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CONTINUITY AND MAXIMUM PRINCIPLE FOR POTENTIALS OF SIGNED MEASURES

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The classical theorem of EVANS-VASILESCO states that a Newtonian potential $U\mu$ of a positive measure μ with compact support K is continuous provided its restriction to K is continuous [8], [19].

On the occasion of the "5. Tagung über Probleme und Methoden der Mathematischen Physik" in Karl-Marx-Stadt (1973), Prof. B.-W. SCHULZE advanced in a discussion the following problem: Does the theorem extend to the case of potentials of signed measures?

Using fine topology arguments we prove the following

Theorem 1'. *Let μ be a signed measure with support K in the m-dimensional euclidean space R^m $(m > 2)$ and let $U\mu$ be finite in R^m. If the restriction of $U\mu$ to K is continuous on K, then the potential $U\mu$ is continuous in the whole space.*

It is known from the classical potential theory that for every Newtonian potential of a positive measure μ with compact support K the following maximum principle of MARIA-FROSTMAN [16], [9] holds:

$$\sup_{x\in R^m} U\mu(x) = \sup_{x\in K} U\mu(x) .$$

An extension of this important property to the case of potentials of signed measures is contained in the following theorem ($[z]^+$ and $[z]^-$ denote respectively the positive and negative parts of a number z).

Theorem 2'. *If μ is a signed measure with support $K \subset R^m$ and $U\mu$ is finite in R^m, then*

$$[\inf_{x\in K} U\mu(x)]^- = \inf_{x\in R^m} U\mu(x) \leqq \sup_{x\in R^m} U\mu(x) = [\sup_{x\in K} U\mu(x)]^+ .$$

In fact, we establish the above results as a consequence of theorems proved below in the context of Brelot's axiomatics of harmonic spaces in which a somewhat stronger form D^* of the axiom of domination is fulfilled. It should be noted here that D^* is in particular true for a class of elliptic partial differential equations investigated in connection with the axiomatic potential theory in [12], [13], [3] and [14].

In what follows we shall suppose that X is a strong harmonic space in the sense of [2] in which the following axiom D^* is satisfied:

D^*: *A finite potential p with compact support $S(p)$ is continuous provided its restriction to $S(p)$ is continuous.*

We are going to show that the axiom D of domination (see [7], Chap. 9, [10], Chap. II) is fulfilled in X. Indeed, suppose that p is a locally bounded potential on X such that its restriction to $S(p)$ is continuous and fix $x_0 \in S(p)$. It is sufficient to verify that p is continuous at x_0. Choose a relatively compact neighborhood U of x_0. By Satz 5.1.4 of [2] there are potentials p_1, p_2 such that p_1 is harmonic in the complement of $\bar{U}$, p_2 is harmonic in U and $p = p_1 + p_2$. Then $S(p_1) \subset \bar{U} \cap S(p)$ is compact and the restriction of p_1 to $S(p_1)$ is continuous. By D^*, in particular, p_1 is continuous at x_0. Since p_2 is continuous at x_0, the same is true for p.

(Note that D does not imply D^* as shown by an example in [6], Corollary 1.2.)

By the result of KÖHN-SIEVEKING [15], X is an elliptic space and since the Brelot convergence axiom is satisfied ([2], Satz 1.5.6), each component of X is a harmonic space in the sense of the axiomatics developed by M. BRELOT (see [5]).

As for the axiom D^*, note that it is fulfilled in the Greenian case and more generally in the case A_2 of Brelot's axiomatics (see [10], Theorem 10.15 and Section 2.7). In particular, D^* is true in any strong harmonic space associated to partial differential equations of elliptic type investigated in [12] (see théorème 36.2), [3] (see p. 12), [13] (cf. p. 222) and [14] (cf. p. 338). For the validity of D^* in the classical case of the Laplace equation for domains having a Green function see [11], Theorem 6.20. In particular, Theorems 1′, 2′ follow immediately from Theorems 1, 2 below and the Riesz representation theorem for potentials.

