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A QUANTITATIVE EXTENSION OF THE PERRON-FROBENIUS THEOREM FOR DOUBLY STOCHASTIC MATRICES

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To the memory of Professor VLADIMÍR KNICHAL

Introduction. Let A be an irreducible stochastic matrix of order n. It follows from the theory of nonnegative matrices that -1 cannot be a proper value of A unless the imprimitivity index [2] of A is even. If the index of imprimitivity of A is even, there exists a subset M of the set $N = \{1, 2, \ldots, n\}$ such that $a_{ik} = 0$ for $i \in M$, $k \in M$ and for $i \in N \setminus M$, $k \in N \setminus M$. In other words the matrix can be brought (by the same permutation of rows and columns) to a block diagonal form of the following type:

$$A = \begin{bmatrix} 0 & A_{12} \\ A_{21} & 0 \end{bmatrix} \tag{1}$$

Therefore -1 is not a proper value of A if A cannot be brought to the aforementioned form. It is to be expected that a quantitative refinement of this statement can be obtained: if we introduce a characteristic measuring how far the given matrix is from matrices of the above form (1) then it is conceivable that the distance of all eigenvalues of A from -1 would be bounded from below by a number depending on that characteristic. It is the purpose of the present paper to estimate the distance of the proper values of A from -1 in terms of two numbers μ and σ the first being a measure of irreducibility, the second a number measuring how close the given matrix is to matrices of the form (1). The measure of irreducibility has been used in [1].

1. PRELIMINARIES

Notation. Let n be a natural number, $n > 2$. We shall denote by $\mathscr{S}$ the set of all symmetric stochastic matrices of order n. The letter N will stand for the set of all natural numbers $\leqq n$. A matrix is a mapping from $N \times N$ into the reals, the value of the mapping A at the point $[i, k]$ being denoted by a_{ik}. If A and B are two matrices, we write $A \geqq B$ if $a_{ik} \geqq b_{ik}$ for all $i, k \in N$. Vectors are column vectors of length n, e

will be the vector defined by $e^T = (1, 1, \ldots, 1)$. If A is a given nonnegative matrix, we define

$$\mu(A) = \min \{ \sum_{M, N \setminus M} a_{ik};\ M \subset N,\ 0 \neq M \neq N \},$$

$$\sigma(A) = \min_M \{ (\sum_{M,M} a_{ik} + \sum_{N \setminus M, N \setminus M} a_{ik});\ M \subset N \}.$$

Observe that if $S = (s_{ik})$ is doubly stochastic then for any M

$$\sum_{M, N \setminus M} s_{ik} = \sum_{N \setminus M, M} s_{ik}.$$

Given an operator T in a linear space E, we denote by $\mathscr{R}(T)$ its range.

Let H be an n-dimensional Hilbert space. If T is a symmetric operator in H, we shall use the notation

$$\lambda_1(T) \geqq \lambda_2(T) \geqq \ldots \geqq \lambda_n(T)$$

for the proper values of T arranged in this order.

We shall use the following inequalities between the numbers $\lambda_i(T)$.

(1,1) *Let H be an n-dimensional Hilbert space and let P be an orthogonal projector in H such that the dimension of its range is r. Let M_1 and M_2 be two symmetric operators in H such that $PM_1P = PM_2P$. Then*

$$\lambda_r(M_1) \geqq \lambda_n(M_2).$$

Proof. Denote by $\mathscr{E}_r$ the set of all r-dimensional subspaces of H. We have then

$$\lambda_r(M_1) = \max_{E \in \mathscr{E}_r} \min_{x \in E, |x| = 1} (M_1 x, x) \geqq \min_{x \in \mathscr{R}(P), |x| = 1} (M_1 x, x) = \min_{|x|=1} (PM_1Px, x) =$$

$$= \min_{|x|=1} (PM_2Px, x) = \min_{x \in \mathscr{R}(P), |x|=1} (M_2 x, x) \geqq \min_{|x|=1} (M_2 x, x) = \lambda_n(M_2).$$

(1,2) **Corollary.** *Let H be an n-dimensional Hilbert space, let M_1 and M_2 be two symmetric operators in H the difference of which has rank h. Then*

$$\lambda_{n-h}(M_1) \geqq \lambda_n(M_2).$$

Proof. Let P be the orthogonal projector on the space $(\mathscr{R}(M_1 - M_2))^\perp$. Apply the preceding lemma.

2. SYMMETRIC STOCHASTIC MATRICES

(2,1) **Lemma.** *Let S be the matrix of order n defined as*

$$S = \begin{bmatrix} 1, & 1, & 1, & 1 \\ 0, & 1, & 1, & 1 \\ 0, & 0, & 1, & 1 \\ & & & \\ 0, & & & 1 \end{bmatrix}$$

Suppose that $A, B \in \mathscr{S}$ and that

$$S^T A S \geqq S^T B S$$

(the inequality being taken elementwise). Then $(Ay, y) \geqq (By, y)$ for each vector y such that $y_1 \geqq y_2 \geqq \ldots \geqq y_n$.

