}], \tag{4.1}$$

$p_r(x)$ being a polynomial in x of degree r, a being arbitrary. $H_n(x, a, p_r(x))$ reduces to the Gould-Hopper polynomial for $p_r(x) = px^r$. In terms of the polynomial $F_n(x, a, m, p_r(x))$, we have

$$F_n(x, a, 0, p_r(x)) = (-1)^n H_n(x, a, p_r(x)) . \tag{4.2}$$

The operational formula for the relevant polynomial is

$$T^n_{a+1-n}\{e^{-p_r(x)}\} = x^{2n} e^{-p_r(x)} F_n(x, a, 0, p_r(x)) . \tag{4.3}$$

Putting $m = 0$ in (3.17) and using (4.2), we get immediately the generating function; for the Gould-Hopper polynomial, to be

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(x, a, p_r(x)) = x^{-a}(x - t)^a e^{p_r(x)} e^{-p_r(x-t)} . \tag{4.4}$$

Again, making use of (4.3), we have from (3.16), the general result

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(x, a + mn, p_r(x)) = \frac{(1 + v)^{a+1}}{1 - (m - 1) v} e^{p_r(x)} e^{-p_r[x(1+v)]} , \tag{4.5}$$

where $v = -(t/x)(1 + v)^m$, m being a constant. For $m = 0$, (4.5) reduces to (4.4). From (4.5), we get generating functions for the generalized Gould-Hopper polynomials of different orders.

Again, putting $b = 0$ and $f(x) = e^{-p_r(x)}$ in (3.15) and using (4.3), we get the general result

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(x, -a - (m - 2) n, p_r(x)) = \tag{4.6}$$

$$= \frac{A(t/x, a, m - 2, 0)}{1 - B(t/x, m - 2)} e^{p_r(x)} . \exp\left\{-p_r\left[x\left\{1 + \frac{B(t/x, m - 2)}{1 - B(t/x, m - 2)}\right\}\right]\right\},$$

m being a constant.

Again, using (2.5) and (4.3), we get the relation

$$F_n(x, a, 0, p_r(x)) = x^{-mn} F_n(x, a - mn, m, p_r(x)) , \tag{4.7}$$

where m is a constant. From (4.7), we get readily the relation between the generalized Gould-Hopper polynomial and other general polynomials.

5. Generalized Laguerre Polynomials. Elsewhere [7], we considered a generalization of the Laguerre polynomial, defined by

$$(5.1) \qquad T_{rn}^{(a)}(x) = \frac{1}{n!} x^{-a} e^{p_r(x)} D^n[x^{n+a} e^{-p_r(x)}],$$

a being a constant, $p_r(x)$ being a polynomial in x of degree r. $T_{rn}^{(a)}(x)$ reduces to the generalized Laguerre polynomial of CHATTERJEA [3] for $p_r(x) = px^r$ and to the Laguerre polynomial for $p_r(x) = x$. In terms of the polynomial $F_n(x, a, m, p_r(x))$, we have

$$(5.2) \qquad F_n(x, a, 1, p_r(x)) = n!\, T_{rn}^{(a)}(x).$$

The operational formula for the generalized Laguerre polynomial is

$$(5.3) \qquad T_{a+1}^{n}\{e^{-p_r(x)}\} = n!\, x^n e^{-p_r(x)}\, T_{rn}^{(a)}(x),$$

$p_r(x)$ being a polynomial in x of degree r, a being a constant.

Putting $m = 1$ in (3.17), and using (5.2), we get immediately the generating function for the generalized Laguerre polynomial

$$(5.4) \qquad \sum_{n=0}^{\infty} t^n\, T_{rn}^{(a)}(x) = (1 - t)^{-a-1}\, e^{p_r(x)} e^{-p_r[x(1-t)^{-1}]}.$$

Again, making use of (5.3), we have from (3.16), the general generating function

$$(5.5) \qquad \sum_{n=0}^{\infty} t^n\, T_{rn}^{(a+(m-1)n)}(x) = \frac{(1 + v)^{a+1}}{1 - (m - 1)\, v}\, e^{p_r(x)} e^{-p_r[x(1+v)]},$$

where $v = t(1 + v)^m$, m being a constant. For $m = 1$, (5.5) reduces to (5.4). From (5.5) we get the generating functions for the generalized Laguerre polynomials of different orders.