If $U \subset X$, then ∂U is the boundary of U in X, while the symbol $\partial_f U$ stands for the fine boundary of U (that is, the boundary of U in the fine topology on X). We shall use the following result of B. FUGLEDE [10], which was in the classical case proved under certain restrictive conditions by M. Brelot [4].

Proposition. *Let u be a harmonic function on an open set $U \subset X$, let p be a finite potential on X and $M \subset X$ a polar set. If $u \geqq -p$ on U and*

$$\operatorname{fine}\lim_{x \to y} u(x) \geqq 0$$

for any $y \in \partial_f U - M$, then $u \geqq 0$ on U.

For the proof we refer to the more general Theorem 9.1 in [10]. We remark only that by Theorem 10.15 in [10], any finite potential is semibounded and by Theorem 8.7 in [10], every harmonic function is finely harmonic.

We also make use of the following property of any finite potential p on X:

(1) $$\hat{R}_p^{S(p)} = p \quad \text{on} \quad X\,,$$

which follows from Lemma 6.8 in [10].

Let us denote by $\mathscr{P}^*$ the set of all differences of two finite potentials on X. If $v \in \mathscr{P}^*$, then $S(v)$ (= the support of v) is the complement of the maximal open set on which v is harmonic. It should be noted here that any $v \in \mathscr{P}^*$ is finely continuous.

Lemma. *Suppose that $v \in \mathscr{P}^*$. Then there are finite potentials p, q such that $v = p - q$ and $S(p) \cup S(q) \subset S(v)$.*

Proof. Denote $K = S(v)$ and $U = X - K$. By the hypothesis there are two finite potentials v_1, v_2 such that $v = v_1 - v_2$. Put $p = \hat{R}_{v_1}^K$, $q = \hat{R}_{v_2}^K$, $w = p - q$. Then p, q are finite potentials harmonic on U ([2], Korolar 2.3.5), so that $S(p) \cup S(q) \subset S(v)$ and the function w is, of course, finely continuous. Since for $i = 1, 2$ the set $\{x \in K;\ \hat{R}_{v_i}^K(x) < v_i(x)\}$ is polar ([7], Corollary 9.2.3, Theorem 9.1.1, Corollary 6.3.6), there is a polar set M such that for any $x \in K - M$ the equality $w(x) = v(x)$ holds. Consider now on U the harmonic function $u = w - v$. Obviously,

$$-q - v_1 \leqq u \leqq p + v_2$$

and for any $x \in \partial_f U - M$

$$\operatorname{fine\,lim}_{y \to x} u(y) = 0\,.$$

By the Proposition, $u = 0$ on U. We see that the finely continuous function $v - w$ vanishes on $X - M$. Since polar sets are nowhere dense in the fine topology ([7], Proposition 6.2.3), $v = w = p - q$ everywhere on X.

The proof is complete.

Theorem 1. *Let $v \in \mathscr{P}^*$ and let the restriction of v to $S(v)$ be continuous. Then v is continuous on X.*

Proof. Write $v = p - q$ where p, q have the property described in the Lemma. Put $U = X - S(v)$, $f = v \mid \partial U$ (= the restriction of v to ∂U) and $u' = v \mid U$. Note that U is resolutive ([7], Theorem 2.4.2) and since

(2) $$|f| \leqq p + q\,,$$

f is resolutive by Proposition 2.4.1 and Corollary 2.4.1 in [7]. Of course, $|H_f^U| \leqq p + q$. Let us denote by M the set of all points at which the set $S(v)$ is thin. Then $M \subset \partial U$ and M is exactly the set of all non-regular points of U. Consequently, M is a polar set ([7], Corollary 9.2.3, Theorem 9.1.1). Since v is finely continuous on X, we have for any $x \in \partial_f U$

(3) $$\operatorname{fine\,lim}_{y \to x} u'(y) = f(x)\,.$$

In view of the fact that f is continuous on ∂U and all points of $\partial U - M$ are regular, we have for any $x \in \partial U - M$

