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LATTICE-ORDERED GROUPS WITH RANK ONE COMPONENTS

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1. INTRODUCTION

Let Γ be a root system (i.e., a po-set such that for each $\gamma \in \Gamma$, $\{\alpha \in \Gamma \mid \alpha \geqq \gamma\}$ is a chain), and let $V = V(\Gamma, A)$ be the group of all functions from Γ into the o-group A whose support satisfies ACC. A component v_γ of $v \in V$ is *maximal* if $v_\gamma \neq 0$ and $v_\alpha = 0$ for all $\alpha > \gamma$. We define $v \in V$ to be positive if each maximal component is positive. Then V is a lattice ordered group (l-group) and $\Sigma(\Gamma, A) = \{v \in V \mid$ support of v is finite$\}$ is an l-subgroup of V. A maximal chain in Γ is called a *root*, and $F(\Gamma, A) = \{v \in V \mid$ support of v lies on only finitely many roots$\}$ is also an l-subgroup of V. Throughout, Z, Q, R will always denote the additive groups of integers, rationals and reals, respectively.

If G is an l-group, a convex l-subgroup M of G is called *regular* if it is maximal without some element $g \in G$. M is a *value* of g. G is called *finite valued* if each $g \in G$ has only a finite number of values. The set $\{G_\gamma \mid \gamma \in \Gamma\}$ of all regular subgroups of G is a root system with respect to inclusion. If G is abelian, then there is an l-isomorphism of G into $V(\Gamma, R)$ [1]. Throughout this paper, Γ will denote the index set for the set of regular subgroups of G. For the rest of this section, all groups are assumed to be abelian.

For each $\gamma \in \Gamma$, let G^γ denote the convex l-subgroup of G which covers G_γ. An l-group G is *generalized discrete* if each G^γ/G_γ is cyclic, and has *rank one components* if each G^γ/G_γ has rank one. RIBENBOIM [10] proved that a generalized discrete o-group can be embedded in $V(\Gamma, Z)$. In [7], HILL and MOTT give a counter-example to Ribenboim's theorem and then show that if G is a countable generalized discrete o-group, it can be embedded in $\Sigma(\Gamma, Z)$. We extend this and other results of Hill and Mott from o-groups to l-groups. If G is a countable generalized discrete l-group with a finite basis, then G can be embedded in $\Sigma(\Gamma, Z)$. If G is generalized discrete, finite valued and Γ satisfies the DCC, then $G \cong \Sigma(\Gamma, Z)$.

An o-group is *hereditary discrete* if each subgroup is discrete. In [11]. SANKAREN and VENKATARAMAN prove that an hereditary discrete o-group is generalized discrete, but we show by a counter-example that this is false. This example is also a counter-example to some other theorems in [11]. For instance; the authors claim that if Γ satisfies the DDC, then an o-group G is isomorphic to $\Sigma(\Gamma, Z)$, if and only if G is hereditary discrete. Hill and Mott supply the correct theorem. An o-group is *ω-discrete* if every l-homomorphic image is discrete. Then an o-group G is ω-discrete if and only if $G \cong \Sigma(\Gamma, Z)$ where Γ satisfies the DCC [7].

An element $0 < s$ in an l-group G is called *basic* if the convex l-subgroup $G(s)$ that is generated by s is an o-group. A basis for G is a maximal pairwise disjoint subset that consists of basic elements. Let us define an l-group G to be *ω-discrete* if each l-homomorphic image has a basis of atoms. We show that G is ω-discrete and finite valued if and only if $G \cong \Sigma(\Gamma, Z)$ and Γ satisfies the DCC.

From the theory developed by MARTINEZ in [8] we deduce that a representable l-group G contains a convex l-subgroup $\mathcal{O}(G)$ ($\mathcal{C}(G)$) with rank one (cyclic) components that contains every other such convex l-subgroup, and is invariant under alls l-automorphisms of G. We compute $\mathcal{O}(G)$ and $\mathcal{C}(G)$ for various well known l-groups.