Proof. There exists a nonnegative vector w such that $y - y_n e = Sw$. Now $(Ay, y) = (A(Sw + y_n e), Sw + y_n e) = (ASw, Sw) + (Ay_n e, y_n e) + (ASw, y_n e) + (Ay_n e, Sw)$. Since $Ae = A^T e = e$, the last expression becomes $(S^T ASw, w) + ny_n^2 + y_n(Sw, e) + y_n(e, Sw)$. Similarly, $(By, y) = (S^T BSw, w) + ny_n^2 + 2y_n(Sw, e)$. Since w is a nonnegative vector, we have $(S^T ASw, w) \geqq (S^T BSw, w)$ whence $(Ay, y) \geqq (By, y)$.

(2,2) **Lemma.** *Let A be a doubly stochastic matrix of order n. Let S be the matrix of order n from lemma (2,1).*

Denote by B the matrix $S^T AS$. Then for each $i, k \in N$

$$b_{ik} = v_i^T A v_k = i + k - n + w_i^T A w_k = \tfrac{1}{2}(v_i^T A v_k + w_i^T A w_k + i + k - n)$$

where $v_i^T = (1, \ldots, 1, 0, \ldots 0)$ the number of ones being i and $w_i = e - v_i$.

Proof. Since $S = (v_1, v_2, \ldots, v_n)$, we have $S^T AS = B = (b_{ik})$ where $b_{ik} = v_i^T A v_k$. Observe now that

$$v_i^T A v_k = (e - w_i^T)\, A(e - w_k) =$$
$$= n - (n - i) - (n - k) + w_i^T A w_k = i + k - n + w_i^T A w_k$$

so that

$$b_{ik} = i + k - n + w_i^T A w_k \tag{2}$$

as well as

$$2b_{ik} = v_i^T A v_k + w_i^T A w_k + i + k - n\,. \tag{3}$$

(2,3) **Definition.** *Let σ and μ be two positive numbers. Let us denote by $\mathscr{A}(\sigma, \mu)$ the set of matrices*

$$\mathscr{A}(\sigma, \mu) = \{A \in \mathscr{S};\ \sigma(A) \geqq \sigma,\ \mu(A) \geqq \mu\}.$$

(2,4) **Lemma.** *Suppose that n is even, $n = 2m$ and that $\sigma + 2\mu \leqq 2$. Set*

$$A(\sigma, \mu) = \begin{bmatrix} Z, & (M(\mu) - Z)\, P \\ P(M(\mu) - Z), & PZP \end{bmatrix}$$

where Z, $M(\mu)$, P are the following matrices of order m.

$$Z = \begin{bmatrix} 0, & 0, & 0, & 0 \\ 0, & 0, & 0, & 0 \\ & & & \\ 0, & 0, & 0, & 0 \\ 0, & 0, & 0, & \frac{1}{2}\sigma \end{bmatrix},$$

$$M(\mu) = \begin{bmatrix} 1 - \frac{1}{2}\mu, & \frac{1}{2}\mu, & 0, & & 0 \\ \frac{1}{2}\mu, & 1 - \mu, & \frac{1}{2}\mu, & & 0 \\ & & & & \\ 0, & & & \frac{1}{2}\mu, & 1 - \frac{1}{2}\mu \end{bmatrix},$$

$$P = \begin{bmatrix} 0, & 0, & 0, & 1 \\ 0, & 0, & 1, & 0 \\ & & & \\ 1, & 0, & 0, & 0 \end{bmatrix}.$$

Then $\sigma(A(\sigma, \mu)) = \sigma$ *and* $\mu(A(\sigma, \mu)) = \mu$ *so that* $A(\sigma, \mu) \in \mathscr{A}(\sigma, \mu)$. *Also,* $S^T A S \geqq S^T A(\sigma, \mu) S$ *for each* $A \in \mathscr{A}(\sigma, \mu)$.

Proof. The matrix $A(\sigma, \mu)$ is symmetric and nonnegative since $Z \geqq 0$ and $M(\mu) - Z \geqq 0$. Also $A(\sigma, \mu) e = e$ since $P\tilde{e} = \tilde{e}$ ($\tilde{e}$ m-dimensional) and $M(\mu) \tilde{e} = \tilde{e}$. Denote, for a moment, $A(\sigma, \mu) = Q = (q_{ik})$. Then

$$\sigma(Q) = \min_{M \subset N} \Big(\sum_{\substack{i \in M \\ k \in M}} q_{ik} + \sum_{\substack{i \in N \setminus M \\ k \in N \setminus M}} q_{ik} \Big) = \sigma,$$

since for any $M \subset N$ we have

$$\sum_{\substack{i \in M \\ k \in M}} q_{ik} + \sum_{\substack{i \in N \setminus M \\ k \in N \setminus M}} q_{ik} \geqq \sum_{i \in N} q_{ii} = \sigma;$$

at the same time, for $M_0 = \{1, 2, \ldots, m\}$, we have

$$\sum_{\substack{i \in M_0 \\ k \in M_0}} q_{ik} + \sum_{\substack{i \in N \setminus M_0 \\ k \in N \setminus M_0}} q_{ik} = \sigma.$$