(4) $$f(x) = \lim_{y \to x} H_f^U(y) ,$$

and, consequently,

(5) $$\text{fine}\lim_{y \to x} H_f^U(y) = f(x) , \quad x \in \partial_f U - M .$$

Since on X

$$-2(p + q) \leqq u' - H_f^U \leqq 2(p + q) ,$$

we conclude from (3) and (5) by the Proposition that $u' = H_f^U$ on U and it follows from (4) that for any $x \in \partial U - M$

(6) $$\lim_{\substack{y \to x \\ y \in U}} v(y) = v(x) .$$

It remains to investigate the points of M. Fix an $x \in M$ and recall that $S(p) \cup S(q) \subset \subset S(v)$, so that $X - U \supset S(p) \cup S(q)$. Since $\{x\} \subset M$ is a polar set, we obtain ([7], Corollary 6.2.1)

$$\hat{R}_p^{X-U}(x) = \hat{R}_p^{X-(U \cup \{x\})}(x)$$

and (1) yields

$$\hat{R}_p^{X-U}(x) \geqq \hat{R}_p^{S(p)}(x) = p(x) .$$

Since evidently $\hat{R}_p^{X-(U \cup \{x\})}(x) \leqq p(x)$, we conclude

$$\hat{R}_p^{X-(U \cup \{x\})}(x) = p(x) ,$$

analogous equality being true for q. Consequently,

$$\int f \, \mathrm{d}\varepsilon_x^{X-(U \cup \{x\})} = \int (p - q) \, \mathrm{d}\varepsilon_x^{X-(U \cup \{x\})} =$$

$$= \hat{R}_p^{X-(U \cup \{x\})}(x) - \hat{R}_q^{X-(U \cup \{x\})}(x) = p(x) - q(x) = f(x) .$$

Since $|f|$ is dominated by a potential (see (2)) we obtain by Corollary 7.2.6 in [7]

$$\lim_{\mathfrak{U}} H_f^U = f(x)$$

for any ultrafilter $\mathfrak{U}$ on U converging to x. It follows that

$$f(x) = \lim_{y \to x} H_f^U(y) = \lim_{\substack{y \to x \\ y \in U}} v(y)$$

and (6) holds for any $x \in \partial U$. By the hypothesis for any $x \in \partial U$ we have

(7) $$\lim_{\substack{y \to x \\ y \in S(v)}} v(y) = v(x)$$

and we conclude easily from (6) and (7) that v is continuous on X.

The proof is complete.

Theorem 2. *Suppose that constant functions are harmonic. If $v \in \mathscr{P}^*$, then*

$$[\inf_{x \in S(v)} v(x)]^- = \inf_{x \in X} v(x) \leqq \sup_{x \in X} v(x) = [\sup_{x \in S(v)} v(x)]^+ .$$

Proof. Since $v \in \mathscr{P}^*$ implies $-v \in \mathscr{P}^*$ it is sufficient to establish the equality

$$\sup_{x \in X} v(x) = [\sup_{x \in S(v)} v(x)]^+ . \tag{8}$$

Let p, q be finite potentials such that $v = p - q$. Put $k = \sup_{x \in S(v)} v(x)$ and suppose that $k \leqq 0$. Then

$$\operatorname{fine\,lim}_{y \to x, y \in X - S(v)} v(y) = v(x) \leqq 0 , \quad x \in \partial_f(X - S(v))$$

and since $v \leqq p$, we conclude by the Proposition that $v \leqq 0$ on X. Hence if $k_1 = \sup_{x \in X} v(x) \leqq 0$, then $p + q \geqq -v \geqq -k_1$ on X and $k_1 = 0$, because $p + q$ is a potential and constant functions are harmonic. On the other hand, if $k_1 > 0$, then the above reasoning shows that $k > 0$. Let us consider the function $u = k - v$. We have

$$\operatorname{fine\,lim}_{y \to x, y \in X - S(v)} u(y) = u(x) \geqq 0 , \quad x \in \partial_f(X - S(v))$$

and $u = k - (p - q) \geqq -p$. By the Proposition, $v \leqq k$ on $X - S(v)$ and (8) is proved.