Final notation: if $\{A_\delta \mid \delta \in \Delta\}$ is a set of l-groups then $\Sigma_\Delta A_\delta$ and $\Pi_\Delta A_\delta$ will denote the cardinal sum and product, respectively, of the A_δ. For l-groups A and B, $A \boxplus B$ denotes their cardinal sum.

2. THE CLASSES $\mathcal{O}$ AND $\mathcal{C}$

Let $\mathcal{O}$ (resp. $\mathcal{C}$) denote the class of all the normal valued l-groups with rank one (resp. cyclic) components. Thus if $G \in \mathcal{O}$ (resp. $G \in \mathcal{C}$), then for each $\gamma \in \Gamma$, $G_\gamma \lhd G^\gamma$ and G^γ/G_γ has rank one (resp. is cyclic). Hill and Mott [7] investigate the abelian o-groups in $\mathcal{C}$ and Mott [9] examines the abelian l-groups in $\mathcal{C}$. Using their terminology, we call a group in $\mathcal{C}$ *generalized discrete.*

Proposition 2.1. *$\mathcal{O}$ and $\mathcal{C}$ are closed with respect to l-subgroups, l-homomorphic images and cardinal sums, but not cardinal products.*

Proof. Mott [9] shows that the proposition holds for abelian generalized discrete l-groups. If H is an l-subgroup of $G \in \mathcal{O}(\mathcal{C})$ and N is a value in H of $0 < h \in H$, then there exists a value M of h in G with $M \cap H = N$. Let M^* be the convex l-subgroup of G which covers M. Then if $N^* = M^* \cap H$, N^* covers N and $N^*/N = (M^* \cap H)/(M \cap H) = (M^* \cap H)/(M \cap M^* \cap H) \cong (M + (M^* \cap H))/M \subset M^*/M$. Thus, since M^*/M is of rank one (cyclic), so is N^*/N, and so $H \in \mathcal{O}(\mathcal{C})$.

Let C be an l-ideal of $G \in \mathcal{O}(\mathcal{C})$ and let $\bar{T}$ be a regular subgroup of G/C. Then $\bar{T} = T/C$ where T is regular in G. If $\bar{T}^*$ covers $\bar{T}$, $\bar{T}^* = T^*/C$ where T^* covers T and hence $\bar{T}^*/\bar{T} \cong T^*/T$ has rank one (is cyclic). Thus $G/C \in \mathcal{O}(\mathcal{C})$.

Let $\{G_i \mid i \in I\}$ be a set of non-zero l-groups. If M is a regular subgroup of $\Sigma_I G_i$, then M is of the form $K \boxplus \Sigma_{I'} G_i$, $I' = I \setminus \{j\}$, where K is regular in G_j. Clearly then $\Sigma_I G_i \in \mathcal{O}(\mathcal{C})$ if and only if each $G_i \in \mathcal{O}(\mathcal{C})$.

Finally, $\prod_{i=1}^{\infty} Z \notin \mathcal{O}$ since it contains a copy of the free abelian l-group on two generators, F_2, and $Z \oplus Z\zeta$ with the natural order is an l-homomorphic image of F_2 (ζ any transcendental element of R). Thus F_2, and hence $\prod_{i=1}^{\infty} Z$, does not belong to $\mathcal{O}(\mathcal{C})$.

Suppose that the l-group G satisfies:

(F) Each disjoint bounded subset of G is finite.