Let us show now that $\mu(Q) = \mu$. Clearly Q is "tridiagonal with respect to the second diagonal" and has the form

$$q_{ik} = \tfrac{1}{2}\mu \quad \text{if} \quad i \neq k \quad \text{and} \quad i + k = n + 2 \quad \text{or} \quad i + k = n,$$

$$q_{ii} = \tfrac{1}{2}\sigma \quad \text{if} \quad 2i = n + 2 \quad \text{or} \quad 2i = n,$$

$$q_{m,m+1} = q_{m+1,m} = 1 - \tfrac{1}{2}\mu - \tfrac{1}{2}\sigma,$$

$$q_{ik} = q_{ki} = 1 - \mu \quad \text{if} \quad i + k = n + 1, \quad 1 < i < m,$$

$$q_{1n} = q_{n1} = 1 - \tfrac{1}{2}\mu, \quad q_{ik} = 0 \quad \text{otherwise}.$$

Thus, for $M = \{1, n\}$ we have

$$\sum_{\substack{i \in M \\ k \in N \setminus M}} q_{ik} = q_{1,n-1} + q_{n,2} = \mu$$

so that

$$\mu(Q) \leqq \mu\,. \tag{4}$$

Suppose that there exists a set $M \subset N$, $\emptyset \neq M \neq N$ such that

$$\sum_{\substack{i \in M \\ k \in N \setminus M}} q_{ik} < \mu\,. \tag{5}$$

Since Q is symmetric, we have

$$\sum_{\substack{i \in M \setminus N \\ k \in M}} q_{ik} < \mu \quad \text{as well}\,.$$

Hence we can assume that $1 \in M$.

Let s be the minimal integer for which at least one of the integers s and $n - s + 1$ does not belong to M. Such integer s exists since $M \neq N$ and $1 \leqq s \leqq m$. Let us show that exactly one of the integers s and $n - s + 1$ belongs to M. This is clear if $s = 1$. If $s > 1$ then $s \notin M$ and $n - s + 1 \notin M$ would imply

$$\sum_{\substack{i \in M \\ k \notin M}} q_{ik} \geqq q_{s-1,n-s+1} + q_{n-s+2,s} = \tfrac{1}{2}\mu + \tfrac{1}{2}\mu = \mu\,,$$

in contradiction with (5).

Since the renumbering $k \leftrightarrow n + 1 - k$ does not change Q, we can assume that $s \in M$, $n - s + 1 \notin M$. Clearly $s < m$ since otherwise $M = N - \{m + 1\}$ and $\sum_{i \neq m+1} q_{i,m+1} = 1$. Now we intend to prove the following assertion: if $s \leqq r \leqq n - s + 1$ then r belongs to M if and only if $r - s$ is even. The proof will be by induction with respect to $p(r) = \min(r - s, n - r - s + 1)$.

The assertion is true if $p = 0$. Now let $p > 0$ and suppose the assertion proved for $p - 1$. Let t be the integer defined as follows

$$t = n - r + 2 \quad \text{if} \quad r - s < n - r - s + 1\,,$$

$$t = n - r \qquad \text{if} \quad r - s > n - r - s + 1\,.$$

(Since n is even, $r - s = n - r - s + 1$ is not possible.) Observe that, in the first case, $2r < n + 1$ and that $2r > n + 1$ in the second case. It follows that $t \neq r$ since $t = r$ implies $2r = n + 2$ in the first case and $2r = n$ in the second case.

We shall show now that the pair r, t either is contained in M or is disjoint with M.

To see that suppose that $r \in M$, $t \in N \setminus M$.

If $s = 1$, then

$$\sum_{\substack{i \in M \\ k \in N \setminus M}} q_{ik} \geqq q_{1n} + q_{rt} = 1 - \tfrac{1}{2}\mu + \tfrac{1}{2}\mu = 1 \geqq \mu \,,$$

in contradiction with (5). If $s > 1$ then

$$\sum_{\substack{i \in M \\ k \in N \setminus M}} q_{ik} \geqq q_{s-1,n-s+1} + q_{s,n-s+1} + q_{rt} = \tfrac{1}{2}\mu + 1 - \mu + \tfrac{1}{2}\mu = 1 \geqq \mu \,,$$

a contradiction as well. Similarly, the assumption $r \in N \setminus M$, $t \in M$ leads to a contradiction in the same manner.

Thus either both r and t belong to M or neither of them belongs to M.

For $p(t)$ we obtain the following estimates:
in the first case

$$p(t) \leqq n - t - s + 1 = r - s - 1 = p(r) - 1$$

and in the second case

$$p(t) \leqq t - s = n - r - s = p(r) - 1 \,.$$

According to the induction hypothesis t belongs to M if and only if $t - s$ is even. Since $r - t$ is even, the same is true for r.