Corollary. *If $v \in \mathscr{P}^*$ vanishes on $S(v)$, then $v = 0$ on X.*

The problem arises whether any $v \in \mathscr{P}^*$ satisfying the hypotheses of Theorem 1 is necessarily a difference of two continuous potentials. The following example shows that this is not the case even if we require in addition that v is a difference of two bounded potentials with compact support. In this example, X is the harmonic space associated to the Laplace equation in R^m, $m > 2$.

Example. Choose strictly positive numbers $c_n, \varrho_n, \varrho'_n$ in such a way that $\varrho'_n < \varrho_n$, $c_n \searrow 0$, $\varrho'_k/\varrho_k \to 1$,

$$\sum_{n=1}^{\infty} \left(\frac{\varrho_n}{c_n}\right)^{m-2} < 1 ,$$

$c_k - \varrho_k > c_{k+1} + \varrho_{k+1}$ for any k and $c_n/|c_l - c_n| \leqq 2$ provided $l \neq n$. (We may put $c_n = 2^{-n}$, $\varrho_n = \alpha(n!)^{2-m}$, $\varrho'_n = (n/(n+1))\, \varrho_n$, where $\alpha > 0$ is sufficiently small.) Put

$$z_n = [c_n, 0, \ldots, 0] \in R^m , \quad v_n^+ = [c_n + \varrho_n, 0, \ldots, 0] , \quad v_n^- = [c_n - \varrho_n, 0, \ldots, 0]$$

and denote by Ω_n and Ω'_n the ball with centre z_n and radius ϱ_n and ϱ'_n, respectively. Define

$$p_n(x) = \begin{cases} (\varrho_n/|x - z_n|)^{m-2} & \text{for } x \notin \Omega_n \\ 1 & \text{for } x \in \Omega_n , \end{cases}$$

$$p_n'(x) = \begin{cases} (\varrho_n/|x - z_n|)^{m-2} & \text{for } x \notin \Omega_n' \\ (\varrho_n/\varrho_n')^{m-2} & \text{for } x \in \Omega_n' , \end{cases}$$

$$p = \sum_{n=1}^{\infty} p_n , \quad p' = \sum_{n=1}^{\infty} p_n' .$$

Clearly, p and p' is a Newtonian potential of a positive measure ν and ν' with support in $\{0\} \cup \bigcup_{n=1}^{\infty} \partial\Omega_n$ and $\{0\} \cup \bigcup_{n=1}^{\infty} \partial\Omega_n'$, respectively. Since

$$p(0) = p'(0) < 1 , \quad p(z_n) \geqq p_n(z_n) = 1 , \quad p'(z_n) \geqq p_n'(z_n) > 1$$

and $z_n \to 0$, we conclude that p and p' are not continuous at 0. Fix now $y \in \partial\Omega_k$ and put $Q = \{k - 1, k, k + 1\}$. If $n < k - 1$, then

$$|y - z_n|^{m-2} \geqq |v_k^+ - z_n|^{m-2} \geqq |z_{k-1} - z_n|^{m-2} \geqq c_n^{m-2} \, . \, 2^{2-m} ,$$

while for $n > k + 1$

$$|y - z_n|^{m-2} \geqq |v_K^- - z_n|^{m-2} \geqq |z_{k+1} - z_n|^{m-2} \geqq c_n^{m-2} \, . \, 2^{2-m} .$$

We see that

$$\sum_{n \notin Q} p_n(y) = \sum_{n \notin Q} \left(\frac{\varrho_n}{|y - z_n|} \right)^{m-2} \leqq 2^{m-2} \sum_{n \notin Q} \left(\frac{\varrho_n}{c_n} \right)^{m-2} .$$