Again, putting $b = 0$ and $f(x) = e^{-p_r(x)}$ in (3.15) and using (5.3), we get the result

$$(5.6) \qquad \sum_{n=0}^{\infty} t^n\, T_{rn}^{(-a-(m-1)n)}(x) = \frac{A(-t, a, m - 2, 0)}{1 - B(-t, m - 2)}\, e^{p_r(x)} \exp\left[\!\left[-p_r \left[x \left\{ 1 + \frac{B(-t, m - 2)}{1 - B(-t, m - 2)} \right\} \right] \right]\!\right],$$

m being a constant. The results in (5.5) and (5.6) are the generalizations of the results of BROWN [1] and CARLITZ [2].

Again, using (2.5) and (5.3), we get the relation

$$(5.7) \qquad F_n(x, a, 1, p_r(x)) = x^{(1-m)n}\, F_n(x, a - (m - 1)\, n, m, p_r(x)),$$

where m is a constant. From (5.7), we get readily the relation between the generalized Laguerre polynomial and other general polynomials.

6. The Bessel Polynomials. The operational representation for the generalized Bessel polynomials [6], is easily seen to be

$$(6.1)\qquad T^n_{a+1+n}\{e^{-b/x}\} = b^n e^{-b/x}\, y_n(x, a+2, b)\,.$$

Again, from (2.5), we have for $p_r(x) = -b/x$ and $m = 2$

$$(6.2)\qquad T^n_{a+1+n}\{e^{-b/x}\} = e^{-b/x}\, F_n(x^{-1}, a, 2, -b/x)\,,$$

and hence

$$(6.3)\qquad F_n(x^{-1}, a, 2, -b/x) = b^n\, y_n(x, a+2, b)\,.$$

Again, making use of (6.1), we get immediately from (3.16), the following generating function for the Bessel polynomials:

$$(6.4)\quad \sum_{n=0}^{\infty} \frac{t^n b^n}{n!}\, y_n(x, a+(m-2)\,n+1, b) = \frac{(1+v)^{a+1}}{1-(m-1)\,v} \exp\left\{\frac{bv}{x(1+v)}\right\},$$

where we have put $p_r(x) = -b/x$ in (3.16) and where $v = xt(1+v)^m$, m being a constant. The relation in (6.4) gives us immediately, generating functions for Bessel polynomials of different orders for different values of m.

Again, putting $b = 0$ and $f(x) = e^{-b/x}$ in (3.15), and using (6.1), we get the result

$$(6.5)\quad \sum_{n=0}^{\infty} \frac{t^n b^n}{n!}\, y_n(x, -a-mn+1, b) = \frac{A(-xt, a, m-2, b)}{1-B(-xt, m-2)} \exp\left\{\frac{b}{x}\, B(-xt, m-2)\right\},$$

m being a constant.

7. The Jacobi Polynomials. It is easily seen that the operational representation for the Jacobi polynomial is given by

$$(7.1)\qquad T^n_{a+1}\{(1-x)^{n+b}\} = n!\,(1-x)^b\, x^n\, P_n^{(a,b)}(1-2x)\,.$$

Putting $b = 0, f(x) = (1-x)^b$, $m = 1$ in (3.6), we get the result of Feldheim [4]

$$(7.2)\qquad \sum_{n=0}^{\infty} t^n\, P_n^{(a,b-n)}(x) = (1-t)^b \left[1 - \frac{t}{2}(1+x)\right]^{-a-b-1}.$$

Also, from (3.6), we have for $b = 0$, $f(x) = (1-x)^b$, the generating function for the Jacobi polynomials

$$(7.3)\quad \sum_{n=0}^{\infty} t^n\, P_n^{(a+(m-1)n,b-n)}(x) = \frac{(1+v)^{a+1}}{1-(m-1)\,v}\left[1 - \frac{(1-x)(1+v)}{2}\right]^b \left(\frac{1+x}{2}\right)^{-b},$$

where $2v = (1+x)(1+v)^m\, t$, m being a constant.

Again, if we put $b = 0, f(x) = (1 - x)^b$ in (3.15), we get the result

(7.4) $$\sum_{n=0}^{\infty} t^n P_n^{(-a-(m-1)n, b-n)}(x) =$$

$$= \left(\frac{1+x}{2}\right)^{-b} \frac{A\left(\frac{-t}{2}(1+x), a, m-2, 0\right)}{1 - B\left(-\frac{t}{2}(1+x), m-2\right)} \left[1 - x\left\{\frac{1}{1 - B\left(-\frac{t}{2}(1+x), m-2\right)}\right\}\right]^b .$$

It is interesting to note that similar results for other polynomials can be found by the method outlined above.

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Author's address: Lucknow University, Lucknow, India.