Consequently if $m - s$ is even we have

$$\sum_{\substack{i \in M \\ k \notin M}} q_{ik} \geqq q_{s-1,n-s+1} + q_{s,n-s+1} + q_{m,m+1}$$

where the first summand is missing if $s = 1$; if $m - s$ is odd, the third summand is replaced by $q_{m+1,m}$, the right-hand side is equal to $1 - \frac{1}{2}\mu + 1 - \frac{1}{2}\mu - \frac{1}{2}\mu \geqq 1 \geqq \mu$. In both cases we obtain a contradiction which completes the proof of the equality $\mu(Q) = \mu$.

To prove the last assertion, consider an arbitrary $A \in \mathscr{A}(\sigma, \mu)$ and apply the equalities (2,2) to the matrices $C = S^T A(\sigma,\mu)\, S = (c_{ik})$ and $B = S^T A S = (b_{ik})$.
It follows easily that

$$\begin{aligned}
c_{ik} &= 0 \quad \text{if} \quad i + k < n \,, \\
c_{ik} &= \tfrac{1}{2}\mu \quad \text{if} \quad i + k = n \,, \quad i \neq k \,, \\
c_{ik} &= \tfrac{1}{2}\sigma \quad \text{if} \quad i + k = n \,, \quad i = k \,, \\
c_{ik} &= i + k - n \quad \text{if} \quad i + k > n \,.
\end{aligned}$$

Thus, $b_{ik} \geqq c_{ik}$ if $i + k < n$. If $i + k > n$, we have, by (2,2)

$$b_{ik} = i + k - n + w_i^T A w_k \geqq i + k - n = c_{ik} \,.$$

It remains to prove $b_{ik} \geqq c_{ik}$ if $i + k = n$.

However, then

$$2b_{ik} = v_i^T A v_k + w_i^T A w_k \text{ by } (2,2) .$$

If $i = k$, then $i = m$, whence

$$2b_{mm} = \sum_{\substack{i \in M \\ k \in M}} a_{ik} + \sum_{\substack{i \in N \setminus M \\ k \in N \setminus M}} a_{ik}$$

for $M = \{1, 2, \ldots, m\}$ so that

$$2b_{mm} \geqq \sigma(A) \geqq \sigma = 2c_{mm} .$$

Now let $i < k$, so that $i < m$. Then, A being symmetric,

$$2b_{i,n-i} = v_i^T A v_{n-i} + w_i^T A w_{n-i} = v_i^T A v_{n-i} + w_{n-i}^T A w_i \geqq$$

$$\geqq v_i^T A(v_{n-i} - v_i) + w_{n-i}^T A(w_i - w_{n-i}) = v_i^T A(w_i - w_{n-i}) + w_{n-i}^T A(w_i - w_{n-i}) =$$

$$= (v_i^T + w_{n-i}^T)\, A(e - v_i - w_{n-i}) =$$

$$= \sum_{\substack{p \in M \\ q \in N \setminus M}} a_{pq} \quad \text{where} \quad M = \{1, \ldots, i, n - i + 1, \ldots, n\} .$$

Since $\emptyset \neq M \neq N$, we have $2b_{i,n-i} \geqq \mu(A) \geqq \mu = 2c_{i,n-i}$ and the proof is complete.

(2,5) **Lemma.** *Let m be a natural number and let $n = 2m$. For each $A \in \mathscr{A}(\sigma, \mu)$ the minimal eigenvalue of $A(\sigma, \mu)$ does not exceed the minimal eigenvalue of A:*

$$\lambda_n(A) \geqq \lambda_n(A(\sigma, \mu)) .$$

Proof. Let λ be the minimal eigenvalue of A and let x be a vector such that $Ax = \lambda x$ and $(x, x) = 1$. Let Q be a permutation matrix such that $x = Qy$ where $y_1 \geqq y_2 \geqq \ldots \geqq y_n$. We have then $\lambda = (Ax, x) = (AQy, Qy) = (Q^T AQy, y)$. It is easy to check that $Q^T AQ \in \mathscr{A}(\sigma, \mu)$ again so that, by lemma (2,4),

$$S^T Q^T AQS \geqq S^T A(\sigma, \mu)\, S .$$

It follows from lemma (2,1) that

$$(Q^T AQy, y) \geqq (A(\sigma, \mu)\, y, y) .$$

We have thus $\lambda \geqq (A(\sigma, \mu)\, y, y)$ and $(y, y) = (x, x) = 1$. It follows that λ cannot be smaller than the minimum of the quadratic form $(A(\sigma, \mu)\, y, y)$ for vectors y with $(y, y) = 1$, which is, indeed, the minimal eigenvalue of $A(\sigma, \mu)$.

(2,6) **Lemma.** *Let m be a natural number, let $0 \leqq \mu \leqq 1$ and denote by $M(\mu)$ the matrix*

$$M(\mu) = \begin{bmatrix} 1-\frac{1}{2}\mu, & \frac{1}{2}\mu, & & & \\ \frac{1}{2}\mu, & 1-\mu, & \frac{1}{2}\mu, & & \\ & & \ddots & & \\ & & \frac{1}{2}\mu, & 1-\mu, & \frac{1}{2}\mu \\ & & & \frac{1}{2}\mu, & 1-\frac{1}{2}\mu \end{bmatrix}.$$

The eigenvalues of the matrix $M(\mu)$ are $\mu_k = 1 - \mu + \mu \cos(k-1)\pi/m$, $k = 1, \ldots, m$. The corresponding eigenvectors are $z^k = (\cos(k-1)\pi/2m$, $\cos 3(k-1)\pi/2m, \ldots, \cos(2m-1)(k-1)\pi/2m)^T$. In particular, $\mu_2 + \mu_m = 2(1-\mu) \geqq 0$.