Then G is normal valued and has a basis $\{s_\lambda \mid \lambda \in \Lambda\}$. Let $s'_\lambda = \{g \in G \mid |g| \wedge s_\lambda = 0\}$. If $\sigma : G \to \Sigma_\Lambda G/s'_\lambda$ is defined by $(g\sigma)_\lambda = s'_\lambda + g$, then σ is an isomorphism. Thus

Corollary. *If G satisfies* (F), *then $G \in \mathcal{O}$ if and only if each $G/s'_\lambda \in \mathcal{O}$ and $G \in \mathcal{C}$ if and only if each $G/s'_\lambda \in \mathcal{C}$.*

Now if we consider the class of totally ordered groups with rank one components (cyclic components) and consider the corresponding "hyper-kernels" that Martinez introduces in [8], then from his Theorem 1.5 we get the following:

Theorem 2.2. *A representable l-group G contains a convex l-subgroup $\mathcal{O}(G)(\mathcal{C}(G))$ with rank one* (*cyclic*) *components that contains every other such convex l-subgroup. Moreover, $\mathcal{O}(G)$ ($\mathcal{C}(G)$) is left invariant by the l-automorphisms of G.*

Actually, the set of all convex l-subgroups with rank one (cyclic) components is a complete sublattice of the lattice of all convex l-subgroups of G with $\mathcal{O}(G)$ ($\mathcal{C}(G)$) as the largest element. In section 4 we compute $\mathcal{O}(G)$ and $\mathcal{C}(G)$ for various well-known l-groups.

Proposition 2.3. *$G \in \mathcal{O}$ if and only if $G(g) \in \mathcal{O}$ for each $g \in G$, and $G \in \mathcal{C}$ if and only if $G(g) \in \mathcal{C}$ for each $g \in G$.*

Proof. If $G \in \mathcal{O}$, then $G(g) \in \mathcal{O}$ for each $g \in G$ by Proposition 2.1. For the converse, let G_γ be a value of $0 < g \in G$ and G^γ the convex l-subgroup covering G_γ. Then $G(g)$ covers $G_\gamma \cap G(g)$ and $G_\gamma \cap G(g)$ is a value of g in $G(g)$. Thus

$$\frac{G(g)}{G(g) \cap G_\gamma} = \frac{G(g) \cap G^\gamma}{G_\gamma \cap G^\gamma \cap G(g)} \cong \frac{G_\gamma + (G^\gamma \cap G(g))}{G_\gamma} \subseteq G^\gamma/G_\gamma.$$

If $0 < x \in G^\gamma$, then $G_\gamma + x < G_\gamma + ng$, some $n > 0$ and hence $G_\gamma + x = G_\gamma + x \wedge ng$. Thus $x = h + x \wedge ng \in G_\gamma + (G^\gamma \cap G(g))$, and so $G^\gamma = G_\gamma + (G^\gamma \cap G(g))$.

The proof for $\mathcal{C}$ is similar.

Proposition 2.4. *For a finite valued l-group G, the following are equivalent.*

(a) *$G \in \mathcal{O}$.*

(b) *Each archimedean o-subgroup is of rank one.*

(c) *Each l-subgroup of G that is generated as a group by two elements belongs to $\mathcal{O}$.*

(d) *Each finitely generated l-subgroup of G belongs to $\mathcal{O}$.*

Proof. By Proposition 2.1, (a) implies (b), (c) and (d). Suppose that G^γ/G_γ is not of rank one. Then there exist $x, y \in G^\gamma \setminus G_\gamma$ such that γ is their only value and $G_\gamma + x$ and $G_\gamma + y$ are independent in G^γ/G_γ. If $m^2 + n^2 \neq 0$, then $mx + ny \in G^\gamma \setminus G_\gamma$ and, in fact, γ is the only value of $mx + ny$. Thus $mx + ny$ must be positive or negative and hence $[x] + [y]$ is an archimedean o-group that is not of rank one and a two generated l-subgroup of G that is not in $\mathcal{O}$.

The next Proposition shows that $\prod_{i=1}^{\infty} Z_i$ satisfies (b), but as we have seen $\prod_{i=1}^{\infty} Z_i \notin \mathcal{O}$. Thus the hypothesis of finite valued can not be discarded. In section 4 we show that the corresponding proposition with $\mathcal{C}$ instead of $\mathcal{O}$ is not valid.