It follows easily that p is a bounded potential continuous at any point of $\bigcup_{n=1}^{\infty} \partial\Omega_n$. One establishes analogously that p' is a bounded potential continuous at any point of $\bigcup_{n=1}^{\infty} \partial\Omega_n'$. Putting $\mu = \nu - \nu'$, we obtain a signed measure with support $K = \{0\} \cup \bigcup_{n=1}^{\infty} \partial\Omega_n \cup \bigcup_{n=1}^{\infty} \partial\Omega_n'$. If $\mu = \mu^+ - \mu^-$ is the Jordan decomposition of μ, then obviously $\mu^+ = \nu$, $\mu^- = \nu'$ and the potential $U\mu$ is continuous at any point of $K - \{0\}$. We are going to prove that $(U\mu) \mid K$ is continuous at 0. For $y \in \bigcup_{k=1}^{\infty} \partial\Omega_k$ we have $U\mu(y) = 0$, while $U\mu(z) = 1 - (\varrho_k/\varrho_k')^{m-2}$ for $z \in \partial\Omega_k'$. Since $U\mu(0) = 0$ and $\varrho_k/\varrho_k' \to 1$, the continuity of $(U\mu) \mid K$ at 0 is obvious.

Suppose that $U\mu = U\mu_1 - U\mu_2$, where μ_1, μ_2 are positive measures with continuous potentials. Then $\mu = \mu_1 - \mu_2$ by the unicity theorem and $\mu_1 \geqq \mu^+$, $\mu_2 \geqq \mu^-$ by the minimal property of the Jordan decomposition (see [18], 6.14). Since any potential of a positive measure is lower semicontinuous, the potential $U\mu^+ = U\mu_1 - U(\mu_1 - \mu^+)$ is also upper semicontinuous. Consequently, $U\mu^+$ is a continuous potential, which is a contradiction.

Remarks. 1. In [1] (p. 354) an example of a bounded continuous potential w (in R^2) with the following property is given: If u, v are subharmonic functions such that $w = u - v$ in R^2, then both u and v are unbounded at the origin.

2. Suppose that G_1 is a continuous function on $R^m \times R^m$ ($m > 2$) and put

$$G(x, y) = |x - y|^{2-m} + G_1(x, y), \quad x \neq y,$$
$$G(x, x) = +\infty.$$

With any signed measure μ with compact support we associate the potential $G\mu$ defined by

$$G\mu(x) = \int G(x, y) \, d\mu(y)$$

for those x for which the integral is meaningful. Observing that the potential $G_1\mu$ (defined in the obvious way) is continuous on R^m, we deduce immediately from Theorem 1′ the following

Proposition. *Let μ be a signed measure with compact support K and let $G\mu$ be finite on R^m. Then the potential $G\mu$ is continuous on the whole space, provided its restriction to K is continuous.*

This proposition applies for example to the potentials corresponding to the Helmholz equation in R^3:

$$\Delta u + \lambda^2 u = 0 \quad (\lambda \in R^1).$$

Indeed, the kernel is given (up to a constant multiple) by

$$G(x, y) = \frac{\cos \lambda |x - y|}{|x - y|}, \quad x \neq y,$$
$$G(x, x) = +\infty$$

and it is obvious that the function

$$[x, y] \mapsto \frac{\cos \lambda |x - y| - 1}{|x - y|}$$

is extensible to a continuous function G_1 on $R^3 \times R^3$.

3. Theorems 1′ and 2′ were announced in [17].

Added 26. 6. 1974. During the conference on potential theory (Oberwolfach, 16. 6.–22. 6. 1974) Prof. Mokobodzki gave another proof of Theorem 1 based on properties of reduits and specific order. Prof. Fuglede noted that potentials p, q from Lemma can also be constructed as follows: If $v = v_1 - v_2$ where v_1, v_2 are finite potentials and w is the specific infimum of v_1, v_2, then one may put $p = v_1 - w$, $q = v_2 - w$.

Added 4. 6. 1975. A strong domination axiom ($\bar{D}$) equivalent with D^* has recently been investigated by K. Janssen and Nguyen-Xuan-Loc (see Math. Z. *141* (1975), 185—191; Z. Wahrscheinlichkeitstheorie und Verw. Gebiete *31* (1975). 147—155).

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