Proof. Write $M(\mu)$ in the form

$$M(\mu) = (1-\mu)I + \mu C$$

where C is the following matrix

$$C = \frac{1}{2}\begin{bmatrix} 1, & 1, & & & \\ 1, & 0, & 1, & & \\ & 1, & 0, & 1, & \\ & & & \ddots & \\ & & 1, & 0, & 1 \\ & & & 1, & 1 \end{bmatrix}.$$

It is easy to verify that the eigenvalues of C are $\cos(k-1)\pi/m$ and that the corresponding eigenvectors are the z^k.

(2,7) **Lemma.** *We have*

$$\lambda_n(A(\sigma, \mu)) = \lambda_m(2Z - M(\mu)).$$

Proof. Consider the matrix U of order $n = 2m$ defined by

$$U = \frac{\sqrt{2}}{2}\begin{bmatrix} I, & P \\ I, & -P \end{bmatrix}$$

where I is the unit matrix of order m. It is easy to check that $U^T U = U U^T = I$ so that U is orthogonal and that

$$U A(\sigma, \mu) U^T = \begin{bmatrix} M(\mu), & 0 \\ 0, & 2Z - M(\mu) \end{bmatrix}.$$

It follows that the minimal eigenvalue of $A(\sigma, \mu)$ equals the smaller of the two numbers $\lambda_m(M(\mu))$, $\lambda_m(2Z - M(\mu))$. It follows from lemma (2,6) that $\lambda_m(M(\mu)) +$

$+ \lambda_2(M(\mu)) \geqq 0$. Since $\lambda_2(M(\mu)) = -\lambda_{m-1}(-M(\mu))$, we have $\lambda_m(M(\mu)) \geqq$ $\geqq \lambda_{m-1}(-M(\mu))$. Denote by P the projection operator given by the matrix

$$\begin{bmatrix} 1, & & & & \\ & 1, & & & \\ & & \ddots & & \\ & & & 1, & \\ & & & & 0 \end{bmatrix}.$$

Since $P\,M(\mu)\,P = P(M(\mu) - 2Z)\,P$, it follows from lemma (1,1) that $\lambda_{m-1}(-M(\mu)) \geqq$ $\geqq \lambda_m(2Z - M(\mu))$ whence $\lambda_m(M(\mu)) \geqq \lambda_m(2Z - M(\mu))$ so that

$$\lambda_n(A(\sigma, \mu)) = \min\{\lambda_m(M(\mu)), \lambda_m(2Z - M(\mu))\} = \lambda_m(2Z - M(\mu)).$$

The proof is complete.

(2,8) **Lemma.** *Let A be a doubly stochastic matrix of order n. Let $B = \frac{1}{2}(A + A^T)$. Then*

$$\mu(B) = \mu(A),$$

$$\sigma(B) = \sigma(A).$$

Each proper value λ of A satisfies the inequality

$$|\lambda + 1| \geqq \lambda_n(B) + 1.$$

Proof. If $M \subset N$, we have

$$\sum_{M, N \setminus M} a_{ik} = \sum_{N \setminus M, M} a_{ik}$$

since both sides of the equation are equal to $m - \sum_{M,M} a_{ik}$ where m is the cardinality of M. It follows that $\sum_{M, N \setminus M} b_{ik} = \sum_{M, N \setminus M} a_{ik}$ whence $\mu(B) = \mu(A)$. Since clearly

$$\sum_{M,M} b_{ik} = \sum_{M,M} a_{ik},$$

it follows that

$$\sigma(B) = \sigma(A)$$

as well.

Now let y be a unit vector for which $Ay = \lambda y$. We have then

$$\operatorname{Re} \lambda = \operatorname{Re}(Ay, y) = (By, y) \geqq \lambda_n(B)$$

whence

$$|\lambda + 1| \geqq \operatorname{Re}(\lambda + 1) = \operatorname{Re} \lambda + 1 \geqq \lambda_n(B) + 1$$

which completes the proof.

(2,9) **Theorem.** *Let n be an even number, $n = 2m > 2$. Let μ, σ be two positive numbers such that $\sigma + 2\mu \leqq 2$. Let A be a doubly stochastic matrix of order n such that*

$$\mu(A) \geqq \mu\,, \quad \sigma(A) \geqq \sigma\,.$$

Then, for each proper value λ of A the distance of λ from -1 is at least $\frac{1}{2}\mu\varphi(2\sigma/u)$ where φ is defined as follows:

$\varphi(\xi)$ is the minimal eigenvalue of the matrix

$$T(\xi) = \begin{bmatrix} 1+\xi, & -1, & & & & \\ -1, & 2, & -1, & & & \\ & & \ddots & & & \\ & & & -1, & 2, & -1 \\ & & & & -1, & 1 \end{bmatrix}$$

of order m.