An element $0 < s \in G$ is *singular* if $0 < g < s$ implies $g \wedge (s - g) = 0$.

Proposition 2.5. *If G is an archimedean l-group and each positive element of G exceeds a singular element, then each o-subgroup of G is cyclic.*

Proof. In [2] it is shown that G is l-isomorphic to a subdirect product of discrete o-groups, T_λ. So without loss of generality $G \subset \Pi_\Lambda T_\lambda$, where each T_λ is a lexicographic extension of the integers by an abelian o-group, and each singular element in G is a characteristic function on a subset of Λ. If $0 < g \in G$, then $g > s > 0$, where s is singular and since $ns \nless g$ for some $n > 0$, $ns_\lambda > g_\lambda > 0$, some λ. But $s_\lambda = 1$ and so g_λ is an integer.

Let $U \neq 0$ be an o-subgroup of G. If $0 < a, b \in U$, then $na > b$ and $nb > a$ for some $n > 0$, so a and b have the same support in Λ. Pick $0 < u \in U$ and let u_λ be a nonzero integral component of u. In fact, we may choose u so that no other positive element in U has a smaller λ-th component. Then u is the least positive element in U. Indeed, if $\bar{u} \in U$ with $\bar{u} \leqq u$, then $\bar{u}_\lambda = u_\lambda$ and since u and $u - \bar{u}$ have the same support if $u > \bar{u}$, we get $u = \bar{u}$. Thus $U = [u]$; for if $0 < b \in U$, then there exists $n > 0$ such that $b < (n + 1) u$. If n is the smallest such and $b \neq nu$, then $0 < b - nu < u$.

Corollary. *Each o-subgroup of $\Pi_\Delta Z_\delta$ is cyclic.*

Question. If G is archimedean and each o-subgroup is cyclic, then does each positive element in G exceed a singular element?*)

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*) Added in proof. J. Jakubík has sent us an example that shows that the answer is no. Another example is the following. From the proof of Proposition 2.5 it follows that each o-subgroup of a subdirect product of integers is cyclic. Let G be a free abelian l-group on two or more generators. Then G is a subdirect sum of integers so each o-subgroup is cyclic, but G contains no singular elements in fact no bounded elements.

Throughout this section G will denote an abelian l-goup. We first extend Hill and Mott's Proposition 4.5 to l-groups.

Proposition 3.1. *If $G \in \mathcal{C}$ is finite valued and Γ satisfies the DCC, then $G \cong \Sigma(\Gamma, Z)$. In particular, $\Sigma(\Gamma, Q)$ is the divisible hull of G.*

Proof. For each $\gamma \in \Gamma$ pick an element $0 < g^\gamma \in G$ such that G_γ is the only value of g^γ and $G^\gamma/G_\gamma = [G_\gamma + g^\gamma]$. Let τ be the map of $\{g^\gamma \mid \gamma \in \Gamma\}$ into $\Sigma(\Gamma, Z)$ such that $g^\gamma\tau$ is the characteristic function on γ. Then τ can be extended to an l-isomorphism σ of the divisible hull G^d of G onto $\Sigma(\Gamma, Q)$ (see [3], Theorem 4.9). Now clearly $G\sigma \supseteq \Sigma(\Gamma, Z)$, so we need only show $G\sigma \subseteq \Sigma(\Gamma, Z)$. Consider $0 < g \in G$ with value G_δ. Then $G_\delta + g = G_\delta + ng^\delta$ for some $n > 0$ and hence $(g\sigma)_\delta = (ng^\delta\sigma)_\delta = n$. Now the support of $(g - ng^\delta)\,\sigma$ is properly contained in the support of $g\sigma$ which is finite. Then, by induction, $g\sigma \in \Sigma(\Gamma, Z)$.

Corollary I. *If $G \in \mathcal{C}$ has a finite basis and Γ satisfies the DCC, then $G \cong \Sigma(\Gamma, Z) = V(\Gamma, Z)$.*

Corollary II. *For an abelian l-group G, the following are equivalent.*

(1) *$G \cong \Sigma(\Gamma, Z)$ with Γ finite.*

(2) *$G \in \mathcal{C}$ and Γ is finite.*

(3) *$G \in \mathcal{C}$ and has finite rank as an abelian group.*

Proof. Clearly (1) implies (2) and (3), and (2) implies (1) by the Proposition. Assume then that (3) is satisfied. Since disjoint elements are independent, G has a finite basis and so each element is the sum of a finite number of disjoint special elements (i.e., elements with only one value). Since special elements with distinct values are independent, it follows that Γ is finite.

Let G^d be the divisible hull of G. It follows that if G has finite rank, then $G^d \cong \Sigma(\Gamma, T_\gamma)$, where Γ is finite and each T_γ is a finite dimensional rational subspace of R with the natural order. If, in addition, $G \in \mathcal{O}$, then $G^d = \Sigma(\Gamma, Q)$.

It is well known and easy to prove that Γ satisfies the DCC if and only if each prime subgroup of G is regular. If $G \in \mathcal{C}$ is finite valued and Γ satisfies the DCC, then we have shown that $G \cong \Sigma(\Gamma, Z) \cong \Sigma(\Gamma, G^\gamma/G_\gamma)$ and $G^d \cong \Sigma(\Gamma, Q)$. If $G \in \mathcal{O}$ is finite valued and Γ satisfies the DCC then $G^d \cong \Sigma(\Gamma, Q)$, but Example 9 of Section 4 shows that G need not be embeddable in $\Sigma(\Gamma, G^\gamma/G_\gamma)$.

Recall that G is ω-discrete if each l-homomorphic image has a basis of atoms. If G is ω-discrete and M is a prime subgroup, then G/M has a least positive element. Thus M must be regular and hence $G \in \mathcal{C}$ and Γ satisfies the DCC. We next extend Hill and Mott's Proposition 4.6 to l-groups.

Proposition 3.2. *G is ω-discrete and finite valued if and only if $G \cong \Sigma(\Gamma, Z)$ and Γ satisfies the DCC.*

Proof. If G is ω-discrete and finite valued then by Proposition 3.1 and the above remarks, $G \cong \Sigma(\Gamma, Z)$ and Γ satisfies the DCC. Conversely, it is clear that $\Sigma = \Sigma(\Gamma, Z)$ is finite valued. Now let C be a convex l-subgroup of Σ, and let $\Lambda = \{\gamma \in \Gamma \mid$ some $c \in C$ has maximal component $c_\gamma\}$. Then $C = \{v \in \Sigma \mid$ support of $v \subseteq \Lambda\}$. Thus $\Sigma/C \cong \Sigma(\Gamma \setminus \Lambda, Z)$, which has a basis of atoms.

Corollary. *If G is finite valued then the following are equivalent.*

(1) *$G \in \mathscr{C}$ and Γ satisfies the DCC.*

(2) *G is ω-discrete.*

Suppose that $G \in \mathscr{O}$, is countable and finite valued. Then in [5] it is shown that $G^d \cong \Sigma(\Gamma, Q)$.

Question. If $G \in \mathscr{C}$ is countable and finite valued, can G be embedded in $\Sigma(\Gamma, Z)$? Hill and Mott have shown that the answer is yes if G is an o-group. One can use the fact that $G^d \cong \Sigma(\Gamma, Q)$ to recover the Hill-Mott result and also to obtain an affirmative answer to the above question, but the proof is long and quite complicated so we only give the following.