This estimate is sharp in the following sense: for each pair of positive numbers μ, σ such that $\sigma + 2\mu \leqq 2$ there exists a symmetric matrix $A(\sigma, \mu)$ and a proper value λ of $A(\sigma, \mu)$ such that

$$\mu(A(\sigma, \mu)) = \mu\,, \quad \sigma(A(\sigma, \mu)) = \sigma\,, \quad \lambda + 1 = \tfrac{1}{2}\mu\varphi(2\sigma/\mu)\,.$$

The matrix $A(\sigma, \mu)$ may be described as follows:

$$A(\sigma, \mu) = \begin{bmatrix} & & & & & & & & \tfrac{1}{2}\mu, & 1-\tfrac{1}{2}\mu \\ & & & & & & & \tfrac{1}{2}\mu, & 1-\mu, & \tfrac{1}{2}\mu \\ & & & & & & \cdot & & & \\ & & & & & \tfrac{1}{2}\mu, & 1-\mu, & \tfrac{1}{2}\mu, & & \\ & & & & \tfrac{1}{2}\sigma, & 1-\tilde{\mu}, & \tfrac{1}{2}\mu, & & & \\ & & & \tfrac{1}{2}\mu, & 1-\tilde{\mu}, & \tfrac{1}{2}\sigma, & & & & \\ & & \tfrac{1}{2}\mu, & 1-\mu, & \tfrac{1}{2}\mu, & & & & & \\ & & \cdot & & & & & & & \\ \tfrac{1}{2}\mu, & 1-\mu, & \tfrac{1}{2}\mu, & & & & & & & \\ 1-\tfrac{1}{2}\mu, & \tfrac{1}{2}\mu, & & & & & & & & \end{bmatrix}$$

where $\tilde{\mu} = \frac{1}{2}\mu + \frac{1}{2}\sigma$. The number $\frac{1}{2}\mu\varphi(2\sigma/\mu)$ may be estimated from below as follows:

$$\tfrac{1}{2}\mu\varphi(2\sigma/\mu) \geqq \frac{\sigma\mu}{n((n-1)\,\sigma + \mu)}\,.$$

Proof. Denote by B the matrix $B = \frac{1}{2}(A + A^T)$. It follows from lemma (2,8) that $B \in \mathscr{A}(\sigma, \mu)$. Let λ be an arbitrary proper value of A. By (2,8),

$$|\lambda + 1| \geqq \lambda_n(B) + 1\,.$$

It follows from lemma (2,5) and (2,7) that

$$\lambda_n(B) \geqq \lambda_n(A(\sigma, \mu)) = \lambda_m(2Z - M(\mu)).$$

The matrix $2Z - M(\mu)$ may be written in the form

$$2Z - M(\mu) = 2Z - I + \frac{\mu}{2} W = -I + \frac{\mu}{2}\left(\frac{4}{\mu} Z + W\right)$$

where

$$W = \begin{bmatrix} 1, & -1, & & & & \\ -1, & 2, & -1, & & & \\ & & & & & \\ & & & -1, & 2, & -1 \\ & & & & -1, & 1 \end{bmatrix}.$$

Hence $\lambda_m(2Z - M(\mu)) = -1 + (\frac{1}{2}\mu)\,\lambda_m((4/\mu)\,Z + W)$ and clearly $\lambda_m((4/\mu)\,Z + W) = \lambda_m(T(\xi)) = \varphi(\xi)$ for $\xi = 2\sigma/\mu$.

If $\xi > 0$, the matrix $T(\xi)$ is invertible and its inverse may be computed as follows. Since

$$T(\xi) = RR^T$$

where

$$R = \begin{bmatrix} \sqrt{\xi}, & -1, & & & \\ & 1, & -1, & & \\ & & & & \\ & & & 1, & -1 \\ & & & & 1 \end{bmatrix},$$

we have $T(\xi)^{-1} = (R^{-1})^T R^{-1}$. Now

$$R^{-1} = \begin{bmatrix} \frac{1}{\sqrt{\xi}}, & \frac{1}{\sqrt{\xi}}, & & \frac{1}{\sqrt{\xi}} \\ & 1\ , & & 1 \\ & & & \\ & & & 1 \end{bmatrix}.$$

Accordingly, the elements u_{ik} of the matrix $T(\xi)^{-1}$ are given by the formulas

$$u_{ik} = \min(i, k) - 1 + \xi^{-1}.$$

It follows that $\varphi(\xi)^{-1} \leqq |T(\xi)^{-1}|_\sigma$ and this may be estimated, in its turn, by the norm of the matrix $T(\xi)^{-1}$ taken as a linear operator in the n-dimensional affine space equipped with the norm $|x| = \max |x_i|$. This norm equals the maximal row sum of the matrix. The maximum is attained in the last row and equals

$$\tfrac{1}{2}(n-1)\,n + n\xi^{-1}\,.$$

Hence $\varphi(\xi) \geqq 2\xi/[n((n-1)\,\xi + 2)]$ so that $\frac{1}{2}\mu\varphi(2\sigma/\mu) \geqq \sigma\mu/[n((n-1)\,\sigma + \mu)]$. The proof is complete.