Proposition 3.3. *Let $G \in \mathscr{C}$ be countable and have a finite basis. Then G can be embedded in $\Sigma(\Gamma, Z)$.*

Proof. We use induction on the number of elements in a basis for G (or equivalently the number of maximal chains in Γ). Now G is a lexicographic extension of its lex kernel, $A \boxplus B$, by an o-group. If $A \boxplus B = 0$, then G is an o-group and the proposition is true by Theorem 5.1 in [7]. If $A \boxplus B \neq 0$, then $A \neq 0 \neq B$ so by induction A and B can be embedded into $\Sigma(\Gamma(A), Z)$ and $\Sigma(\Gamma(B), Z)$ respectively. Now $G/(A \boxplus B) \in \mathscr{C}$, is ordered and countable and hence free [7]. Thus $G = A \boxplus B \oplus C$, a direct lexicographic extension. C can be embedded into $\Sigma(\Gamma(C), Z)$ and it follows that G can be embedded into $\Sigma(\Gamma, Z)$.

4. EXAMPLES

1. An l-subgroup G of $\prod_{i=1}^{\infty} Z_i$ that is epi-archimedean (i.e., each regular subgroup is maximal), **but $G \notin \mathscr{C}$.** Let H be the l-subgroup of all the functions in $\prod_{i=1}^{\infty} R_i$ with finite range and let α be the l-automorphism of $\prod_{i=1}^{\infty} R_i$ obtained by multiplication by the element $x = (x_i)$ where $x_i = i$. Let G be the l-subgroup of $H\alpha$ consisting of all the integer valued functions. It is shown in [4] that G has the desired properties.

2. *If* $K = \Pi_\Delta Z_\delta$, *then* $\mathscr{C}(K)$ *is the l-group of all bounded functions.*

Proof. Let B be the l-subgroup of all bounded functions. Then B is epi-archimedean and so if C is a prime subgroup of B, B/C is an archimedean o-group with a singular element $C + a$, where a is the characteristic function on Δ. Hence B/C is cyclic and $B \in \mathscr{C}$. If $0 < g \in K \setminus B$, then g is unbounded and it follows, as in example 1, that $K(g)$ contains an l-subgroup that does not belong to $\mathscr{C}$. Hence B is the largest convex l-subgroup of G that belongs to $\mathscr{C}$ and hence $B = \mathscr{C}(K)$.

We next generalize (2). Let G be an abelian l-group and let $X = \{x \in G^+ \mid x$ exceeds only a finite number of disjoint elements$\}$. Then $[X]$, the group generated by X, is the largest l-ideal of G that satisfies property (F) and $[X]^+ = X$. $[X]$ is called the F-ideal of G and is denoted by $F(G)$. (See [3], *p.* 3.30).

3. *If* $V = V(\Gamma, Z)$, *then* $\mathscr{C}(V) = F(V) + B$, *where* B *is the group of all the bounded functions that live on the minimal elements of* Γ.

Proof. B is an l-ideal of V and by (2), $B \in \mathscr{C}$ so $B \subseteq \mathscr{C}(V)$. Let us call an element $\gamma \in \Gamma$ *finite* if $\{\delta \in \Gamma \mid \delta \leqq \gamma\}$ contains only a finite number of roots, and let Γ_f denote the set of all the finite elements in Γ. For each $\gamma \in \Gamma_f$ let $c(\gamma)$ be the characteristic function on γ. Then $V(c(\gamma)) \subseteq \mathscr{C}(V)$ and so $\mathscr{C}(V) \supseteq \bigvee V(c(\gamma)) = F(V)$. Now consider $0 < v \in \mathscr{C}(V)$ with value γ. Then $\gamma \in \Gamma_f$, for otherwise γ exceeds an infinite disjoint subset $\delta_1, \delta_2, \ldots$ of Γ and hence $\mathscr{C}(V)$ contains a copy of $\prod_{i=1}^{\infty} Z_i$ which is impossible by (2). Similarly all but a finite number of the values of v must be minimal elements in Γ. Finally, by (2) there must be a bound for all the maximal components of v. Therefore $v \in F(V) + B$.

4. *If* $V = V(\Gamma, Q)$, *then* $\mathscr{O}(V) = F(V)$. *In particular*, $\mathscr{O}(\Pi_\Delta Q_\delta) = \Sigma_\Delta Q_\delta$.

Proof. As above $\mathscr{O}(V) \supseteq \bigvee V(c(\gamma)) = F(V)$. Suppose (by way of contradiction) that $0 < v \in \mathscr{O}(V) \setminus F(V)$. Then v must exceed an infinite number of disjoint special elements $a_1, a_2, \ldots$ with maximal components $\gamma_1, \gamma_2, \ldots$. Thus since $\mathscr{O}(V)$ is invariant under all l-automorphisms of V, it must contain the group of all functions that live on $\gamma_1, \gamma_2, \ldots$. But this group contains a copy of $\prod_{i=1}^{\infty} Z_i$ and so does not belong to $\mathscr{O}$.

5. An l-group $G \in \mathscr{C}$ for which Γ satisfies the DDC, but G is not ω-discrete. Let G be the group of all periodic sequences of integers. Then by (2), $G \in \mathscr{C}$ and since G is epi-archimedean, Γ satisfies the DCC, but G has no basis.

6. An l-group G which is ω discrete, but G is not finite valued. Let $G \subseteq \prod_{i=1}^{\infty} Z_i$ be the group of all eventually constant sequences, and for each $i = 1, 2, \ldots$, let $G_i =$

$= \{g \in G \mid g_i = 0\}$. Then the G_i together with $\sum_{i=1}^{\infty} Z_i$ are all the regular subgroups of G and since each l-ideal C of G is the intersection of regular subgroups, it follows that G/C has a basis of atoms.

7. a) *$F(\Gamma, Z) \in \mathscr{C}$ and hence so does $\Sigma(\Gamma, Z)$.*

b) *$V(\Gamma, Z) \in \mathscr{C}$ if and only if Γ contains only a finite number of roots.*

Proof. a) Since F is finite valued, then

$$F_\gamma = \{v \in F \mid v_\alpha = 0 \text{ for all } \alpha \geqq \gamma\} \quad (\gamma \in \Gamma)$$

are all the regular subgroups of F and clearly $F^\gamma/F_\gamma \cong Z$.

b) If Γ has only a finite number of roots, then $F(\Gamma, Z) = V(\Gamma, Z)$. Conversely, if Γ has an infinite number of roots then there exists an infinite disjoint subset $\{\gamma_i \mid i \in I\}$ in Γ. Then V contains a copy $\prod_{i=1}^{\infty} Z_i$ and so $V \notin \mathscr{C}$.

8. An hereditary discrete *o*-group that is not generalized discrete. Let $H = (\sum_{i=1}^{\infty} Z_i) \oplus Q$, lexicographically ordered from the right, and let G be the subgroup of H generated by $(1, 0, 0, \ldots; \frac{1}{2})$, $(0, 1, 0, \ldots; \frac{1}{4})$, $(0, 0, 1, 0, \ldots; \frac{1}{8})$, ... Let $O \neq S$ be a subgroup of G and consider $0 \neq s \in S$ with a minimal value $v(s)$ (note that $\Gamma = \{1, 2, 3, \ldots; \infty\}$). If $v(s) = n$, then $s = (s_1, \ldots. s_n, 0, \ldots; 0)$, with $s_n \neq 0$. The set of all such elements in S together with 0 is a subgroup of S – in fact, it is a convex cyclic subgroup of S and so S is discrete. If $v(s) = \infty$, then the map $(x_1, \ldots; y) \to y$ is an o-isomorphism of S into $U = \{m/2^n \mid m, n \in Z\}$. Now a straightforward computation shows that a non-cyclic subgroup of U is of the form $U \cdot m$ where m is a fixed odd positive integer. Thus, if S is not cyclic, it contains elements $(x_1; \frac{1}{2}m)$, $(x_2; \frac{1}{4}m), \ldots$, where the $x_j \in \sum_{i=1}^{\infty} Z_i$. Since $m \neq 0$, $x_1 \neq 0$. Also it follows that $2x_2 = x_1$, $4x_3 = x_1, \ldots$. Thus x_1 is divisible by all powers of 2; which is impossible.