3. MATRICES OF ODD ORDER

In this section we intend to prove analogous results for matrices of odd order. Here the situation is different in that the characteristic σ cannot assume values smaller than 1. We shall estimate the distance $|\lambda + 1|$ in terms of the irreducibility characteristic μ only.

(3,1) **Lemma.** *Let A be a doubly stochastic matrix of order $n > 1$. If n is odd then $\sigma(A) \geqq 1$.*

Proof. Let M_1, M_2 be a nontrivial decomposition of N so that $N = M_1 \cup M_2$ and $M_1 \cap M_2$ is void. Since n is odd, one of the two sets has a greater number of elements than the other. Hence we may suppose that card $M_1 >$ card M_2. If we write $\sigma_1, \sigma_2, \sigma$ respectively for the sums $\sum\limits_{M_1,M_1}, \sum\limits_{M_2,M_2}, \sum\limits_{M_1,M_2}$ of elements of A, we have

$$\sigma_1 + \sigma = \text{card}\, M_1\,,$$

$$\sigma_2 + \sigma = \text{card}\, M_2\,.$$

Hence $\sigma_1 + \sigma_2 \geqq \sigma_1 - \sigma_2 \geqq \text{card}\, M_1 - \text{card}\, M_2 \geqq 1$. Since M_1, M_2 was an arbitrary decomposition, we have $\sigma(A) \geqq 1$.

(3,2) **Lemma.** *Let $0 \leqq \mu \leqq 1$ and let $\mathscr{A}(\mu) = \{A \in \mathscr{S},\ \mu(A) \geqq \mu\}$. Denote by $A(\mu)$ the matrix*

$$A(\mu) = \begin{bmatrix} & & & & \frac{1}{2}\mu, & 1-\frac{1}{2}\mu \\ & & & \frac{1}{2}\mu, & 1-\mu, & \frac{1}{2}\mu \\ & & & & & \\ & \frac{1}{2}\mu, & 1-\mu, & \frac{1}{2}\mu, & & \\ 1-\frac{1}{2}\mu, & \frac{1}{2}\mu, & & & & \end{bmatrix}.$$

Then $A(\mu) \in \mathscr{A}(\mu)$ and $S^TAS \geqq S^T A(\mu)\, S$ for each $A \in \mathscr{A}(\mu)$.

Proof. Denote by q_{ik} the elements of the matrix $A(\mu)$. Obviously, $q_{ik} \geqq 0$, $\sum\limits_k q_{ik} = 1$ so that $A(\mu) \in \mathscr{S}$.

We shall prove now that $\mu(A(\mu)) = \mu$. Clearly if $M_0 = \{1, n\}$ then

$$\sum_{M_0, N \setminus M_0} q_{ik} = \mu$$

so that $\mu(A(\mu)) \leqq \mu$. If $\mu = 0$ we have $\mu(A(0)) = 0$. Assume thus that $\mu > 0$ and that there is a non-void proper subset M of N such that

$$(6) \qquad \sum_{M, N \setminus M} q_{ik} < \mu\,.$$

Since $A(\mu) \in \mathscr{S}$, it follows that

$$\sum_{N \setminus M, M} q_{ik} = \sum_{M, N \setminus M} q_{ik} < \mu$$

as well. Hence we can assume that $\frac{1}{2}(n + 1) \in M$. Let s be the minimal index, $1 \leqq s \leqq \frac{1}{2}(n + 1)$ such that all indices $s, s + 1, \ldots, n - s + 1$ belong to M. Since $M \neq N$, $s \geqq 2$ and exactly one of the indices $s - 1$, $n - s + 2$ belongs to M since otherwise $\sum_{M, N \setminus M} q_{ik} \geqq q_{n-s+1,s-1} + q_{s,n-s+2} \geqq \mu$, a contradiction. Since $A(\mu)$ remains unchanged if we perform the renumbering $j \to n + 1 - j$, we can assume $s - 1 \in M$, $n - s + 2 \notin M$. Since then the sum $\sum_{M, N \setminus M} q_{ik}$ contains the entries $q_{s-1,n-s+2} = 1 - \mu$ and $q_{s,n-s+2} = \frac{1}{2}\mu$, we have necessarily, by (6), $s - 2 \notin M$, $n - s + 3 \in M$, $s - 3 \in M$, $n - s + 4 \notin M$ etc. However, $q_{1n} = 1 - \frac{1}{2}\mu$ is then also contained in the sum which is thus greater than or equal to $1 - \mu + \frac{1}{2}\mu + 1 - \frac{1}{2}\mu = 2 - \mu \geqq \mu$, a contradiction with (6). Thus $\mu(A(\mu)) = \mu$.

Denote by $C = (c_{ik})$ the matrix $S^T A(\mu) S$. It is easy to check that

$$\begin{aligned} c_{ik} &= 0 \quad \text{if} \quad i + k < n\,, \\ c_{ik} &= \tfrac{1}{2}\mu \quad \text{if} \quad i + k = n\,, \\ c_{ik} &= i + k - n \quad \text{if} \quad i + k > n\,, \quad i, k = 1, \ldots, n\,. \end{aligned}$$

Let us assume that A is a symmetric stochastic matrix such that $\mu(A) \geqq \mu$. Denote by $B = (b_{ik})$ the matrix $S^T A S$.

We intend to show that $B \geqq C$. Clearly $b_{ik} \geqq c_{ik}$ if $i + k < n$. If $i + k > n$, we have, by (2,2),

$$b_{ik} = i + k - n + w_i^T A w_k \geqq i + k - n = c_{ik}\,.$$

Now let $i + k = n$. Since n is odd, we can assume $i < k$, so that $i \leqq \frac{1}{2}(n - 1)$. By (2,2) and the symmetry of A,

$$\begin{aligned} 2b_{ik} &= v_i^T A v_k + w_i^T A w_k = v_i^T A v_{n-i} + w_{n-i}^T A w_i \geqq \\ &\geqq v_i^T A(v_{n-i} - v_i) + w_{n-i}^T A(w_i - w_{n-i}) = \\ &= v_i^T A(w_i - w_{n-i}) + w_{n-i}^T A(w_i - w_{n-i}) = (v_i^T + w_{n-i}^T)\, A(e - v_i - w_{n-i}) = \\ &= \sum_{M, N \setminus M} a_{pq} \quad \text{where} \quad M = \{1, \ldots, i, n - i + 1, \ldots, n\}\,. \end{aligned}$$

Since $\emptyset \neq M \neq N$ it follows that

$$2b_{i,n-i} \geqq \mu(A) \geqq \mu = 2c_{i,n-i}\,.$$

The proof is complete.

(3,3) **Lemma.** *Let n be a natural number. Then the minimal eigenvalue $\lambda_n(A)$ of any matrix $A \in \mathscr{A}(\mu)$ does not exceed $\lambda_n(A(\mu))$.*

Proof. Let λ be the minimal eigenvalue of A and let x be a vector such that $Ax = \lambda x$ and $(x, x) = 1$. Let Q be a permutation matrix such that $x = Qy$ and

$$y_1 \geqq y_2 \geqq \ldots \geqq y_n\,.$$

We have then

$$\lambda = (Ax, x) = (AQy, Qy) = (Q^T AQy, y)\,.$$

It is easy to check that $Q^T AQ \in \mathscr{S}$ and that $\mu(Q^T AQ) = \mu(A) \geqq \mu$. Hence $Q^T AQ \in \mathscr{A}(\mu)$. It follows from lemmas (3,2) and (2,1) that

$$(Q^T AQy, y) \geqq (A(\mu)\, y, y)\,.$$

We have thus $\lambda \geqq (A(\mu)\, y, y)$ and $(y, y) = (x, x) = 1$. It follows that $\lambda \geqq \lambda_n(A(\mu))$.

(3,4) **Theorem.** *Let n be an odd number, $n > 1$. Let $0 \leqq \mu \leqq 1$. Let A be a doubly stochastic matrix of order n such that $\mu(A) \geqq \mu$. Then each proper value λ of A satisfies the inequality*

$$|\lambda + 1| \geqq \mu(1 - \cos \pi/n)\,.$$

This estimate is sharp in the following sense: for each μ, $0 \leqq \mu \leqq 1$, there exists a doubly stochastic matrix $A(\mu)$ with $\mu(A(\mu)) = \mu$ and a proper value λ of $A(\mu)$ such that

$$\lambda + 1 = \mu(1 - \cos \pi/n)\,;$$

this matrix $A(\mu)$ is

$$A(\mu) = \begin{bmatrix} & & & & \frac{1}{2}\mu, & 1 - \frac{1}{2}\mu \\ & & & \frac{1}{2}\mu, & 1 - \mu, & \frac{1}{2}\mu \\ & & & & & \\ & \frac{1}{2}\mu, & 1 - \mu, & \frac{1}{2}\mu, & & \\ 1 - \frac{1}{2}\mu, & \frac{1}{2}\mu, & & & & \end{bmatrix}.$$

Proof. First of all, it is easy to verify that the number $\lambda = -1 + \mu(1 - \cos \pi/n)$ is an eigenvalue of $A(\mu)$ corresponding to the eigenvector $[\cos \pi/2n, \cos 3\pi/2n, \ldots, \cos (2n - 1)\, \pi/2n]$.

Now, let A be an arbitrary doubly stochastic matrix of order n such that $\mu(A) \geqq \mu$. Let $G = \frac{1}{2}(A + A^T)$. According to (2,8), $\mu(G) = \mu(A) \geqq \mu$ and each proper value λ of A satisfies the inequality

$$|\lambda + 1| \geqq \lambda_n(G) + 1 .$$

In order to complete the proof, it suffices to observe that $G \in \mathscr{A}(\mu)$ and to apply lemma (3,4).

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