9. An hereditary discrete o-group of rank 2 that is not generalized discrete. Let G be the subgroup of $Q \oplus Q$ that is generated by $\{(n/p_n, 1/p_n) \mid p_n \text{ is the } n\text{-th prime}\}$. Then a rather complicated computation shows that each rank one subgroup is cyclic. In particular, G is indecomposable, for otherwise $G \cong Z \oplus Z$. Define $(a, b) \in G$ to be positive if $b > 0$, or $b = 0$ and $a > 0$. Then G is an o-group and $Z \times 0$ is the only proper convex l-subgroup. Note $V(\Gamma, G^\gamma/G_\gamma) \cong Z \oplus S$, where S is the group of all rationals with square-free denominators. Now any rank two subgroup of $Z \oplus S$ is decomposable, so G can't be embedded in $V(\Gamma, G^\gamma/G_\gamma)$ (See [3] for another proof of this).

Finally, each archimedean subgroup of G is rank one and hence cyclic. If T is a non-archimedean subgroup, then it must contain an element of the form $(t, 0)$ with $t \neq 0$ and so have a cyclic convex subgroup. Thus G is hereditary discrete.

10. A lexicographic extension of Z by Q that is hereditary discrete. In [6] FUCHS and LOONSTRA describe how to construct a torsion free group G of rank two such that each subgroup of rank one is cylic and each torsion free quotient group of rank one is divisible. Pick $0 \neq g \in G$ and let C be the pure subgroup of G that is generated by g. Then C is cyclic and $G/C \cong Q$. Now G is indecomposable and so cannot be embedded into $Z \oplus Q$. If we order G so that C is the proper convex subgroup of G, then G is herediatary discrete, but not generalized discrete.

Note also that this is a counterexample to a theorem of Ribenboim [10] that asserts that G can be embedded in $V(\Gamma, G^{\gamma}/G_p) \cong Z \oplus Q$.

Hill and Mott [7] show that Ext $(Q, Z) = R$ and hence there exists a non-splitting extension of Z by Q. This also contradicts Ribenboim's theorem. The example by Fuchs and Loonstra is more constructive: they also show that Z is the endomorphism ring of G and G/nG is cyclic of order n for each positive integer n.

References

[1] *Conrad, P., Harvey, J.* and *Holland, C.*: "The Hahn embedding theorem for abelian lattice ordered groups", Trans. Amer. Math. Soc. *108* (1963), 143—169.

[2] *Conrad, P.* and *McAlister, D.*: "The completion of a lattice ordered group", J. Australian Math. Soc. *9* (1969), 182—208.

[3] *Conrad, P.*: Lattice Ordered Groups, Tulane University (1970) New Orleans.

[4] *Conrad, P.*: "Epi-archimedean groups" to appear Czech. Math. J.

[5] *Conrad, P.*: "Countable vector lattices" to appear Bul. Australian Math. Soc.

[6] *Fuchs, L.* and *Loonstra, F.*: "On the cancellation of modules in direct sums over Dedekind domains", Indagationes Math. *33* (1971), 163—169.

[7] *Hill, P.* and *Mott, J.*: "Embedding theorems and generalized discrete ordered abelian groups", Trans. Amer. Math. Soc. *175* (1973) 283—297.

[8] *Martinez J.*: "Archimedean-like classes of lattice ordered groups", Trans. Amer. Math. Soc. *186* (1973) 33—49.

[9] *Mott, J.*: "Generalized discrete *l*-groups", (Preprint).

[10] *Ribenboim, P.*: "Sur les groups totalment ordonnés et l'arithmétique des anneaux des valuation", Summa Brasil Math. *4* (1958) 1—64.

[11] *Sankaran, N.* and *Venkataraman, R.*: "A generalization of the ordered group of integers", Math. Zeit. *79* (1962) 21—31.

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